GLOBAL SOLUTIONS TO THE NONLINEAR
HYPERBOLIC PROBLEM

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ABSTRACT

We prove the existence and uniqueness of global regular solutions to the mixed problem for the nonlinear hyperbolic equation with nonlinear damping.

\[ u_{tt} - a(u)u + |u_t|^{\rho} u_t = f(x,t) \text{ in } (0,1) \times (0,T) = Q, \]
\[ u(0,t) = u(1,t) = 0, \]
\[ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \]

Where \( a(u) \geq a_0 > 0, \rho > 1 \). No restrictions on a size of \( u_0, u_1, f \) are imposed.

It is well-known that quasilinear hyperbolic equations, generally speaking, do not have regular solutions for all \( t > 0 \). Their solutions can blow up at a finite period of time. See examples of such singularities in [1, 3]. On the other hand, it was observed that adding a linear damping to the nonlinear hyperbolic equations one can expect the existence of global regular solutions provided initial conditions and right-hand side have sufficiently small appropriate norms, [2]. Moreover, in [3] was shown that the presence of the nonlinear damping allows to prove the existence of regular solutions for the equation

\[ K(u)u_{tt} - \Delta u + |u_t|^{\rho} u_t = f, \quad K > 0. \]

without restrictions on a size of the initial data and \( f \).

Later, using the idea of [3], we proved in [4] the existence of regular solutions for the damped Carrier equation.

\[ u_{tt} - M \left( u(t)^2 \right) \Delta u + \alpha |u_t|^{\rho} u_t = f \]

without smallness conditions for the initial data and \( f \).

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Here, we continue to exploit this idea and consider the following nonlinear mixed problem.

\[ u_{tt} - a(u) u_t |u_t|^\rho u_t = f(x,t) \quad \text{in} \quad Q = (0,1) \times (0,T), \quad (1) \]

\[ u(0,t) = u(1,t) = 0, \quad (2) \]

\[ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad (3) \]

where \( a(u) \) is a smooth positive function.

Unlike the Carrier equations, (1) has local nonlinearities: the function \( a(u) \) depends on a solution; and the function \( M \left( \int u(t)^2 \right) \) depends on the \( L_2 \)-norm of it. This difference makes study of (1)-(3) more complicated and forces us to consider only the one-dimensional case. Nevertheless, the basic technique is similar to one used in the case of the Carrier equation in [4]. Under natural conditions for \( a(u) \), we prove the existence and uniqueness of regular solutions to (1)-(3) without any restrictions on a size of \( u_0, u_1, f \).

**Assumptions.**

A1. \( a(u) \in C^1(R); \ a(u) \geq a_0 > 0. \)

A2. \( |a_u| \leq Aa(u). \)

A3. \( 1 < \rho, \)

Where \( a_0, A \) are positive constants.

In the sequel, we use standard notations for functional spaces, see [5].

**Theorem.** Let \( T \) be an arbitrary positive number; \( u_0 \in H^2(0,1) \cap H^1_0(0,1) \) and A1-A3 hold. The for any \( f \) such that \( f, f_t \in L^2 \), there exists a unique regular solution to (1)-(3), \( u(x,t) \):

\[ u \in L^\infty(0,T;H^2(0,1) \cap H^1_0(0,1)), \]

\[ u_t \in L^\infty(0,T;H^1_0(0,1)), \]

\[ u_{tt} \in L^\infty(0,T;L^2(0,1)). \]
The scheme of the proof.

The assumption A1 allows to rewrite (1) in the equivalent form

\[
\frac{1}{a(u)} u_a - u + \frac{u_t}{a(u)} u_t = f
\]  

(4)

Obviously, solutions to (4), (2), (3) are also solutions to (1)-(3). Equation (4) is similar to the class of quasilinear hyperbolic equations studied in [4] with exception of the damping that can degenerate when \( a(u) \to \infty \). Also, the coefficient of \( u_{tt} \), \( \left( \frac{1}{a(u)} \right) \) can be zero. It means that (4) is the degenerated hyperbolic equation. Moreover, dependence of the damping term of \( u \) and \( u_t \) brings more difficulties to analysis of (4), (2), (3). Nevertheless, we can employ in our case the technique developed in [4].

Approximate solutions to (4), (2), (3) will be constructed by the Faedo-Galerkin method with the special basis. Let \( w_j(x) \) be eigen-functions of the problem

\[
w_{jxx} + \lambda_j w_j = 0 \text{ in } (0, 1),
\]

\[
w_j(0) = w_j(1) = 0.
\]

Then for \( \varepsilon > 0 \)

\[
u^N_{\varepsilon}(x, t) = \sum_{j=1}^{N} g^N_j(t) w_j(x),
\]

where unknown functions \( g_j(t) \) are solutions to the following Cauchy problem

\[
\left( \frac{1}{a(u^N_{\varepsilon})} \right) u^N_{\varepsilon t} + \left( \frac{u^N_{\varepsilon t}}{a(u^N_{\varepsilon})} \right) u^N_{\varepsilon t} + \left( \frac{u^N_{\varepsilon t}}{a(u^N_{\varepsilon})} \right) u^N_{\varepsilon t} = \left( \frac{1}{a(u^N_{\varepsilon})} \right) w_j(t).
\]

(6)
\[ g_j^N(0) = \alpha_j = (u_i, w_j), \]
\[ g_j^N(0) = \beta_j = (u_1, w_j), \quad j = 1, ..., N. \]  

(7)

Here \((u, v)(t) = \int u(x, t) v(x, t) dx\).

The system of nonlinear ordinary differential equations (6) is not solved with respect to \(g_j^N\), but it can be transformed to a normal system of ODE due to the fact that the matrix \(I + \left(\sum_{a, \epsilon N} \frac{1}{a} w_j, w_i\right)\), \(i, j = 1, ..., N\), is positive for \(\epsilon > 0\), see A1.

Hence, the Cauchy problem (6), (7) has solutions \(g_j^N\) at some interval \((0, T_N)\), and we need a priori estimates in order to prolongate solutions to the interval \((0, T)\) and to pass to the limits when \(\epsilon \to 0\) and \(N \to \infty\).

The First Estimate

Multiplying (6) by \(g_j^N\) and using A1-A3, after some calculations we come to the inequality

\[
\int \left( \frac{|u_{xu}^N(x, t)|^2}{a(u_{xu}^N)} + |u_{xx}^N(x, t)|^2 \right) dx + \int_0^t \int \frac{|u_{a}^N|^p+2}{a(u_{a}^N)} dx dt \leq \\
C_1 \left( \|u_0\|_{H_0^1(0, 1)}, \|u_1\|_{L^3(0, 1)}, \|f\|_{L^3(Q)} \right),
\]

(8)

where C does not depend on \(\epsilon, N, t\).

From here and from A1.

\[
\sup_{t \in (0, T)^{\epsilon=0.1}} \max_{x \in (0, 1)} |u(x, t)| \leq C_2.
\]

(9)
This imply
\[ a_0 \leq a(u) \leq M < \infty \]  
(10)
Where \( C_2, M \) do not depend on \( \epsilon, N, T \).

The Second Estimate

Taking the derivative of (6) with respect to \( t \), multiplying the result by \( g_{\mu \tau}^N \), after standard transformations we obtain

\[
\int_0^1 \left( |u_{\epsilon \tau}^N(x, t)|^2 + |u_{\epsilon \tau}^N(x, t)|^2 \right) dx \leq 
\]

\[
C_3 \left( |u_0|_{H^2(0,1)} + |u_1|_{H^2(0,1)} + \|f\|_{H^1(0, T, L^3(\mathbb{R}))} \right),
\]

where \( C_3 \) does not depend on \( \epsilon, N, t \).

Finally, taking into account (5) and estimates (8)-(11), we get

\[
\int_0^1 |u_{\epsilon \tau}|_{XX} (x, t)^2 \, dx \leq C_4 \]

(12)

With (8)-(12) it is easy to pass to the limits in (6) when \( N \to \infty, \epsilon \to 0 \), hence, to prove the existence of regular solutions to (4), (2), (3) and, consequently, to (1)-(3). Uniqueness may be proved in the usual way.

Theorem is proved.

Remark. The function \( a(u) \) can depend on \( x, t \).
REFERENCES

1. ROZHDESTVENSKII, B.L., YANENKO, N. N., Systems of quasilinear equations and their applications to gas dynamics, Providence, AMS, (1983).


