GLOBAL EXISTENCE OF SOLUTIONS FOR THE DEGENERATE WAVE EQUATIONS OF KIRCHHOFF TYPE WITH NONLINEAR DISSIPATIVE TERM OF VARIABLE COEFFICIENT

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ABSTRACT. In this paper we investigate the global existence and decay of solutions to a degenerate wave equations with nonlinear dissipative term of variable coefficient.

1. INTRODUCTION

The objective of this paper is to study the global existence and the decay property of the nonlinear system:

\[
(P) \quad \begin{align*}
  u'' - M \left( \int_{\Omega} |\nabla u|^2 dx \right) \Delta u + a(x)g(u') &= 0 \quad \text{in} \quad Q = \Omega \times ]0, T[ \\
  u &= 0 \quad \text{in} \quad \sum = \Gamma \times ]0, T[ \\
  u(x, 0) &= u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u_1(x) \quad \text{in} \quad \Omega
\end{align*}
\]

where $\Omega$ is a bounded open domain in $\mathbb{R}^N$ ($N \geq 1$) with a smooth boundary $\Gamma$, $T > 0$, $M(s) = s$, $\forall s \geq 0$, $\Delta$ is the Laplace operator, $g$ and $a$ are functions satisfying suitable conditions.

Existence of global solutions to the system $(P)$ has been investigated by many authors (ef. [1], [2], [4], [7], [8], etc) for different and positive constant, with a positive or non-negative function. Mochizuki [5] investigated the nondegenerate problem with dissipative term $a(x, t)u'$. Our purpose in this time is to prove the global existence and decay rate of solution for the case: $M(s) = s$ and $a(x)$ is a positive function.

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2. PRELIMINARIES

In the sequel $L^p(\Omega)$, $1 \leq p < \infty$ will denote the collection of $L$-functions which are $p$th-integrable over $\Omega$. For $m \in \mathbb{N}$, the space $H^m(\Omega)$ is the Sobolev class of the functions of the spatial variable $x$ which along with their first $m$ derivates belong to $L^2(\Omega)$ (See, for example Medeiros & Milla Miranda [3]) and the closure in $H^m(\Omega)$ of the space $D(\Omega)$ of the test functions on $\Omega$ is denoted by $H^m_0(\Omega)$ the inner product and norm of $L^2(\Omega)$ are represented by $(\cdot, \cdot)$ and $| \cdot |$ respectively.

Let $X$ be a Banach space, $T > 0$ or $T = +\infty$ and $1 \leq p \leq \infty$, denote by $L^p(0, T; X)$ the Banach space of all measurable functions $u : [0, T] \rightarrow X$ such that $t \mapsto |u(t)|_X$ is in $L^p(0, T)$, with norm

$$
|u|_{L^p(0, T; X)} = \left( \int_0^T |u(t)|^p_X dt \right)^{1/p},
$$

if $1 \leq p < \infty$, and if $p = \infty$, then

$$
|u|_{L^\infty(0, T; X)} = \text{ess sup}|u(t)|_X.
$$

We use the following well-known lemmas without the proof in this paper:

Lemma 2.1. (Sobolev - Poincaré) If $u \in H^1_0(\Omega)$ then $u \in L^q(\Omega)$ and the inequality

$$
|u|_q \leq C_q |\nabla u|
$$

holds, where $q$ is a number satisfying $1 \leq q \leq \frac{2N}{N-2}$ if $N > 2$ and $1 \leq q < \infty$ if $N = 2$ and $1 \leq q \leq \infty$ if $N = 1$.

Lemma 2.2. (Nakao [6]) Let $\phi(t)$ be a nonnegative bounded function on $[0, \infty[$ satisfying

$$
\sup_{t \leq s \leq t+1} \phi(t)^{1+r} \leq k_0 (\phi(t) - \phi(t+1))
$$

for $r > 0$ and $k_0 > 0$. Then

$$
\phi(t) \leq C(1+t)^{-\frac{1}{r}}, \ \text{for all } t \geq 0
$$

where $C > 0$ is a positive constant depending on $\phi(0)$ and other known constants.
3. THE MAIN RESULT

Theorem 3.1. Let \( g : \mathbb{R} \to \mathbb{R} \) be a non-decreasing continuous function such that

\[
g(0) = 0 \tag{3.1}
\]
\[
g'(s) \geq \tau > 0 \tag{3.2}
\]
\[
|g(s)| \leq C_0 |s|^q \tag{3.3}
\]

\( C_0 \) and \( \tau \) are two positive constants and \( q > 1 \) is such that \( (N-2)q \leq N+2 \).

The function \( a \) satisfies

\[
a \in W^{1,\infty}(\Omega) \tag{3.4}
\]
\[
a(x) \geq a_0 > 0, \quad \forall x \in \Omega \tag{3.5}
\]

Let \( u_0 \in H^1_0(\Omega) \cap H^2(\Omega) \) with \( u_0(x) \neq 0, \forall x \in \Omega \) and \( u_1 \in H^1_0(\Omega) \cap L^2(\Omega) \), then exist \( \epsilon_0 > 0 \) with the following property:

For each \( \{u_0, u_1\} \) satisfying

\[
2 \left( \frac{\|\nabla u_1\|^2}{\|\nabla u_0\|^2} + |\Delta u_0|^2 \right) < \epsilon_0 \tag{3.6}
\]

then exists only one solution \( u : \Omega \times [0, T[ \to \mathbb{R} \) such that

\[
u \in L^\infty(0, T; H^1_0 \cap H^2) \tag{3.7}
\]
\[
u' \in L^\infty(0, T; H^1_0) \tag{3.8}
\]
\[
u'' \in L^\infty(0, T; L^2) \tag{3.9}
\]

\[
\frac{d}{dt}(u'(t), w) - \frac{d}{dt} \left( \frac{\|\nabla u(t)\|^2}{\|\nabla u_0\|^2} + |\Delta u(t)|^2 \right) + (a(x)g(u'), w) = 0, \tag{3.10}
\]

\( \forall w \in H^1_0(\Omega) \), in the sense of \( D'(0,T) \)

\[
u(0) = u_0, \quad u'(0) = u_1 \tag{3.11}
\]
\[
|\nabla u(t)| > 0, \quad \forall t \in [0, +, \infty[ \tag{3.12}
\]

Proof. We will use the Faedo-Galerkin method's.
We consider \( \{w_j\}_{j \in \mathbb{N}} \) an orthonormal basis of \( H_0^1(\Omega) \cap H^2(\Omega) \) and denote by \( V_m = [w_1, \ldots, w_m] \) the subspace of \( H_0^1(\Omega) \cap H^2(\Omega) \) spanned by the first vectors of \( \{w_j\}_{j \in \mathbb{N}} \).

We seek \( u_m(t) \) in the form

\[
u_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j
\]

such that, for all \( w \) in \( V_m \), \( u_m(t) \) satisfies the approximate equation

\[
(u_m''(t), w) = \left( m^{-1} + |\nabla u_m(t)|^2 \right) (\Delta u_m(t), w) + (a(x)g(u_m'(t)), w) = 0 \tag{3.13}
\]

with the following initial conditions

\[
u_m(0) = u_{0,m} \rightarrow u_0 \text{ in } H_0^1(\Omega) \cap H^2(\Omega) \tag{3.14}
\]

\[
u'_m(0) = u_{1,m} \rightarrow u_1 \text{ in } H_0^1(\Omega) \cap L^2(\Omega) \tag{3.15}
\]

Using (3.3) we deduce from (3.25) that \( (g(u_{1m})) \) is bounded in \( L^2(\Omega) \). Under these conditions, the system (3.13) - (3.15) has a local solution \( u_m(t) \) over the interval \([0, T_m]\). We shall see that \( u_m(t) \) can be extended for all \( t \geq 0 \).

A priori Estimative I

For \( w = 2u_m'(t) \) in (3.13) we find

\[
\frac{d}{dt} \left\{ |u_m'(t)|^2 + m^{-1} |\nabla u_m(t)|^2 + \frac{1}{2} |\nabla u_m(t)|^4 \right\} + 2\int_{\Omega} a(x)g(u_m'(t))u_m'(t)dx = 0
\]

Integrate in \([0, t], t < T_m\), to obtain

\[
|u_m'(t)|^2 + \frac{1}{2} |\nabla u_m(t)|^4 + 2\int_0^t \int_{\Omega} a(x)g(u_m'(s))u_m'(s)ds \leq |u_1|^2 + |\nabla u_0|^2 + |\nabla u_0|^4 \tag{3.16}
\]

It follows that

\[
|u_m'(t)| \leq k \tag{3.17}
\]

\[
|\nabla u_m(t)| \leq k
\]
Then we extend the approximate solution \( u_m(t) \) to the interval \([0, T]\) for any \( 0 < T < \infty \).

From now on we denote by \( C \) various constants independent of \( m \) and \( t \) in \([0, T]\).

Also it follows from (3.16), (3.3) and (3.5) that

\[
\int_0^t \int_{\Omega} a(x) g(u_m') u_m' \, dx \, dt \leq C 
\]

(3.18)

\[
\int_0^t \int_{\Omega} |q(u_m')|^{\frac{q+1}{q}} \, dx \, dt \leq C
\]

(3.19)

A priori Estimative II

Putting \( w = -2 \Delta u_m' \) in (3.13) we have

\[
\frac{d}{dt} \left\{ |\nabla u_m'(t)|^2 + (m^{-1} + |\nabla u_m(t)|^2) |\Delta u_m(t)|^2 \right\} +
\]

\[
+ 2 \left( \nabla \left( a(x) g(u_m') \right), \nabla u_m' \right) = \left( \frac{d}{dt} |\nabla u_m(t)|^2 \right) |\Delta u_m(t)|^2
\]

Let us define

\[
F_m(t) = \frac{|\nabla u_m'(t)|^2}{m^{-1} + |\nabla u_m(t)|^2} + |\Delta u_m(t)|^2 = f_m(t) + |\Delta u_m(t)|^2
\]

A simple computation shows that

\[
F_m'(t) = -2(a(x) \nabla g(u_m'), \nabla u_m') - (g(u_m') \nabla a(x), \nabla u_m')
\]

\[
= \frac{2(\nabla u_m(t), \nabla u_m'(t)) \nabla u_m'(t)}{(m^{-1} + |\nabla u_m(t)|^2)^2}
\]

(3.20)

But:

\[
(a \nabla g(u'), \nabla u') = (a g'(u') \nabla u', \nabla u') \geq \tau a_0 |\nabla u'|^2
\]

(3.21)

and using (3.3), the Sobolev embedding and (3.4) give

\[
|\nabla (\Delta u', \nabla u')| \leq C_0 C_{2q} |a|_{1, \infty} |\nabla u'|^{q+1}
\]

(3.22)

From (3.22), (3.21), in (3.20) we get
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\[ F_m'(t) \leq -2\tau a_0 \frac{|\nabla u_m'|^2}{m^{-1} + |\nabla u_m|^2} + 2C_0C_2q|a|_{1,\infty} \frac{|\nabla u_m'|^{q+1}}{m^{-1} + |\nabla u_m|^2} \]
\[ + 2 \left[ \frac{|\nabla u_m'|^2}{m^{-1} + |\nabla u_m|^2} \right]^{3/2} \]  
(3.23)

That is:

\[ F_m'(t) \leq 2 \left[ f_m^{1/2}(t) + \delta_0 f_m^{\frac{q-1}{2}} - \delta \right] f_m(t) \]  
(3.24)

where \( \delta_0 = C_0C_2q|a|_{1,\infty}(1 + k)^{\frac{q-1}{2}}, \delta = \tau a_0. \)

Integrating (3.24) from 0 to \( t \) we have

\[ F_m(t) \leq 2 \int_0^t \left( f_m^{1/2}(s) + \delta_0 f_m^{\frac{q-1}{2}}(s) - \delta \right) f_m(s) ds + F_m(0) \]  
(3.25)

Now, since \( F_m(0) \rightarrow F(0) \), it follows of (3.6) that

\[ F_m(0) < \epsilon_0 \]  
(3.26)

for sufficiently large \( m. \)

We shall prove that

\[ f_m^{1/2}(t) + \delta_0 f_m^{\frac{q-1}{2}}(t) < \frac{\delta}{2}, \quad \forall t \in [0, \infty[ \]  
(3.27)

for \( \epsilon_0 = \min \left\{ \left( \frac{\delta}{4} \right)^2, \left( \frac{\delta}{4\delta_0} \right)^{\frac{q-1}{2}} \right\} \). In fact

\[ f_m^{1/2}(0) \leq f_m^{1/2}(0) < \epsilon_0^{1/2} \leq \frac{\delta}{4} \]

\[ \delta_0 f_m^{\frac{q-1}{2}}(0) \leq \delta_0 F_m^{\frac{q-1}{2}}(0) < \delta_0 \epsilon_0^{\frac{q-1}{2}} \leq \frac{\delta}{4} \]

and thus we have

\[ f_m^{1/2}(0) + \delta_0 f_m^{\frac{q-1}{2}}(0) < \frac{\delta}{2} \]

Suppose, then, that (3.27) does not hold for all \( t \geq 0. \)

Because of the continuity of \( f_m(t) \), there is \( t^* > 0 \) such that

\[ f_m^{1/2}(t) + \delta_0 f_m^{\frac{q-1}{2}}(t) < \frac{\delta}{2} \quad \text{for} \quad 0 \leq t < t^* \]  
(3.28)
\[ f_m^{1/2}(t^*) + \delta_0 f_m^{2-1}(t^*) = \frac{\delta}{2} \tag{3.29} \]

(3.25), (3.26) and (3.28) gives:

\[ F_m(t^*) \leq F_m(0) < \epsilon_0 \]

This inequality yields

\[ f_m^{1/2}(t^*) + \delta_0 f_m^{2-1}(t^*) < \frac{\delta}{2} \]

a contradiction to (3.19). Hence (3.27) is true.

From (3.25) and (3.17) we obtain

\[ F_m(t) + \delta \int_{0}^{t} \frac{\left| \nabla u_m(s) \right|^2}{m^{-1} + \left| \nabla u_m(s) \right|^2} ds \leq C \]

wich implies

\[ \left| \nabla u_m(t) \right| \leq C \tag{3.30} \]
\[ \frac{\left| \nabla u_m(t) \right|^2}{m^{-1} + \left| \nabla u_m(t) \right|^2} \leq C \tag{3.31} \]
\[ \int_{0}^{t} \frac{\left| \nabla u_m(s) \right|^2}{m^{-1} + \left| \nabla u_m(s) \right|^2} ds \leq C \tag{3.32} \]

A priori Estimation III

Taking \( w = u''_m(t) \) in (3.13) and choosing \( t = 0 \) we get

\[ \left| u''_m(0) \right| \leq \left( \frac{1}{m} + \left| \nabla u_{0m} \right|^2 \right) \left| \Delta u_{0m} \right| + \left| a_{1,\infty} \right| g(u_{1m}) \]

hence \( u''_m(0) \) is bounded in \( L^2(\Omega) \). Next, by differentiation of (3.13) and putting \( w = 2u''_m(t) \) we find

\[ \frac{d}{dt} \left\{ \left| u''_m(t) \right|^2 + \left( m^{-1} + \left| \nabla u_m(t) \right|^2 \right) \left| \nabla u'_m(t) \right|^2 \right\} + 2 \int_{\Omega} a(x) g'(u_m) (u''_m) dx \]

\[ = 2 \left( \nabla u_m, \nabla u'_m \right) \left| \nabla u'_m(t) \right|^2 + 4 \left( \nabla u_m, \nabla u'_m \right) \int_{\Omega} (\Delta u_m) u''_m dx \]

\[ \leq \left| \nabla u_m \right| \left| \nabla u'_m \right|^3 + 4 \left| \nabla u_m \right| \left| \nabla u'_m \right| \left| u''_m \right| \left| \Delta u_m \right| \leq C + C \left| u''_m(t) \right|^2 \]

where we have used the priori estimatives I and II.

Integrating from 0 to \( t \) we have, using the Gronwall's Inequality:
Passage to the limit.

The proof is essentially included in [8]. For completeness however we shall see the convergence of dissipative term. By applying the Dunford-Pettis and Banach-Bourbaki theorems we conclude from (3.17) - (3.19), (3.30) - (3.32) and (3.33), replacing the sequence $u_m$ with a subsequence if needed, that

$$\begin{align*}
|u_m''(t)| & \leq C \quad (3.33) \\

\text{weak-star in } L^\infty(0,T; H_0^1 \cap H^2) & \quad (3.34) \\

\text{weak-star in } L^\infty(0,T; H_0^1) & \quad (3.35) \\

\text{weak-star in } L^\infty(0,T; L^2) & \quad (3.36) \\

\text{almost everywhere in } Q & \quad (3.37) \\

g(u_m') & \to \chi \quad \text{weak in } L^{\frac{2+1}{v}}(\Omega) \quad (3.38)
\end{align*}$$

$$\begin{align*}
|\nabla u_m|^2 \Delta u_m & \to \psi \quad \text{weak-star in } L^\infty(0,T; L^2) \\

\text{for all } v \in L^{q+1}(0,T; H_0^1(\Omega)) \\

\text{in fact, from (3.18) and Fatou's Lemma } u'g(u') & \in L'(Q). \text{ This yield } g(u') \in L^1(Q). \\

\text{On the other hand (3.37) and the continuity of } g \text{ we deduce that} \\

g(u_m') & \to g(u') \quad \text{a.e. in } Q
\end{align*}$$

Let $E \subseteq Q$ and set

$$
E_1 = \{(x,t) \in E : g(u_m'(x,t)) \leq |E|^{-1/2}\}; \quad E_2 = E - E_1
$$

when $|E|$ is the measure of $E$.

If $h(r) = \inf\{|x| : x \in \mathbb{R} \text{ and } |g(x)| \geq r\}$ then we have

$$
\int_E |g(u_m')|dxdt \leq |E|^{1/2} + [h(|E|^{-1/2})]^{-1} \int_{E_2} |u_m'g(u_m')|dxdt
$$

Applying (3.18) we have that
From Vitali's convergence theorem we get

\[ g(u'_m) \to g(u') \quad \text{in} \quad L^1(Q) \]

Hence we have

\[
\int_0^T \int_{\Omega} a(x) |g(u'_m) - g(u')| dx dt \leq |a|_{1,\infty} |g(u'_m) - g(u')|_{L^1(Q)} \to 0
\]
as \( m \to \infty \).

So we get that

\[ a(x)g(u'_m) \to a(x)g(u') \quad \text{in} \quad L'(Q) \]

and from (3.38)

\[ a(x)g(u'_m) \to a(x)g(u') \quad \text{weak-star in} \quad L^{\frac{q+1}{q}}(Q) \]

this implies (3.39).

We now prove that \( |\nabla u(t)| > 0 \) for all \( t \geq 0 \). We need the following lemma

**Lemma.** If \( v : [-T, T] \to H^1_0(\Omega) \cap H^2(\Omega) \) is a weak solution of

\[
\begin{align*}
v''(t) - |\nabla v(t)|^2 \Delta v(t) + a(x)g(v'(t)) &= 0, \quad -T \leq t \leq T \\
v(0) &= 0, \quad v'(0) = 0
\end{align*}
\]

then \( v(t) = 0, \text{ for } t \in [-T, T] \).

**Proof.** Multiplying with \( 2v'(t) \) we have

\[
\frac{d}{dt} \left\{ |v'(t)|^2 + \frac{1}{2} |\nabla v(t)|^4 \right\} + 2 \int_{\Omega} a(x)g(v'(t))v'(t) dx = 0
\]

and integrating in \([0, t]\), using (3.2), gives

\[
|v'(t)|^2 + \frac{1}{2} |\nabla v(t)|^4 \leq 2a_0 |v(t)| \int_0^t |v'(s)|^2 ds
\]

Gronwall’s Lemma assures \( v'(t) = 0 \) and \( v(t) = 0 \) for all \( t \in [-T, T] \).

This concludes the proof of this lemma. \( \square \)
We now turn to the proof of $\nabla u(t) > 0, \forall t \geq 0$. Suppose that there exists a number $T > 0$ such that $\nabla u(T) = 0$. Since the a priori estimates imply that $\frac{|\nabla' u(T)|}{|\nabla u(T)|}$ is bounded, then

$$|\nabla' u(T)| \leq C|\nabla u(T)| = 0.$$  

Hence, the above lemma implies that $u(t) = 0$, for $0 \leq t \leq T$, which contradicts $u_0(x) \neq 0$.

The uniqueness is a consequence of the monotonicity of $g$ and Gronwall's Inequality. We shall omit the proof. Since it can be obtained in a standard way.

**Theorem 3.2. (Energy Decay)** In addition to (3.1) - (3.3), assume that

$$g(s) \leq C_1|s| \text{ if } |s| \leq 1$$

(3.40)

then the total energy

$$E(t) = |u'(t)|^2 + \frac{1}{2}|\nabla u(t)|^4$$

satisfies

$$E(t) \leq \frac{C_2}{(1 + t)^2}, \text{ for all } t \geq 0$$

where $C_1$ and $C_2$ are positive constants.

**Proof.** Taking the scalar product of the first equation of (P) with $2u'$ and integrating over $\Omega$ we obtain

$$E'(t) + 2 \int_{\Omega} a(x)u'g(u')dx = 0$$

(3.41)

Integrating (3.41) over $[t, t + 1]$ we get

$$2 \int_{t}^{t+1} \int_{\Omega} a(x)u'g(u')dxds = E(t) - E(t + 1) \equiv D(t)^2$$

From this we obtain

$$\int_{t}^{t+1} |u'(s)|^2ds \leq CD(t)^2$$

(3.43)
By the Mean Value theorem there exist two points \( t_1 \in \left[ t, t + \frac{1}{4} \right] \) and \( t_2 \in \left[ t + \frac{3}{4}, t + 1 \right] \) such that

\[
|u'(t_i)|^2 \leq CD(t)^2, \quad i = 1, 2
\]

(3.44)

thus, multiplying of the first equation in (P) by \( u \) and integrating it over \( \Omega \times ]t_1, t_2[ \), we have from (3.42) - (3.44) and Lemma 2.1

\[
\int_{t_1}^{t_2} |\nabla u(s)|^4 \, ds = \int_{t_1}^{t_2} |u'(s)|^2 \, ds - (u'(t_1), u(t_1)) + (u'(t_2), u(t_2))
\]

\[
- \int_{t_1}^{t_2} (a(x)g(u'), u) \, ds
\]

\[
\leq C(D(t)^2 + D(t)E(t)^{1/4}) = A(t)^2
\]

(3.45)

From (3.43) and (3.45) we conclude that

\[
E(t_2) \leq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} E(s) \, ds \leq CA(t)^2
\]

therefore, we obtain

\[
E(t) = E(t_2) + 2 \int_t^{t_2} \int_\Omega a(x)g(u')u' \, dx \, ds
\]

\[
\leq CA(t) \leq C\{D(t)^2 + D(t)E(t)^{1/4}\}
\]

Using Young's inequality, noting that \( D(t)^2 \leq E(t) \leq E(0) \) we get

\[
\sup_{t \leq s \leq t+1} E(s)^{3/2} \leq CD(t)^2 = C(E(t) - E(t + 1))
\]

Hence, lemma 2.2 gives

\[
E(t) \leq C(1 + t)^{-2}, \quad \forall t \geq 0
\]

this ends the proof of theorem. \( \square \)
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5. REFERENCES


