GLOBAL EXISTENCE AND EXPONENTIAL DECAY TO THE WAVE EQUATION WITH LOCALIZED FRICTIONAL DAMPING*

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ABSTRACT.- In this paper we show the global existence and the exponential decay of solutions of the one-dimensional wave equation with localized frictional damping.

1. INTRODUCTION

In this paper we study the existence of global smooth solution to the wave equations when the dissipation given by the frictional term is effective only in a part of the domain. Moreover we show that this local dissipation is strong enough to produce uniform decay of the solutions. Let us consider a string of length $L$. The mathematical model which defines the deformations of the string is given by

\[ u_{tt} - \left[ \sigma \left( u_x \right)_x \right] + a(x)u_t = 0 \quad \text{in} \quad [0, L] \times [0, \infty), \quad (1.1) \]

with initial data

\[ u(x, 0) = u_0(x) \quad u_t(x, 0) = u_1(x) \quad \text{in} \quad [0, L] \quad (1.2) \]

and boundary conditions

\[ u(L, t) = u(0, t) = 0 \quad \forall t \geq 0 \quad (1.3) \]

where by $u = u(x, t)$ we denote the displacement, $\sigma$ is a function satisfying

\[ \sigma \in C^3(\mathbb{R}) \quad \text{such that} \quad \sigma'(0) > 0 \quad \text{and} \]

\[ a \in C^1(0, L) \quad a \geq 0, \quad a(x) > 0 \quad \text{in} \quad [I_0, L] \quad \text{where} \quad 0 \leq I_0 < L. \quad (1.5) \]

It is well known by now that in non linear elasticity when $a = 0$, the equation is conservative and the solution will blow up in a finite time, no matter how small and regular the initial data is. On the other hand, when the dissipation is effective in the whole domain, then when the initial data is taken small enough there exists only one global smooth solution of the system.

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The problem we consider here is intermediary between these two cases. The natural questions under in our case is about the strongness of the dissipation. We can ask, is the local dissipation strong enough to secure estimates and prevent blow up? The mains result of this paper is to give a positive answer to that question.

We will prove that the problem (1.1) - (1.5) is well posed, that is if \( a \geq 0 \), with \( a \) zero outside of a boundy of 0 or \( L \), the problem has a global smooth solution which decays exponentially as time goes to infinity, provided the initial data is small enough.

The main result of this paper is summarized in the following Theorem.

**Theorem 1.1 (Main Result)** Let us suppose that the initial data satisfies

\[
\|u_0\|_{H^3(0,L)}^2 + \|u_1\|_{H^2(0,L)}^2 + \|u_t(0)\|_{H^1(0,L)}^2 < \epsilon^2
\]

for any \( \epsilon > 0 \) and small enough, where \( u_t(0) = \sigma(u_{0,x})u_{0,xx} - a(x)u_t \), the there exists a global smooth solution of problem (1.1) - (1.5) satisfying \( u \in C([0,\infty); H^3) \cap C^1([0,\infty); H^2) \cap C^2([0,\infty); H^1) \), which decays exponentially as time goes to infinity.

The method we use is based on the multiplicative technique. We introduce a new multiplicator which allows us to get the global estimate that we need to show the global existence result.

The main idea is to construct a Liapunov functional \( L \) satisfying

\[
\frac{d}{dt} L(t) \leq -\kappa L(t)
\]

where \( \kappa \) is a positive constant.

**2. EXISTENCE AND ASYMPTOTIC BEHAVIOUR**

Our first step is to establish the local existence result.

**Theorem 2.1 (Local Existence)** Let us take initial data satisfying

\[
u_0 \in H^3(0,L) \cap H^1_0(0,L), \quad u_1 \in H^2(0,L) \cap H^0_0(0,L)
\]

then, there exists \( T > 0 \) and a function

\[
u \in X := C(0,T; H^3) \cap C^1(0,T; H^2) \cap C^2(0,T; H^1)
\]

satisfying problem (1.1) - (1.5).
Proof.- The main idea is to use the fixed point Theorem. To do this we define

\[ W_\mu (0, T) := \left\{ w \in C^0 (0, T; H^2), w \in C^i (0, T; H^{3-i}) \mid i = 1, 2 \right\} \]

\[ w (x, 0) = u_0 (x), \ w_t (x, 0) = u_t (x), \ \| w \|_W \leq \mu \]

\[ T : W_\mu \rightarrow W_\mu \]

\[ v \rightarrow Tv = u \]

where \( u \) is the solution of

\[
\begin{cases}
    u_{tt} - \sigma' (v_x) u_{xx} + a (x) u_t = 0 \\
    u (0) = u_0 \in H^2 (0, L), \ u_t (0) = u_t \in H^1 (0, L) \\
    u (L, t) = u (0, t) = 0
\end{cases}
\]

To show the global existence, it is enough to show that

\[ \lim_{t \rightarrow t_{\text{fin}}} \| u (t) \|_X < C \]

Here we follow similar techniques as in [7]. First study the linearized problem.

\[ u_{tt} - \sigma' (v_x) u_{xx} + a (x) u_t = f \quad \text{in} \quad ]0, L[ \times ]0, T[ \]  

(2.1)

where

\[ f = \left[ \sigma' (u_x) - \sigma' (0) \right] u_{xx} \]

In general we consider

\[ U_{tt} - \sigma' (0) U_{xx} + a (x) U_t = F \quad \text{in} \quad ]0, L[ \times ]0, T[ \]  

\[ U (L, t) = U (0, t) = 0 \quad \forall t \geq 0 \]  

(2.2)

(2.3)

where by \( U, F \) we are denoting \( (U, F) = \{(u, f), (u_t, f_t), (u_{tt}, f_{tt})\} \).

Under the above notations the energy associated is

\[ E (t, U) = \frac{1}{2} \int_0^L U_t^2 + \sigma' (0) U_x^2 \ dx. \]  

(2.4)

Multiplying (2.2) by \( U_t \), performing an integration by parts and using the boundary conditions we get

\[ \frac{dE (t, U)}{dt} = - \int_0^L a (x) U_t^2 \ dx + \int_0^L FU_t \ dx \]  

(2.5)
Observation 1 The Lemmas 2.1, 2.2, 2.3 that will be enunciated, are valid in the cases

\[(U, F) = \{(u, f), (u_t, f_t), (u_{tt}, f_{tt})\}\].

Let us introduce the following functionals

\[
I(t, U) = \int_0^L \alpha(x)U U_t \, dx + \frac{1}{2} \int_0^L a(x)\alpha(x)|U|^2(x) \, dx
\]

\[
J(t, U) = \int_0^L U_t q(x)U_x \, dx
\]

\[
L(t, U) = NE(t, U) + I(t, U) + J(t, U)
\]

where

\[
\alpha \in C^2([0, L]), \quad \alpha(0) < 0, \quad \alpha(L) > 0, \quad \alpha' \leq 0,
\]

\[
q \in C^1(0, L), \quad q(0) = q(L) = 0 \quad \text{and} \quad \alpha, q \text{ satisfying (2.6)}
\]

\[
\alpha(x) + \frac{1}{2} q'(x) \geq C > 0 \quad \text{and} \quad \exists \alpha_0 > 0 \text{ such that}
\]

\[
\alpha_0 \alpha(x) - \alpha(x) + \frac{1}{2} q'(x) \geq D > 0
\]

The above conditions over \(\alpha\) imply that there exists \(L^* \in (0, L_0)\) such that \(\alpha(L^*) = 0\), therefore we have

\[
q'(x) = \begin{cases} 
\alpha(x) + C & \text{in } [L^*, L_0 + \delta_1] \\
-\alpha(x) + C & \text{in } [0, L^*] \cup [L_0 + \delta_2, L]
\end{cases}
\]

such that \(q' \in C(0, L)\). Since \(0 < \delta_1 < \delta_2\) taking \(\int_0^L q'(x) \, dx = 0\).

Lema 2.1 Let \(a, \sigma, \alpha\) be as (1.5), (1.4), (2.6) and \(U\) be the strong solution of (2.2) - (2.3) then the following inequalities follows

\[
\frac{dt}{dt} L(t, U) \leq -\sigma'(0) \int_0^L a(x)|U_x|^2 \, dx + \int_0^L a(x)|U|^2 \, dx + \int_0^L a(x)U F \, dx \tag{2.7}
\]

Proof.- Multiplying (2.2) by \(\alpha(x)U\) and integrating over \([0, L]\), we find that

\[
\frac{1}{2} \frac{d}{dt} \int_0^L a(x)\alpha(x)|U|^2 \, dx = \int_0^L \alpha(x) F U \, dx - \int_0^L \alpha(x)UU_t \, dx + \sigma'(0) \int_0^L \alpha(x)UU_{tt} \, dx \tag{2.8}
\]
Since
\begin{equation}
\int_0^L \alpha(x) U U_{xx} \, dx = - \int_0^L \alpha'(x) U U_x \, dx - \int_0^L \alpha(x) |U_x|^2 \, dx \tag{2.9}
\end{equation}

\begin{equation}
= \frac{1}{2} \int_0^L \alpha''(x) |U|^2 \, dx - \int_0^L \alpha(x) |U_x|^2 \, dx
\end{equation}

Using (2.9) and \( \alpha''(x) \leq 0 \), (2.8) will be
\begin{equation}
\frac{1}{2} \frac{d}{dt} \int_0^L \alpha(x) U U_x \, dx \leq \int_0^L F \alpha(x) U \, dx - \int_0^L \alpha(x) U U_{tt} \, dx - \sigma'(0) \int_0^L \alpha(x) |U_x|^2 \, dx.
\end{equation}

Since
\begin{equation}
\frac{d}{dt} \int_0^L \alpha(x) U U_t \, dx = \int_0^L \alpha(x) |U_t|^2 \, dx + \int_0^L \alpha(x) U U_{tt} \, dx
\end{equation}

Summing up the two above relations our conclusion follows.

\textbf{Lemma 2.2} Let \( a, \sigma, \alpha \) be as in (1.5), (1.4) and \( U \) be the strong solution of (2.2) - (2.3) then the following inequality holds
\begin{equation}
\frac{d}{dt} J(t, U) = \frac{\sigma'(0)}{2} \left\{ |U_x(t, L)|^2 - |U_x(0, t)|^2 \right\} \tag{2.10}
\end{equation}

\begin{equation}
- \frac{1}{2} \int_0^L \varphi'(x) \left\{ |U_t|^2 + \sigma'(0) |U_x|^2 \right\} \, dx - \int_0^L a(x) U \varphi(x) U_x \, dx
\end{equation}

\begin{equation}
+ \int_0^L F \varphi(x) U_x \, dx
\end{equation}

for any \( \varphi \in C^1(0, L) \).

\textbf{Proof.} - Multiplying equation (2.2) by \( \varphi(x) U_x \) and integrating over \([0, L]\) we find
\begin{equation}
\int_0^L U_t \varphi(x) U_x \, dx = \sigma'(0) \int_0^L U_{xx} \varphi(x) U_x \, dx - \int_0^L a(x) \varphi(x) U U_x \, dx + \int_0^L F \varphi(x) U_x \, dx
\end{equation}

integrating by parts
\begin{equation}
= - \frac{\sigma'(0)}{2} \int_0^L \varphi'(x) |U_x|^2 \, dx + \frac{\sigma'(0)}{2} \int_0^L \varphi(x) |U_x|^2 \, dx - \int_0^L a(x) \varphi(x) U U_x \, dx + \int_0^L F \varphi(x) U_x \, dx \tag{2.11}
\end{equation}
But

\[ \int_0^L U_t \varphi(x) U_x \, dx = \frac{d}{dt} \int_0^L U_t \varphi U_x \, dx - \int_0^L U_t \varphi(x) U_x \, dx \]

integrating by parts

\[ \frac{d}{dt} \int_0^L U_t \varphi U_x \, dx + \frac{1}{2} \int_0^L \varphi'(x) |U_t|^2 \, dx \]  \hspace{1cm} (2.12) \]

From relations (2.11) and (2.12), our conclusion follows.  

**Lemma 2.3** Let \( a, \sigma, \alpha, q \) be as in (1.5), (1.4), (2.6) and let us take \( N \) large enough, then there exists \( \delta > 0 \) such that

\[ \frac{dL(t,U)}{dt} \leq -\delta E(t,U) + \int_0^L F \{ NU_t + \alpha(x)U + q(x)U_x \} \, dx \]  \hspace{1cm} (2.13) \]

Moreover, there exist positive constants \( k_1, k_2 \) such that

\[ k_1 E(t,U) \leq L(t,U) \leq k_2 E(t,U) \]  \hspace{1cm} (2.14) \]

**Proof.** From (2.5), (2.7), (2.10) and since \( q(0) = q(L) = 0 \) (that is \( b(t) = 0 \)) we have

\[ \frac{dL(t,U)}{dt} \leq \]

\[ -N \int_0^L a(x) |U_t|^2 \, dx - \sigma'(0) \int_0^L \alpha(x) |U_x|^2 \, dx - \frac{1}{2} \int_0^L q'(x) \{ |U_t|^2 + \sigma'(0) |U_x|^2 \} \, dx \]  \hspace{1cm} (2.15) \]

\[ + \int_0^L \alpha(x) |U_t|^2 \, dx - \int_0^L a(x) U_t q(x) U_x \, dx + \int_0^L F \{ NU_t + \alpha(x)U + q(x)U_x \} \, dx \]

Let us find some estimates to \( I_1, I_2 \), where \( I_1 = I_{11} + I_{12} \)

\[ I_1 = -\int_0^L \left[ Na(x) - \alpha(x) + \frac{1}{2} q'(x) \right] |U_t|^2 \, dx - \sigma'(0) \int_0^L \left[ \alpha(x) + \frac{1}{2} q'(x) \right] |U_x|^2 \, dx \]

\[ I_2 = -\int_0^L \left( \frac{1}{\sqrt{2\varepsilon}} \sqrt{aU_t} \right) \left( \sqrt{2\varepsilon} \sqrt{aqU_x} \right) \, dx \]
Summing up $I_1$ with $I_2$ we get

$$I_1 + I_2 \leq - \int_0^L \left( N - \frac{1}{4\varepsilon} \right) a(x) - \alpha(x) + \frac{1}{2} q'(x) \right] U_x^2 \, dx - \int_0^L \sigma'(0) \left[ \alpha(x) + \frac{1}{2} q'(x) \right] U_x^2 \, dx + \varepsilon \int_0^L a q^2 |U_x|^2 \, dx.$$

Since,

$$I_3 = \int_{I_0}^L a(x) q^2 |U_x|^2 \, dx \leq \left( \max_{x \in [I_0, L]} |q(x)|^2 \right) \int_{I_0}^L U_x^2 \, dx$$

we can write

$$I_1 + I_2 \leq \int_0^L \left( N - \frac{1}{4\varepsilon} \right) a(x) - \alpha(x) + \frac{1}{2} q'(x) \right] U_x^2 \, dx - \int_0^L \sigma'(0) \left[ \alpha(x) + \frac{1}{2} q'(x) \right] U_x^2 \, dx$$

$$- \int_0^L a(x) q^2 |U_x|^2 \, dx + \varepsilon \int_{I_0}^L U_x^2 \, dx$$

(2.16)

where $\alpha$ and $q$ must satisfy $\alpha(x) + \frac{1}{2} q'(x) \geq C > 0$ in $[0, L]$. Let us take, $\varepsilon$ such that $\varepsilon < \frac{1}{M_0} \sigma'(0)$ and $N$ such that

$$N > \frac{1}{4\varepsilon} \quad \text{and} \quad \left[ \left( N - \frac{1}{4\varepsilon} \right) a(x) - \alpha(x) + \frac{1}{2} q'(x) \right] \geq D > 0.$$

where

$$C = \min_{x \in [0, L]} \left\{ \alpha(x) + \frac{1}{2} q'(x) \right\}$$

$$D = \min_{x \in [0, L]} \left\{ \left( N - \frac{1}{4\varepsilon} \right) a(x) - \alpha(x) + \frac{1}{2} q'(x) \right\}$$

Under the above notations we can rewrite inequality (2.16) in the following way
\[ l_1 + l_2 \leq -D \int_0^L |U_t|^2 \, dx - (\sigma'(0)C - \varepsilon M_0) \int_{t_0}^L |U_x|^2 \, dx - \sigma'(0)C \int_{t_0}^L |U_x|^2 \, dx \tag{2.17} \]

\[ \leq -D \int_0^L |U_t|^2 \, dx - (\sigma'(0)C - \varepsilon M_0) \int_{t_0}^L |U_x|^2 \, dx \]

\[ \leq -2 \min \left\{ D, C - \frac{\varepsilon M_0}{\sigma'(0)} \right\} \frac{1}{2} \int_0^L |U_t|^2 + \sigma'(0)|U_x|^2 \, dx \]

\[ \leq -2 \min \left\{ D, C - \frac{\varepsilon M_0}{\sigma'(0)} \right\} E(t, U) \]

We finish the proof of (2.13) using relation (2.17) in (2.15). To show (2.14), we proceed as follows:

\[ \left| \int_0^L \alpha(x)UU_t \, dx \right| \leq |\alpha| \frac{1}{2} \int_0^L |U|^2 + |U_t|^2 \, dx \tag{2.18} \]

\[ \left| \int_0^L a(x)\alpha(x)|U|^2(x) \, dx \right| \leq |\alpha| \frac{1}{2} \int_0^L |U|^2 \, dx \tag{2.19} \]

\[ \left| \int_0^L q(x)UU_t \, dx \right| \leq |q| \frac{1}{2} \int_0^L |U_x|^2 + |U_t|^2 \, dx \tag{2.20} \]

From (2.18), (2.19), (2.20) we have

\[ \left| \int_0^L \alpha(x)UU_t \, dx + \frac{1}{2} \int_0^L a(x)\alpha(x)|U|^2 \, dx + \int_0^L q(x)UU_t \, dx \right| \]

\[ \leq |\alpha| \frac{1}{2} \int_0^L 2|U|^2 + |U_t|^2 \, dx + |q| \frac{1}{2} \int_0^L |U_x|^2 + |U_t|^2 \, dx \]

Using Poincare's inequality we have

\[ |R| \leq |\alpha| \frac{1}{2} \int_0^L 2C_p |U_x|^2 + |U_t|^2 \, dx + |q| \frac{1}{2} \int_0^L |U_x|^2 + |U_t|^2 \, dx \]

\[ \leq b_2 E(t, U) \tag{2.21} \]
where $b_2$ is a positive constant. From (2.21) it follows that
\[ \int_0^L \alpha(x) UU_i dx + \frac{1}{2} \int_0^L a(x) \alpha(x) |U|^2 dx + \int_0^L q(x) U_i U_i dx \leq b_2 E(t, U) \] (2.22)

Recalling the definition of $L(t, U)$ and using (2.22) we arrive at
\[ L(t, U) \leq \{ N + b_2 \} E(t, U) \]

on the other hand, using (2.21) we get
\[ L(t, U) \geq \{ N - b_2 \} E(t, U) \]

Taking $N$ such that $N > b_2$ our conclusion follows.

Let us define the following functional
\[ M(t) = \| u(t) \|_{H^2(0, L)}^2 + \| u_t(t) \|_{H^2(0, L)}^2 + \| u_{tt}(t) \|_{H^2(0, L)}^2 \]

To obtain the global existence result we use the following hypotheses
\[ M(0) = \| u_0 \|_{H^2(0, L)}^2 + \| u_1 \|_{H^2(0, L)}^2 + \| u_{tt}(0) \|_{H^2(0, L)}^2 < \varepsilon^2 \]

for $\varepsilon > 0$ small enough. Since the local solution is continuous it follows that there exists $t_0 > 0$ such that
\[ M(t) < \varepsilon^2 \quad \forall t \in [0, t_0] \]

Let us define
\[ T^* = \sup \{ t_1, M(t) < d \varepsilon^2, t \in [0, t_1] \}, \]

and let us suppose that $T^* < \infty$. Therefore
\[ M(t) < d \varepsilon^2 \quad \forall t \in [0, T^*]. \] (2.23)

Using Gagliardo Niremberg's inequality we get
\[ |u_x(x, t)| \leq c_1 \varepsilon. \] (2.24)
By the mean value theorem, (2.23) and the continuity of $\sigma^*$ we have

$$|\sigma'(u_x) - \sigma'(0)| \leq c_2 \varepsilon.$$  (2.25)

Using Gagliardo Niremberg's inequality and (2.23) we get

$$|u_{xx}| < c \varepsilon, \quad |u_{xx}| < c \varepsilon.$$  (2.26)

From equation (2.1) we have

$$\int_0^L |u_{xx}|^2 \, dx \leq 2c_3 \int_0^L |u_t|^2 + |u|^2 \, dx$$  (2.27)

$$\int_0^L |u_{xxx}|^2 \, dx \leq 3c_3^2 \varepsilon^2 \int_0^L |u_{tt}|^2 + |u_t|^2 + |u|^2 \, dx$$  (2.28)

where

$$c_3 := \left\{ \frac{1}{\sigma'(0) - \varepsilon c_2} \right\}^2$$

$$\hat{c} := \left\{ c \varepsilon \max_{[a,b]} |\sigma''(x)| \frac{1}{\sigma'(0) - \varepsilon c_2} + 1 \right\}.$$

Taking $\varepsilon > 0$ such that

$$\sigma'(0) > \varepsilon c_2.$$

Let us define

$$L_1(t) = L(t, u) + \frac{N}{2} \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] |u_{xx}|^2 \, dx$$

$$L_2(t) = L(t, u_t) + \frac{N}{2} \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] |u_{xxt}|^2 \, dx$$

$$L_3(t) = L(t, u_{tt}) + \frac{N}{2} \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] |u_{xxt}|^2 \, dx$$

$$S(t) = L_1(t) + L_2(t) + L_3(t)$$

**Lemma 2.4** With initial data (1.6) and $u$ local solution of problem (1.1) - (1.5) there exists positive constants $k_3$ and $k_4$ such that
where

\[ k_3 = k_1 - \frac{N}{2} c_2 \frac{\varepsilon}{\sigma'(0)} \]
\[ k_4 = k_2 + \frac{N}{2} c_2 \frac{\varepsilon}{\sigma'(0)} \]

for \( \varepsilon \) small enough.

**Proof.**- This result follows easily from relations (2.14) and (2.25). □

**Lemma 2.5** Under the same hypotheses of Lemma 2.4, there exists a positive constant \( k \) such that

\[ \frac{dS(t)}{dt} \leq -kS(t). \]  

(2.32)

Moreover, there exist positive constants \( k_3 \) and \( k_6 \) such that

\[ k_3 M(t) \leq S(t) \leq k_6 M(t). \]  

(2.33)

**Proof.**-

To show (2.32) we need to estimate the following expression

\[ \int_0^L F \{ NU_t + \alpha U + q U_x \} \, dx \]

where \( (F, U) = \{(f, u), (f_t, u_t), (f_{tt}, u_{tt})\} \),

(i) **Caso** \( (F, U) = (f, u) \)

\[ \int_0^L f u_t = \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] u_x u_t \, dx \]
\[ \quad = -\int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] u_x \, dx + \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] u_t \, dx \]
\[ \quad = -\int_0^L \sigma'(u_x) u_{xu} u_t \, dx - \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] u_t u_x \, dx \]
\[ \quad = -\int_0^L \sigma'(u_x) u_{xu} u_t \, dx - \frac{d}{dt} \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] u_x^2 \, dx + \frac{1}{2} \int_0^L \sigma'(u_x) u_{tt} \, u_x^2 \, dx \]
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From (2.26), \( |\sigma'(u_x)| < M \) and Young's inequality it follows

\[
\leq \frac{M\alpha}{2} \int_0^L |u_t|^2 + |u_x|^2 \, dx + \frac{M\alpha}{2} \int_0^L |u_x|^2 \, dx - \frac{1}{2} \frac{d}{dt} \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] |u_x|^2 \, dx
\]

\[
\leq M\alpha \max \left\{ \frac{2}{\sigma'(0)}, 1 \right\} E(t, u) - \frac{1}{2} \frac{d}{dt} \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] |u_x|^2 \, dx
\]  

(2.34)

From (2.26), (2.27) and Poincare's inequality we have

\[
\int_0^L f\alpha u \, dx \leq \epsilon b_3 \left\{ E(t, u) + E(t, u_t) \right\}
\]  

(2.35)

where \( b_3 \) is a positive constant. From (2.25) and (2.27) we get

\[
\int_0^L f\alpha u \, dx \leq \epsilon b_4 \left\{ E(t, u) + E(t, u_t) \right\}
\]  

(2.36)

where \( b_4 \) is a positive constant.

(ii) Caso \( (F, U) = (f, u_t) \)

\[
\int_0^L f \alpha u \, dx = \int_0^L \sigma'(u_x)u_{x,x}u_t \, dx + \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] u_{x,x}u_t \, dx
\]

Using (2.26) we get

\[
\int_0^L f \alpha u \, dx \leq \frac{M\alpha}{2} \epsilon \int_0^L u_t^2 + |u_t|^2 + \int_0^L \frac{1}{2} \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] u_{x,x}u_t \, dx
\]  

(2.37)

Performing an integration by parts and using (2.26) we have

\[
I_4 \leq \frac{M\alpha}{2} \epsilon \int_0^L |u_t|^2 + |u_x|^2 \, dx + \int_0^L \frac{1}{2} \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] u_{x,x}u_t \, dx
\]  

(2.38)

Sustituting of (2.38) in (2.37), we have

\[
\int_0^L f \alpha u \, dx \leq M\alpha \epsilon \max \left\{ \frac{3}{2\sigma'(0)}, 1 \right\} E(t, u_t) - \frac{1}{2} \frac{d}{dt} \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] |u_x|^2 \, dx
\]  

(2.39)

From (2.26) and Young's inequality we get

\[
\int_0^L f\alpha u_t \, dx \leq \frac{M\alpha}{2} \epsilon |\alpha| \int_0^L |u_{x,x}|^2 + |u_t|^2 \, dx + \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] u_{x,x} \alpha(x) u \, dx
\]  

(2.40)
To estimate $I_2$ we make an integration by parts

$$I_2 = -\int_0^L \left\{ \left[ \sigma'(u_x) - \sigma'(0) \right] \alpha(x) u_t \right\} u_{xx} dx$$

$$= -\int_0^L \sigma''(u_x) \alpha(x) u_t u_x dx - \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] \alpha'(x) u_t u_x dx$$

$$- \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] \alpha(x) u_{xt}^2 dx$$

Using (2.26) and (2.25) once more

$$I_2 \leq \frac{Mc}{2} \varepsilon |\alpha|_\infty \int_0^L |u_{xx}|^2 + |u_t|^2 dx + \frac{c_2}{2} \varepsilon |\alpha'|_\infty \int_0^L |u_{xt}|^2 + |u_t|^2 dx +$$

$$c_2 \varepsilon |\alpha|_\infty \int_0^L |u_{xt}|^2 dx$$

(2.41)

Inserting (2.41) into (2.40) we conclude that

$$\int_0^L f_\alpha u_t dx \leq \varepsilon b_5 \left\{ E(t,u) + E(t,u) \right\}$$

(2.42)

where $b_5$ is a positive constant. Using (2.26), (2.25) and (2.28) we have that

$$\int_0^L f_\alpha u_t dx \leq b_5 \varepsilon \left\{ E(t,u) + E(t,u) + E(t,u) \right\}$$

(2.43)

where $b_5 > 0$.

(iii) Caso $(F, U) = (u_{tt}, u_{tt})$

$$\int_0^L f_\alpha u_{tt} dx = \int_0^L \sigma''(u_x) |u_{tt}|^2 u_{xx} u_m dx + \int_0^L \sigma''(u_x) u_{xx} u_{tt} dx +$$

$$2 \int_0^L \sigma''(u_x) u_{xx} u_{tt} dx + \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] u_{xx} u_{tt} dx$$

Using (2.26), $|\sigma''(u_x)| \leq M_1$ and Young’s inequality

$$\leq M_1 c \varepsilon^2 \int_0^L |u_{xx}|^2 + |u_{tt}|^2 dx + M_1 c \varepsilon \int_0^L |u_{xx}|^2 + |u_{tt}|^2 dx +$$

$$M_1 \varepsilon \int_0^L |u_{xx}|^2 + |u_{tt}|^2 dx + \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] u_{xx} u_{tt} dx$$

(2.44)
Integrating by parts

\[ I_3 = - \int_0^L \left\{ \left[ \sigma'(u_x) - \sigma'(0) \right] u_{tt} \right\} u_{xx} \, dx \]

\[ = - \int_0^L \sigma''(u_x) u_{xx} u_{tt} \, dx - \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] u_{tt} u_{xx} \, dx \]

\[ = - \int_0^L \sigma''(u_x) u_{xx} u_{tt} \, dx - \frac{d}{dt} \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] u_{tt}^2 \, dx \]

\[ + \frac{1}{2} \int_0^L \sigma'(u_x) u_{tt} \left| u_{tt} \right|^2 \, dx \]

Using (2.26) we get

\[ I_3 \leq M c e^2 \int_0^L 2 \left| u_{xx} \right|^2 + \left| u_{tt} \right|^2 \, dx - \frac{1}{2} \frac{d}{dt} \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] u_{tt}^2 \, dx \]  \hspace{1cm} (2.45)

Substitution of (2.45) into (2.44) yields

\[ \int_0^L f_{tu} u_{tt} \leq \varepsilon \max \left\{ M t e^2, M c^3 \right\} \int_0^L \left| u_{xx} \right|^2 + \left| u_{tt} \right|^2 + \left| u_{tx} \right|^2 + 3 \left| u_{tt} \right|^2 \, dx \]

\[ - \frac{1}{2} \frac{d}{dt} \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] u_{tt}^2 \, dx \]

Using (2.27) e (2.28) we get

\[ \int_0^L f_{tu} u_{tt} \leq \varepsilon b_1 \left\{ E(t, u_t) + E(t, u_x) + E(t, u) \right\} \]

\[ - \frac{1}{2} \frac{d}{dt} \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] u_{tt}^2 \, dx \]  \hspace{1cm} (2.46)

where \( b_1 > 0 \). On the other hand, using (2.26), we have

\[ \int_0^L f_{tu} u_{tt} \leq \frac{M t e^2}{2} \left| \alpha \right|_\infty \int_0^L \left| u_{xx} \right|^2 + \left| u_{tt} \right|^2 \, dx + \frac{M c e}{2} \left| \alpha \right|_\infty \int_0^L \left| u_{xx} \right|^2 + \left| u_{tt} \right|^2 \, dx \]

\[ + M c e \left| \alpha \right|_\infty \int_0^L \left| u_{xx} \right|^2 + \left| u_{tt} \right|^2 \, dx + \int_0^L \alpha(x) \left[ \sigma'(u_x) - \sigma'(0) \right] u_{xxn} u_t \, dx \]

\[ \leq I_4 \]
Integrating by parts

\[
I_4 = -\int_0^L \left\{ \alpha(x) \left[ \sigma'(u_x) - \sigma'(0) \right] u_t \right\}_x u_{tt} \, dx
\]

\[
= -\int_0^L \sigma'(u_x) u_{xxx} u_t \alpha(x) u_{tt} \, dx - \int_0^L \alpha(x) \left[ \sigma'(u_x) - \sigma'(0) \right] u_{tx}^2 \, dx
\]

\[-\int_0^L \alpha'(x) \left[ \sigma'(u_x) - \sigma'(0) \right] u_t u_{tt} \, dx
\]

Using (2.26) and (2.25) we get

\[
I_4 \leq \frac{M \epsilon e}{2} |\alpha|_\infty \int_0^L \left| u_{xxx} \right|^2 + |u_t|^2 \, dx + c_2 \epsilon |\alpha|_\infty \int_0^L \left| u_{xxx} \right|^2 \, dx +
\]

\[
\frac{c_2 \epsilon e}{2} |\alpha'|_\infty \int_0^L \left| u_{xxx} \right|^2 + |u_t|^2 \, dx
\]

Substitution of (2.48) into (2.47) yields

\[
\int_0^L f \alpha u_t \, dx \leq 4 \epsilon c_4 \int_0^L \left| u_{xxx} \right|^2 + |u_t|^2 + |u_{xt}|^2 + |u_{xtt}|^2 \, dx
\]

Where

\[
c_4 := \max \left\{ \frac{M_1 \epsilon}{2} |\alpha|_\infty, M_1 |\alpha|_\infty, c_2 |\alpha|_\infty \right\}
\]

Using (2.27) and (2.28) we have

\[
\int_0^L f \alpha u_t \, dx \leq \epsilon b_8 \left\{ E(t, u_t) + E(t, u) + E(t, u) \right\}
\]

for some \( b_8 > 0 \).

Using (2.26) we get

\[
\int_0^L f \alpha q(x) u_{tt} \, dx \leq \frac{M_1 \epsilon}{2} |q|_\infty \int_0^L \left| u_{xxx} \right|^2 + |u_{tt}|^2 \, dx +
\]

\[
M \epsilon |q|_\infty \int_0^L \left| u_{xxx} \right|^2 \, dx +
\]

\[
M \epsilon |q|_\infty \int_0^L \left| u_{xxx} \right|^2 + |u_{tt}|^2 \, dx +
\]

\[
\int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] q(x) \frac{d}{dt} \left\{ \left| u_{tt} \right|^2 \right\} \, dx
\]

\[
:= I_5
\]

(2.50)
Integrating by parts and using \( q(0) = q(L) = 0 \) we get

\[
I_5 = -\frac{1}{2} \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] q(x) u_{xx}^2 \, dx
= -\frac{1}{2} \int_0^L \sigma''(u_x) u_x q(x) u_{xx}^2 \, dx - \frac{1}{2} \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] q'(x) u_{xx}^2 \, dx
\]

From (2.26) and (2.25) we get

\[
I_5 \leq \frac{Mc}{2} \varepsilon |q'_\infty| \int_0^L |u_{xx}|^2 \, dx + \frac{c_2}{2} \varepsilon |q'_\infty| \int_0^L |u_{xx}|^2 \, dx = \frac{Mc}{2} \varepsilon |q'_\infty| \int_0^L |u_{xx}|^2 \, dx + \frac{c_2}{2} \varepsilon |q'_\infty| \int_0^L |u_{xx}|^2 \, dx
\]  

(2.51)

Substitution of (2.51) into (2.50) produces

\[
\int_0^L f_0 q(x) u_{xx} \, dx \leq \varepsilon c_3 \int_0^L \left\{ |u_{xx}|^2 + 5 |u_t|^2 + |u_{xx}|^2 \right\} \, dx
\]

where

\[
c_3 := \max \left\{ \frac{M_{c_2}}{2} |q'_\infty|, Mc |q'_\infty|, \frac{c_2}{2} |q'_\infty| \right\}
\]

Using (2.27) and (2.28) we get

\[
\int_0^L f_0 q(x) u_{xx} \, dx \leq \varepsilon c_3 \max \left\{ 2 c_3, 2 \varepsilon^2 c_3, \frac{5}{\sigma'(0)} \right\}
\]

\[
\{ E(t, u_t) + E(t, u) + E(t, u) \}
\]  

(2.52)

Summing up (2.34), (2.35), (2.36), (2.39), (2.42), (2.43), (2.46), (2.49) and (2.52) we get

\[
\sum_{i=0}^2 \int_0^L f_i \left\{ N u_{t,i} + cu_i + q u_{i,x} \right\} \, dx \leq \varepsilon \sum_{i=0}^2 \gamma_i E(t, u_i) - \frac{N}{2} \frac{d}{dt} \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] u_x^2 \, dx - \frac{N}{2} \frac{d}{dt} \int_0^L \left[ \sigma'(u_x) - \sigma'(0) \right] u_{xx}^2 \, dx
\]

(2.53)

with the following notations
\[
\begin{align*}
    F_0 &= f, \quad u_0 = u \\
    F_1 &= f_t, \quad u_1 = u_t \\
    F_2 &= f_{tt}, \quad u_2 = u_{tt}
\end{align*}
\]

where \( \{\gamma_i\}_{i=0}^2 \) are positive constants depending on \( \varepsilon, c_p, \sigma'(0) \), of the \( L^\infty \) norm of the functions \( \alpha, q, \sigma', q', \sigma''(u_x) \) and \( \sigma''(u_x) \).

Using Lemma 2.3 and (2.53) we have

\[
\frac{dS(t)}{dt} \leq - \sum_{i=0}^{2} \left( \delta - \varepsilon \gamma_i \right) E(t, u_i) \quad (2.54)
\]

where \( \{\delta - \varepsilon \gamma_i\}_{i=0}^2 \) is positive for \( \varepsilon \) small enough. From (2.54), (2.29), (2.30), (2.31) we have

\[
\frac{dS(t)}{dt} \leq - \frac{1}{k_4} \left\{ \left( \delta - \varepsilon \gamma_0 \right) L_4(t) + \left( \delta - \varepsilon \gamma_1 \right) L_2(t) + \left( \delta - \varepsilon \gamma_2 \right) L_3(t) \right\}
\]

\[
\leq - kS(t)
\]

where

\[
k := \frac{1}{k_4} \min_{i=0,1,2} \{\delta - \varepsilon \gamma_i\}
\]

From where (2.32) follows. To show (2.33) we need the following estimates

\[
\int_0^L \left| u_{xx} \right|^2 \, dx \leq k_7 \int_0^L \left| u_{x} \right|^2 + \left| u_t \right|^2 + \left| u_{xx} \right|^2 \, dx \quad (2.55)
\]

\[
\int_0^L \left| u_{xxx} \right|^2 \, dx \leq k_8 \int_0^L \left| u_x \right|^2 + \left| u_{xx} \right|^2 + \left| u_{xxx} \right|^2 + \left| u_t \right|^2 \, dx \quad (2.56)
\]

where \( k_7 \) and \( k_8 \) positive constants. This inequalities follows differentiating relation (2.1) with respect to \( x \) and \( t \), and using the same above reasoning. From the definition of \( S \) and relations (2.29), (2.30), (2.31) we get

\[
S(t) \leq k_4 \{ E(t, u) + E(t, u_t) + E(t, u_{tt}) \}
\]

Then, from (2.55) we can conclude that

\[
S(t) \leq k_6 M(t)
\]
for some $k_0 > 0$. On the other hand, recalling the definition of $S$ and keeping in mind relations (2.29), (2.30), (2.31) we have

$$S(t) \geq k_3 \left\{ E(t, u) + E(t, u_t) + E(t, u_{tt}) \right\}$$

Using (2.27), (2.28), (2.56) and Poincare's inequality we arrive at

$$S(t) \geq k_2 \mathcal{M}(t).$$

for $k_2 > 0$. Therefore (2.33) follows. \[\Box\]

Integrating (2.32) from 0 to $t$ and letting $t \to T^*$ we get

$$S(T^*) < S(0).$$

Applying (2.33) is follows that

$$k_2 \mathcal{M}(T^*) \leq S(T^*) = S(0) \leq k_5 \mathcal{M}(0),$$

that is

$$\mathcal{M}(T^*) \leq \frac{k_6}{k_5} \mathcal{M}(0) < \frac{k_6}{k_5} e^2.$$

Taking $d = \frac{k_6}{k_5}$. There exists $T_i > T^*$ such that $\mathcal{M}(t) < d e^2$ for any $t$ on $[T^*, T_i)$ which implies that there exists $T_i > T^*$ such that $\mathcal{M}(t) < d e^2$ for any $t$ in the interval $[0, T_i)$, which is contradictory to the maximality of $T^*$. Therefore, we conclude that $T^* = \infty$ which implies that the solution is global in $t$. From (2.32) we get the exponential decay of $S$. Using (2.33) we get the exponential decay of $\mathcal{M}$. The proof is now complete.
REFERENCES


