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ENERGY DECAY OF A LINEAR HYPERBOLIC EQUATION WITH LOCALLY DISTRIBUTED DAMPING

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ABSTRACT.- *Decay estimates for the energy are derived for a linear hyperbolic equation with time - dependent coefficients.*

1. INTRODUCTION

Let Ω a bounded domain in $\Re^N$ $(N \geq 1)$ having a smooth boundary $\Gamma = \partial\Omega$.
We fix $x_0 \in R^N$. Let $a(x,t)$ be a nonnegative bounded function.
Let us consider the following linear hyperbolic equation.

$$\left| \begin{array}{ll} K(x,t)\ u'' - \Delta u + a(x,t)\ u' = 0 & \text{in } Q = \Omega\times]0,T[\\ \\ u = 0 & \text{on } \Sigma = \Gamma\times]0,T[\\ \\ u(0) = u_0\ ,\ u'(0) = u_1 & \text{on } \Omega \end{array} \right. \qquad (1.1)$$

Under the assumptions that $K(x,t) \geq h_0 > 0,\ \forall (x,t) \in Q$ and

$$a(x,t) - \left|\frac{\partial K}{\partial t}(x,t)\right| \geq \varepsilon_0 > 0\ ,\ \forall\ x \in \omega$$

Where ω is a neighborhood of $\overline{\Gamma_0}$,

$$\Gamma_0 = \{\ x \in \Gamma : \mathrm{m}(x).\nu(x) \geq 0\}$$

being $\nu(x)$ the unit outward normal at $x \in \Gamma$ and $\mathrm{m}(x) = x - x_0$, we prove the exponential decay of weak solutions to problem (1.1).
The case $K(x,t) = 1$ has been studied by many authors: M. Nakao [6]. L.R. Tcheougoné Tebou [7], P. Martínez [5]. Motived by the above results the authors of the present work investigated the case with time dependent coefficients.

2. PRELIMINARIES

The function spaces we use are all standard and the definition of them is unified. We shall denote the following

$$(u,v)=\int_\Omega u(x)\ v(x)\ dx\ ,\ |u|^2=\int_\Omega |u(x)|^2\ dx$$

$$u'=\frac{\partial u}{\partial t}\ \text{ and }\ R=\max_{x\in\Omega}\ \|x-x_0\|$$

Let us state the general hypothesis

(A.1) $K\in W^{1,\infty}\left(0,\infty;C^1\left(\overline{\Omega}\right)\right),\ K'\in L^1\left(0,\infty,L^\infty(\Omega)\right)$

$K(x,t)\geq h_0>0\ ,\ \forall(x,t)\in Q$

(A.2) $a\in W^{1,\infty}\left(0,\infty;C^{-1}\left(\overline{\Omega}\right)\right)\cap W^{2,\infty}\left(0,\infty;L^\infty(\Omega)\right)$

$a\in L^1\left(0,\infty;L^\infty(\Omega)\right),\ a(x,t)\geq a_0>0$ in Q

(A.3) $a(x,t)-|K'(x,t)|\geq\varepsilon_0>0\ ,\ \forall x\in\omega$

We use the following well known lemma without the proof in this paper.

Lemma 2.1.- *(Haraux [2]) Let* $E:[\ 0,\infty\ [\ \to[\ 0,\infty\ [$ *be a nonincreasing locally absolutely continous funtion such that there are nonnegative constants* β *and* A *with*

$$\int_S^\infty E(t)^{\beta+1}\ dt\leq A\ E(S)\ ,\ \forall\ S\geq 0$$

Then we have

$$E\ (t)\leq\begin{cases} E(0)\ e^{1-\frac{t}{A}}\ ,\ \forall\ t\geq 0\ ,\ \textit{if}\ \ \beta=0\\ \left[A\left(1+\dfrac{1}{\beta}\right)\right]^{1/\beta} t^{1/\beta},\ \textit{if}\ \ \beta>0\end{cases}$$

3. STATEMENT OF THE RESULTS

In this section, we state main theorems.

Theorem 3.1

Let $u_0\in H_0^1(\Omega)\cap H^2(\Omega)$ *and* $u_1\in H_0^1(\Omega)$*. Under assumptions (A.1) – (A.3), for each* $T>0$*, the problem (1.1) admits a unique strong solution*

$$u : \Omega \times \left]\, 0,\, T\, \right[\rightarrow R \text{ such that}$$

$$u_0 \in L^\infty\left(0,\ T;\ H_0^1 \cap H^2\right),\ u' \in L^\infty\left(0,\ T;\ H_0^1\right),\ u'' \in L^\infty\left(0, T;\ L^2(\Omega)\right) \quad (3.1)$$

Now, we present a result on stability of strong solutions wich will be extended to weak solutions .

Theorem 3.2

Let us consider the energy

$$E(t) = \frac{1}{2}\left|\sqrt{K}\, u(t)\right|^2 + \frac{1}{2}\left|\nabla u(t)\right|^2 \quad (3.2)$$

Under hypothesis of theorem 3.1, with $\nabla K\,.\mathrm{m} \geq 0\ \ \forall\ x \in \Omega$ *the energy (3.2) determined by the strong solutions u decays exponentially, that is for some constants* $c > 0$ *and* $r > 0$

$$E(t) \leq C\ E(0)\ e^{-\gamma t}, \quad \forall\, t \geq 0 \quad (3.3)$$

Proof

First we remark that is sufficient to prove the estimate (3.3) where the initial conditions verify

$$\{\ u_0,\ u_1\} \in \left(H_0^1(\Omega) \cap H^2(\Omega)\right) \times H_0^1(\Omega)$$

Then an easy density argument give the result for all initial condition in $H_0^1(\Omega) \times L^2(\Omega)$

We verify that (1.1) is a dissipative problem.

Theorem 3.3

Suppose that $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ *and that assumptions (A1), (A2) hold. Then (1.1) has a unique weak solution* $\mathrm{u} : \Omega \times \left]\, 0,\, T\, \right[\rightarrow R$ *in the space*

$$C\left(0,\ T;\ H_0^1(\Omega)\right) \cap C^1\left(0,\ T;\ L^2(\Omega)\right)$$

Furthermore, theorem (3.2) holds for the weak solution u.

Lemma 3.1.-

$$\forall\ 0 \leq S < T < \infty :\ E(S) - E(T) = -\int_S^T \int_\Omega \left(a - \frac{1}{2}\ K'\right)\ (u')^2\, dx\, ds$$

Proof.-

We multiply (1.1) by u' and we integrate by parts on $\Omega \times [S, T]$, observing first that

$$\left(Ku'',u'\right)=\frac{1}{2}\frac{d}{dt}\left|\sqrt{K}\,u'\right|^2-\frac{1}{2}\int_\Omega K'(u')^2\,dx.$$

We obtain

$$\int_S^T (Ku''-\Delta u,u')\,ds=\left(\frac{1}{2}\left|\sqrt{K}\,u'\right|^2+\frac{1}{2}|\nabla u|^2\right)\Big|_S^T+\int_S^T\int_\Omega\left(a-\frac{1}{2}K'\right)(u')^2\,dx\,ds$$

Therefore

$$E(T)-E(S)=-\int_S^T\left(\left(a-\frac{1}{2}K'\right),(u')^2\right)ds \tag{3.4}$$

Remark.- We deduce from (3.4) and (A.3) that

$$\int_S^T\left((a-K'),u'^2\right)ds\leq E(S)-E(T) \tag{3.5}$$

Lemma 3.2.- *Let* $q\in\left[W^{1,\infty}(\Omega)\right]\cap\ \alpha\in\mathbb{R}$ *and* $\zeta\in W^{1,\infty}(\Omega)$. *We have the identities*

$$\left(ku',2\,q.\nabla u+\alpha u\right)\Big|_S^T+\int_S^T\int_\Omega(divq-\alpha)\left\{\left|\sqrt{K}\,u'\right|^2-|\nabla u|^2\right\}dx\,dt+ \tag{3.6}$$

$$+\int_S^T\left((a-K')\,u',2q.\nabla u+\alpha u\right)dt+\int_S^T\left(\nabla K.q,u'^2\right)dt$$

$$+2\int_S^T\int_\Omega\frac{\partial q_K}{\partial x_J}\frac{\partial u}{\partial x_J}\frac{\partial u}{\partial x_J}\,dx\,dt=\int_S^T\int_\Gamma q.v\left(\frac{\partial u}{\partial v}\right)^2 d\Gamma\,dt$$

$$\left(Ku',u\right)\Big|_S^T-\int_S^T\int_\Omega\zeta\left\{\left|\sqrt{K}\,u'\right|^2-|\nabla u|^2\right\}dx\,dt+$$

$$+\int_S^T\left(u\nabla u,\nabla\zeta\right)dt+\int_S^T\left((a-K')u',\zeta u\right)dt=0. \tag{3.7}$$

The proof of Lemma 3.2 is based on standard multipliers technique. The interested reader should refer to Lions [4] or Komornik [3].

Throughout the remaining part of this work positive constants will be denoted by C and will change from line to line.

In order to prove (3.3) we proceed in two steps.

Step 1. Appliying (3.5) with $q(x)=m(x),\alpha=n-1$ observing that $div(m)=N$ and using (3.2) we get

$$\left(Ku', 2\,m.\nabla u + (N-1)u\right)\Big|_S^T + \int_S^T \int_\Omega \left(\left| \sqrt{K}\, u' \right|^2 - |\nabla u|^2 \right) dx\, dt$$

$$+ \int_S^T \left((a - K')\, u', 2m.\nabla u + (N-1)u \right) dt + \int_S^T \int_\Omega \nabla K.m\, (u')^2\, dx\, dt$$

$$+ 2\int_S^T \int_\Omega |\nabla u|^2\, dx\, dt = \int_S^T \int_\Gamma m.\nu \left(\frac{\partial u}{\partial \nu} \right)^2 d\Gamma\, dt\, ;$$

$$\begin{aligned} 2\int_S^T E(t)\, dt = &-\left(Ku', 2m.\nabla u + (N-1)\,u\right)\Big|_S^T - \\ &- \int_S^T \left((a - K')u', 2m.\nabla u + (N-1)\,u \right) dt \\ &- \int_S^T \int_\Omega (\nabla K.m)\, u'^2\, dx\, dt + \int_S^T \int_\Gamma m.\nu \left(\frac{\partial u}{\partial \nu} \right)^2 d\Gamma\, dt \end{aligned}$$

Since the energy is nonincreasing, using the result of komornik [3], we obtain

$$\left| -\left(Ku', 2m.\nabla u + (N-1)\,u\right)\Big|_S^T \right| \le C\, E(s) \tag{3.9}$$

$$\begin{aligned} \left| -\int_S^T \left((a - K')u', 2m.\nabla u + (N-1)\,u \right) dt \right| &\le C \int_S^T \left| E'(t) \right|^{1/2} \left| E^{1/2}(t) \right| dt \\ &\le c\, E(S) + \varepsilon \int_S^T E(t) dt\, , \quad \varepsilon > 0 \end{aligned} \tag{3.10}$$

If follows from (3.8) - (3.10) that

$$\int_S^T E(t)\, dt \le C\, E(S) + R \int_S^T \int_{\Gamma_0} \left(\frac{\partial u}{\partial \nu} \right)^2 d\Gamma\, dt \tag{3.11}$$

To estimate the last term in (3.11) we utilize (3.7) with $\zeta = \eta$ where $\eta \in \mathrm{W}^{1,\infty}(\Omega)$ is a function that

$$\left| \begin{array}{lll} 0 \le \eta \le 1 & & \\ \eta = 1 & in\ \hat{\omega} & ; \quad \dfrac{|\nabla \eta|^2}{\eta} \in L^\infty(\omega) \\ \eta = 0 & in\ \Omega \setminus \omega & \end{array} \right.$$

and $\hat{\omega}$ is open set in Ω with $\Gamma_0 \subseteq \hat{\omega} \subsetneqq \omega$. First we have from (3.7)

$$\int_S^T \int_\Omega \eta\, |\nabla u|^2 dx\, dt = (Ku',\eta u)\Big|_S^T - \int_S^T \big((a-K')\,u',\eta u\big)\, dt \tag{3.12}$$
$$+\int_S^T \int_\Omega \eta \left|\sqrt{K}u'\right|^2 dx\, dt \;-\; \int_S^T (u\nabla u, \nabla \eta)\, dt$$

Simple calculations, using Young's ineguality show that

$$\left| -(Ku',\eta u)\right|_S^T \le CE(S) \tag{3.13}$$

$$\left| -\int_S^T \big((a-K')\mathrm{u}',\eta u\big)\, dt \le CE(S) + \varepsilon \int_S^T E(t)\, dt \quad , \quad \varepsilon > 0 \tag{3.14}$$

$$\left| -\int_S^T (u\nabla u, \nabla\eta)\, dt \right| \le C \int_S^T \int_\omega |u|^2\, dt\, dx + \frac{1}{2}\int_S^T \int_\Omega \eta\, |\nabla u|^2 \in dx\, dt \tag{3.15}$$

$$\left| \int_S^T \eta \int_\Omega \text{n} \left|\sqrt{K}u'\right|^2 dx\, dt \right| \le C \int_S^T \int_\omega |u'|^2\, dx\, dt \tag{3.16}$$

Reporting (3.13) – (3.16) in (3.12), we find

$$\frac{1}{2}\int_S^T \int_\Omega \eta |\nabla u'|^2\, dt\, dx \le C\, E(S) + C\int_S^T \int_w \left(|u'|^2 + |u|^2 \right) dx\, dt + \tag{3.17}$$
$$+\varepsilon \int_S^T E(t)\, dt, \qquad \varepsilon > 0$$

Step 2.- We take a vector field $h \in \left[W^{1,\infty}(\Omega)\right]^N$ such that

$$h = \nu \text{ on } \Gamma_0, \;\; h.\nu \ge 0 \; on\, \Gamma \text{ and } \mathrm{h} = 0 \text{ on } \Omega \setminus \hat{w}$$

Choosing $\alpha = 0$ and $\mathrm{q} = \mathrm{h}$ in (3.6), we easily deduce

$$\int_S^T \int_{T_0} \left(\frac{\partial u}{\partial \nu}\right)^2 d\Gamma\, dt \le C \left(E(S) + \int_S^T \int_{\hat{\omega}} |u'|^2 + |\nabla u|^2 \right) dx\, dt + \varepsilon \int_S^T E(s)\, dt \tag{3.18}$$

combining (3.18) and (3.17), we have $\forall\, \varepsilon > 0$

$$\int_S^T \int_{\Gamma_0} \left(\frac{\partial u}{\partial \nu}\right)^2 d\Gamma\, dt \le C \left(E(S) + c\int_S^T \int_\omega |u'|^2 + |\nabla u|^2 \right) dx\, dt + \varepsilon \int_S^T E(t)\, dt \quad , \quad \forall\, \varepsilon > 0 \tag{3.19}$$

We conclude that (3.19) and (3.1) that

$$\int_S^T E(t)\,dt \le C\,E(S) + C\int_S^T \int_{\hat{\omega}} \left(|u'|^2 + |\nabla u|^2\right) dx\,dt \tag{3.20}$$

Now in under to absorb the last theorem of the right hand side of (3.20) we adapt a method introduced in Conrad and Rao [1]. To this end, we consider $z(t) \in H_0^1(\Omega)$ solution of

$$\left| \begin{array}{ll} -\Delta z = x(\omega)u & in \;\; \Omega \\ z = 0 & on \;\; \Gamma \end{array} \right.$$

where $x(\omega)$ is the characteristic function of ω.

It is easy to verify that z' is solution of the problem

$$\left| \begin{array}{ll} -\Delta z' = x(\omega)u' & in \;\; \Omega \\ z' = 0 & on \;\; \Gamma \end{array} \right.$$

A simple computations gives

$$|z| \le C|u|_{L^2(u)} \tag{3.21}$$

$$|z'| \le C|u'|_{L^2(w)} \tag{3.22}$$

$$(\nabla z, \nabla u) = |u|_{L^2(w)} \tag{2.23}$$

Next, we multiply (1.1) by z, integrate by parts on $\Omega \times]S,T[$ and use (3.23), we obtain

$$\int_S^T \int_w |u|^2\,dx\,dt = -(Ku', z)\Big|_S^T - \int_S^T \left(\left(a - K'\right)u', z\right) dt + \int_S^T (Ku', z')\,dt \tag{3.24}$$

Now, we note that

$$|(Ku', z)| \le C\,E(S)$$

$$\left|\int_S^T \left(\left(a - K'\right)u', z\right) dt\right| \le C\,E(S) + \varepsilon \int_S^T E(t)\,dt \quad , \quad \varepsilon > 0$$

$$\left|\int_S^T (Ku', z)\,dt\right| \le C\int_S^T \int_w |u'|^2\,dx\,dt + \varepsilon_1 \int_S^T E(t)\,dt \quad , \quad \varepsilon_1 > 0$$

We deduce from the last three estimates and (3.24) that

$$\int_S^T \int_{\omega} |u'|^2 \, dx\, dt \le C\, E(S) + \varepsilon \int_S^T E(t)\, dt + C \int_S^T \int_{w} |u'|^2 \, dx\, dt \tag{3.25}$$

Inserting (3.25) into (3.20) gives

$$\int_S^T E(t) \;\le\; C\, E(S) + C \int_S^T \int_{w} |u'|^2 \, dx\, dt \tag{3.26}$$

Thanks to (A.3) we have

$$\int_S^T \int_{\omega} |u'|^2 \, dx\, dt \le \frac{1}{\varepsilon_0} \int_S^T \int_{\omega} \left(\mathrm{a} - \frac{1}{2} K' \right) u'^2 \, dx\, dt \le C\, E(S) \tag{3.27}$$

Finally the application of lemma 2.1 implies (3.3).

The proof of theorem 3.2 is completed.

Remark.- The proof of theorem (3.3) is achieved using a density argument.

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