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Multiplicity and Transitoriness of Chaotic Events

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Commonly authors of literature dealing with chaos report a single and truncated chaotic event occurring in the chaotic system they have investigated. This paper reports a multiplicity of chaotic events detected in the non-linear damped and forced oscillator. In order to detect chaos in this oscillator, a Virtual Lab (integrated and interactive computer program) was developed by the author of this report. With this Virtual Lab many chaos simulations were executed and the resulting Poincaré Maps for angles of 0° and 180° were extracted and filtered to avoid event duplicity. It has been found that chaotic events do not last forever; they have a beginning and an end, which means they are transitory. No numerical connection has been detected between the natural frequency of a chaotic oscillator with that of the periodical applied force.
Keywords: Nonlinear, dynamics, chaos, computer simulation, Runge-Kutta, Poincaré maps, numerical methods.

Multiplicidad y transitoriedad de los eventos caóticos

Comúnmente los autores de literatura sobre Caos reportan un solo evento caótico truncado que acontece en los sistemas caóticos que ellos han investigado. Este documento reporta múltiples eventos caóticos detectados en el oscilador no lineal amortiguado y forzado. Con la finalidad de detectar Caos en este oscilador, el autor de este reporte desarrolló un Laboratorio Virtual (software interactivo e integrado), con el cual se ejecutaron muchas simulaciones y también se extrajeron y compararon Mapas de Poincaré para ángulos de 0° y 180°, a fin de evitar duplicidad de eventos. Se encontró que los eventos caóticos no son imperecederos, ellos tienen un inicio y un final, lo cual significa que ellos son transitorios. Tampoco se detectó relación numérica alguna entre los valores de la frecuencia natural del oscilador caótico con aquella de la fuerza periódica aplicada.
Palabras claves: Dinámica no lineal, Caos, simulación computarizada, Runge-Kutta, mapa de Poincaré, métodos numéricos.

The most common -though not the unique- route to Chaos is that of period bifurcation, in which the oscillating system vibrating with a single period, eventually changes and vibrates with two periods -bifurcates-, sometime later these two periods are replaced each by other two new periods, and so on until chaos sets in and it becomes impossible to foresee the next period of oscillation.

Usually the literature research, papers and books, dealing with chaos report a single and partial cascade of period bifurcations, this is, they show only a unique and truncated cascade. This is confusing because the newcomer to chaos land is urged to think that chaos once established is infinite and that chaotic systems experiment only a single and unique chaotic event.

This investigation is about chaos in the nonlinear damped and forced oscillator. It has been found that chaotic events in this system are finite, they do not last forever, it has also been encountered that this oscillator displays many chaotic events. Even more, this research has not found any connection between the natural frequency of the oscillator and the frequency of the applied force. Very likely other systems prone to chaos behave the same way.

Real-life systems likely to experiment chaos are not prepared to vibrate the way a chaotic system does, they usually collapse soon after chaos begins. Mechanical devices commonly present in factories [1,2] collapse soon after the system changes the frequencies they are supposed

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to oscillate with. The human heart is also an oscillator and it is suspected that when it undergoes fibrillation, it has entered chaos. Obviously, the systems just mentioned are not adequate to perform a chaos research, for this reason chaos must be investigated by means of computerized simulation of mathematical models. Currently high speed computers enabled with high resolution computer graphics are of great help in chaos research. Evidently the algorithms developed by researchers play a critical role in these investigations.

A Virtual Lab [3, 4], an interactive and integrated computer program, to investigate chaos in the nonlinear damped and forced oscillator has been developed by the author of this report. This Virtual Lab uses the Runge-Kutta method to numerically solve the differential equation of the above mentioned vibrator. The program has been prepared to execute up to 30 million time steps or iterations.

Interested readers who are not so fond of computer programming, may numerically solve the differential equation dealt with in this report, by means of commercial computer softwares like MathLab, Mathematica, Maple, etc., which are available in the market. It is worthwhile mentioning that a very great amount of the literature on chaos is based on research performed with these commercial softwares. Obviously these investigations are limited by what the softwares are prepared to accomplish.

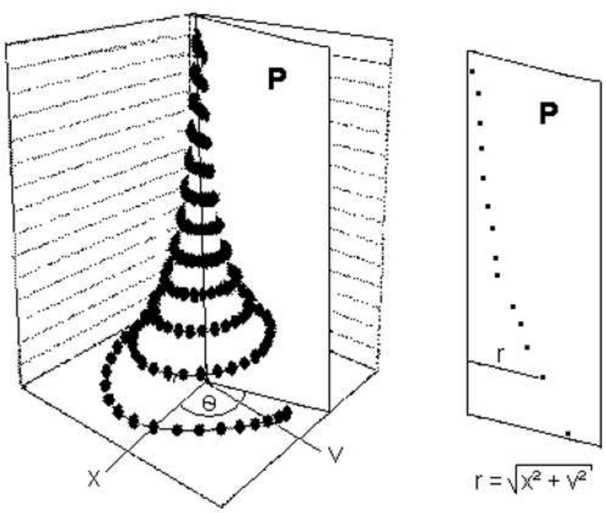

Figure 1: In left, the State Space is the plotting of displacement and velocitiy versus time (vertical axis). The depicted curve corresponds to a damped oscillator, which as time elapses reduces its amplitud and velocity until the ocillator stops. In right side, the sketched plane P is the Poincaré section at some angle θ. The curve intersects the Poincaré section at some points which constitute the Poincaré Map at angle θ.

Brief introduction to Chaos research

This section contains a brief theoretical introduction to the techniques used in this research.

The State Space

The time evolution of a system can be visualized in a State Space [5–7], in the case of a mechanical system this is a tridimensional plotting of displacement $X(t)$ and velocity $V(t)$ versus time, as depicted in Fig. 1. The State Space of a Hamiltonian of energy preserving system is known as a Phase Space. The state space is a geometrical representation of the behavior of the system.

In the simplest cases a state space is presented as the bi-dimensional projection of the evolution of the system on the $X - V$ plane, and if the system is not chaotic the shape of its state space is smooth and understandable, but when the system is chaotic the shape of the state space becomes extremely irregular as time elapses, hence in order to analyze the 3D-version of the state space a Poincaré section, Fig.1, is used.

Attractors

The orbit in the state space of a simple pendulum is an elipse, showing the energy conservation at points when the position and velocity are zeros, respectively. But, with dissipation the orbit reaches a point in the state space after some oscillations. This means, that a system with dissipation the orbit in the state space has been attracted to some central region, in this case a point showing a lower dimension.

Then, an Attractor is the orbit in the state space showing the behavior of a system that settles down to, or is attracted to a point in the long term [5].

The Fig.1 shows the state space of a damped oscillator, which eventually stops, it can be seen that the state space orbit gradually approaches a point and once there, it remains there. Whatever the initial amplitude of this oscillator, its orbits collapse to a point in state space, hence the point is known as an Attractor. If the oscillator were not damped, energy would be conserved and the shape of the orbit would be a loop, circle or ellipse, and this would be the attractor.

Chaotic attractors have a complicated geometry, they are associated to unpredictable motions and they are fractals, i.e., their dimension is fractional. Fractal structures unveil more and more details as they are more and more magnified.

Poincaré maps

Commonly in order to visualize chaos in a given system, the Poincaré sections [5, 6] are extracted and plotted. A

Poincaré section is the 2D plotting, see Fig. 1, of the points where the phase space orbit intersects a surface, usually a plane, at a selected angle chosen from 0° to 360° on the $X-V$ plane. The objective of the Poincaré section is to detect any structure in the attractor, if there is one.

Notice that while the state space is a 3D plotting, the Poincaré section is a 2D one, hence the latter is easier to analyze than the former.

Experimental detection of the oscillation period

It is worthwhile recalling that in the case of a regular oscillator the oscillation period may be experimentally obtained as the distance in the amplitude versus time plotting, between any two consecutive points with the same phase. Obviously, chaotic oscillators are far from regular, however this criterion at the time of inspecting the period is maintained and, in the chaos argot researchers speak of a period doubling cascade.

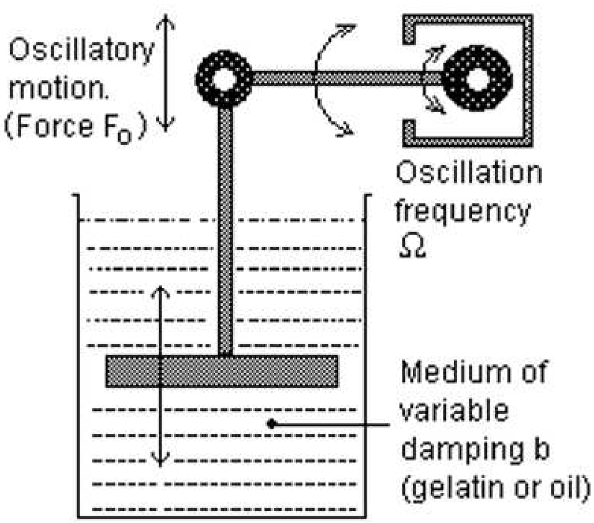

Figure 2: Forced oscillations of a vibrator immersed in a medium of variable damping.

Analyzing chaos simulations

Chaos computer simulations are accomplished by numerically solving the differential equations modeling the systems in which chaos is to be investigated. When using the Runge-Kutta method to solve these equations, two time series are generated [3,4], one being displacement $x(t)$ and the other velocity, $v(t)$, both as a function of time. With the $x(t)$ and $v(t)$ time series a State Space is generated. Since directly analyzing the state space is rather difficult, a Poincare Section at some selected angle is extracted and plotted for analysis [3, 4].

The Chaos research being reported

This research is focused in the chaotic oscillations of a periodically forced nonlinear vibrator immersed in a medium of variable damping, which may be modeled by the mechanical system shown in Fig. 2, and which is mathematically represented by the differential equation

$$\frac{dx^2}{dt^2} + b\frac{dx}{dt} + \omega_0^2 \sin x = F_0 \sin \Omega t \ , \tag{1}$$

where ω_0 is the natural frequency of the oscillator and $Omega$ is the frequency of the applied periodic force F_0.

At simulation time the damping b of the system was kept constant and the applied force was continuously varied, this is equivalent to maintaining the applied force constant and varying the damping.

The Virtual Lab used in this investigation is prepared to show on screen a vibrator oscillating according to the simulation evolution and simultaneously depicting the state space. This feature helps to understand the orbits appearing in state space, which not always are so easy to follow, because for example, sometimes the chaotic oscillator makes weird kicks instead of completing an orbit, some other times the oscillator makes unexpected stops and changes in its motion direction.

The Virtual Lab [3,4] is enabled to detect the Poincaré section at any angle. However it has been experimentally found that the state space orbit not necessarily hits a Poincare section at every angle, it has been observed that for some angles the Poincare sections have very few hits or they are completely empty. With the aim on maintaining a unique frame of reference for all simulations, this research was focused on the Poincaré sections at 0° and at 180°. Previous research had determined that chaotic events display rather rich Poincare sections at these two angles.

Results

It has been encountered that chaotic events do not last forever, they are finite, having a beginning and an end; evidence of this may be appreciated in Figs.3, 4 and 5, where the complete chaotic events are displayed. Also it has been found that the studied system may display many chaotic events. Actually some 25 chaotic events were detected, but after screening them some of them were discarded due to parameter similitudes and all were reduced to the five shown in Fig.5, where to save space only Poincaré Maps at 0° are shown.

The Fig. 3 displays the Poincaré Maps for a chaotic event along 30 million time steps. The Fig. 4 displays these maps for a simulation along 4.7 million time steps. Recalling that the Poincaré Map allows visualizing portions of the state space, it is evident that the phase space orbit is not uniform, the X and V behavior of the system continuously changes as time elapses.

The Figs.3 and 4 displays two period bifurcation cascades for the nonlinear damped and forced oscillator, both are Poincaré Maps, the upper corresponds to 0° and that at the bottom is at 180°. Obviously with higher-resolution computer graphics more details may be appreciated. It is evident that chaotic events are transitory, they have a beginning and an end, it is also evident that the system abandons chaos with the same smoothness it entered.

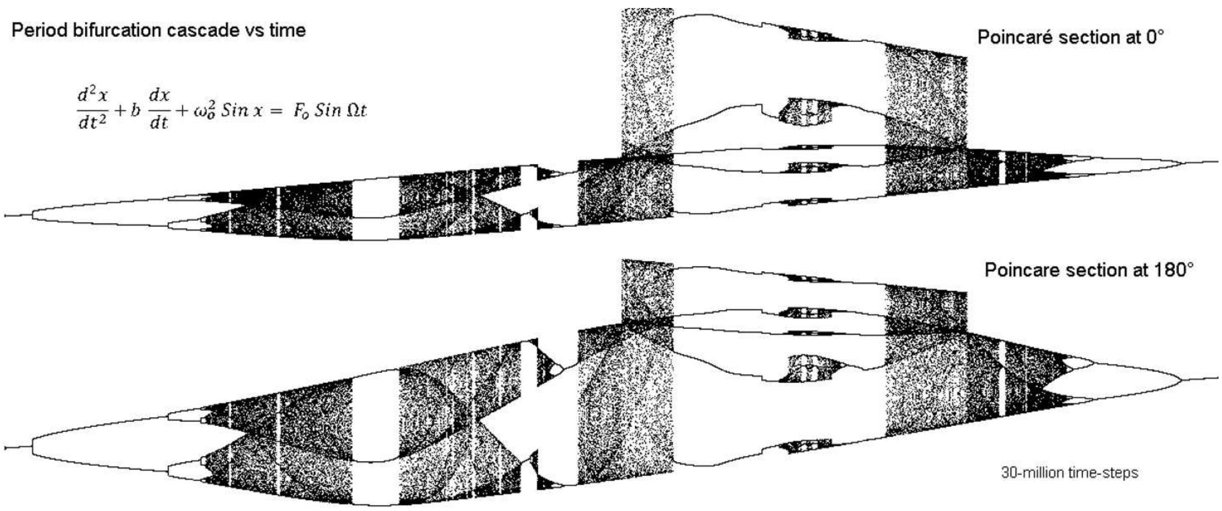

Figure 3: Chaos in the nonlinear damped and forced oscillator. The image shows the results of a simulation along 30 million time steps. Up to three period bifurcations are clearly seen ath the left side. The upper and lower cascades are the Poincaré Sections at 0° and at 180°, respectively. When higher resolution computer graphics are used to display this image, more bifurcations and details are seen with the naked eye. It can be seen that the chaotic event has a beginning, at the left, and an end at the right side, besides this, it is also evident that the system leaves chaos, right side, with the same smoothness it entered into it at the left side.

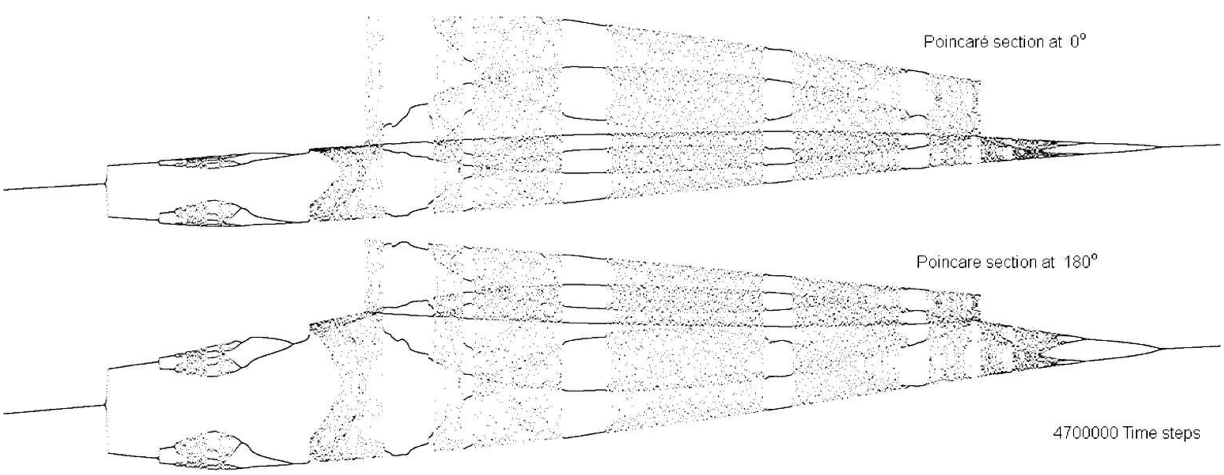

Figure 4: Chaos in the nonlinear damped and periodically forced oscillator. Poincaré Sections for 0° (top) and for 180° (bottom) along 4.7 million time steps. It is evident that chaos does not last forever. When antering chaos there is a series of periodic bifurcations, when abandoning chaos there is a series of period collapses, being in both cases a smooth transition.

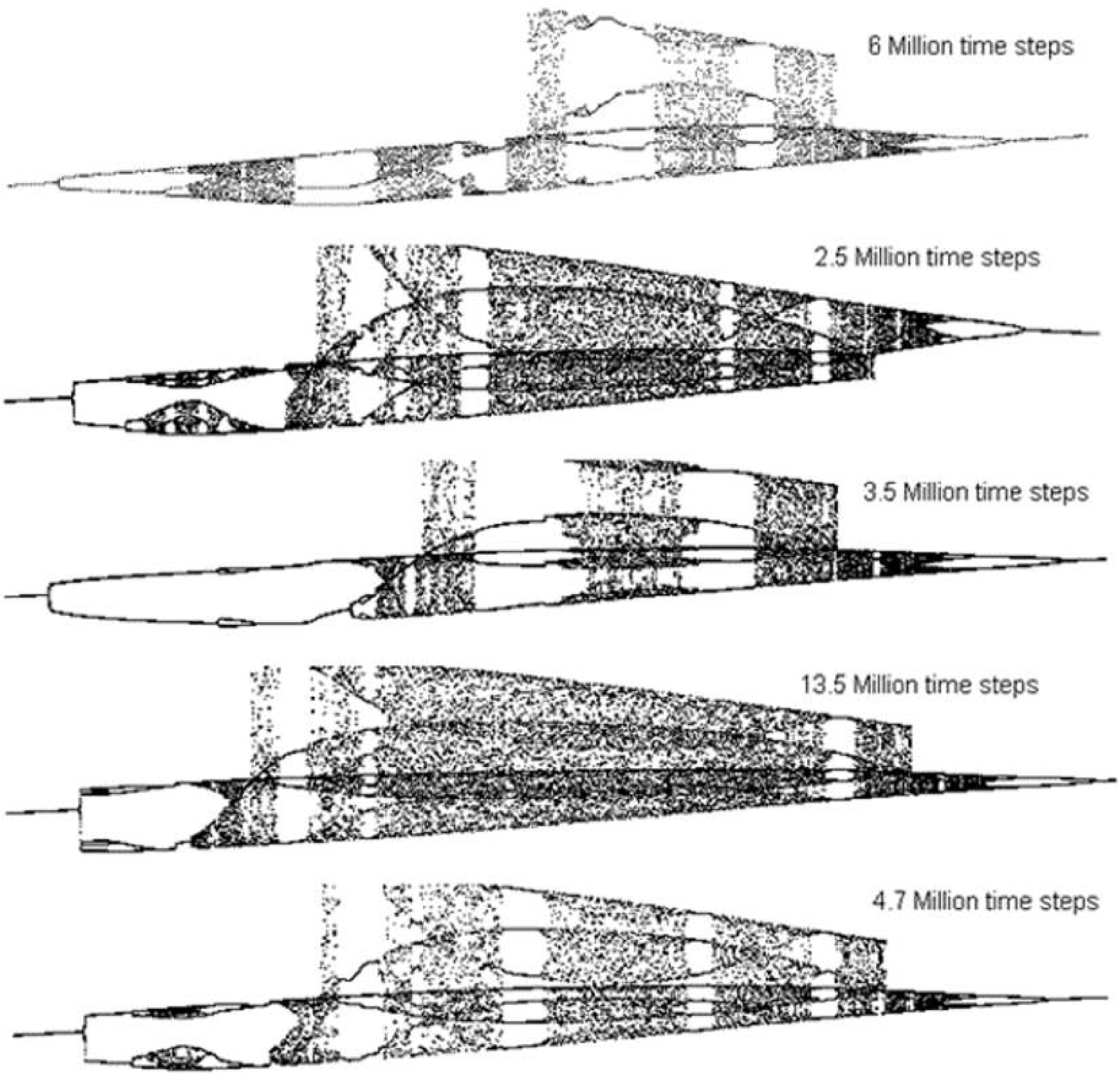

Figure 5: Poincaré Maps at 0° (Periods vs time) for five different chaotic events. As it can be seen, all the events have a beginning and an end and, the system abandons chaos as smoothly as it entered it. In each case the number of time-steps is shown.

For every identified chaotic event the natural frequency of the oscillator was compared with that of the applied force and, no common feature was observed, this is, no connection between them was detected.

Concerning the relationship between the frequencies associated to the chaos events depicted in Fig.5, the relation ω_0/Ω does not show any regularity; their values, from top to bottom, are the following:

$$\frac{\omega_0}{\Omega} = 1.39,\ 2.676,\ 2.49,\ 3.038,\ 2.7452. \qquad (2)$$

In these simulations when the system abandons chaos the oscillation amplitude increases, if not immediately, after some short time, this is because in this simulation the applied force is periodic and every time higher. The same behavior is observed at the beginning of the simulation, before the system begins to bifurcate its period to enter chaos.

Conclusions

Many chaotic happenings were detected in the nonlinear damped and forced oscillator; these were later classified as belonging to five different events. This means that there is a multiplicity of chaotic events. If investigation is continued it is highly probable that more chaotic events will be detected in this system.

It has been found that chaotic events are transitory, they do not last forever, this is they have a beginning and an end; additionally it has been observed that the system leaves chaotic events with the same smoothness it started them. When entering a chaotic event a bifurcation of the period is observed, then each of these bifurcations bifurcates again and again, and then the system bursts into chaos. When abandoning chaos the opposite effect is observed, this is, the system collapses the period cascade by pairs until finally it finishes with a single period. This means that the transition towards chaos is as smooth as the transition out of it.

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