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Structure of the Velocity during a Chaotic Episode

Javier Montenegro Joo*[1,2]

[1] *Facultad de Ciencias Físicas, Universidad Nacional Mayor de San Marcos, Lima 1, Perú*
[2] *VirtualDynamics: Science & Engineering Virtual Labs, Lima, Perú*

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An investigation on the structure of the velocity in the nonlinear damped and forced oscillator, during a chaotic episode has been carried out. It has been found that the extreme values of the velocity show a bifurcating cascade as time elapses, just like the cascade detected in the extremes of oscillation amplitudes versus time, which leads to the very well-known bifurcation cascade of the periods during a chaotic event. It has been encountered that displacement and velocity during a chaotic episode in the investigated system, keep both the same structure. This similar structure detected in displacement and velocity suggests the possibility that these two magnitudes share the same pattern in other chaotic systems.
Keywords: Non-linear oscillations, chaos, displacement, velocity, Poincare Maps.

Estructura de la velocidad durante un episodio caótico

Se ha realizado una investigación sobre la estructura de la velocidad en el oscilador no-lineal forzado y amortiguado durante un episodio caótico. Se ha encontrado que los valores extremos de la velocidad muestran una cascada de bifurcación a medida que transcurre el tiempo, tal como la que se observa en los valores extremos de la amplitud de oscilación versus tiempo, lo cual conduce a la muy bien conocida cascada de bifurcación del periodo de la oscilación durante un episodio caótico. Se ha encontrado que el desplazamiento y la velocidad durante un evento caótico en el sistema investigado, mantienen ambos la misma estructura. La estructura similar detectada en el desplazamiento y la velocidad sugiere la posibilidad de que estas dos magnitudes manifiesten el mismo esquema en otros sistemas caóticos.
Palabras claves: Oscilaciones no-lineales, caos, desplazamiento, velocidad, Mapas de Poincaré.

From elementary oscillations theory, specifically from the simple harmonic motion (SHM), it is known that the velocity and acceleration of an oscillator also oscillate as time elapses. The SHM is a smooth oscillatory motion with smooth velocity and acceleration, where the displacement, the velocity and the acceleration have all the same structure, the three are harmonic functions of time.

Obviously it is intuitively expected that a chaotic oscillator must have both, disordered velocity and disordered acceleration, even more, from the experience with the SHM it is expected that displacement, velocity and acceleration must share the same pattern. When watching with the naked eye and in slow motion [1] the (x, v, t) evolution in State Space, it results evident that the velocity of a chaotic oscillator is rather messy; this may also be appreciated in Fig. 1, which shows a fragment of a chaotic event, in this figure the displacement and velocity are quite disordered, however, the question of the question is whether the velocity during a chaotic event, is simply messy or it has some structure and, if so, what is that structure.

*Director@VirtualDynamics.Org, www.VirtualDynamics.Org

A plotting of acceleration versus time shows a similar behavior. The investigation reported in this paper deals with the structure of the velocity of an oscillator while it undergoes a chaotic episode. The study of the acceleration is left for a future work yet to come.

Velocity of the system

With the aim on visualizing the to-and-from motion of an oscillator, a virtual simulator [1] which shows on computer-screen the motion of a little ball, while motion graphs are also sketched on screen, has been developed by this researcher. As the little ball oscillates this simulator simultaneously depicts curves of displacement, velocity, acceleration and space state.

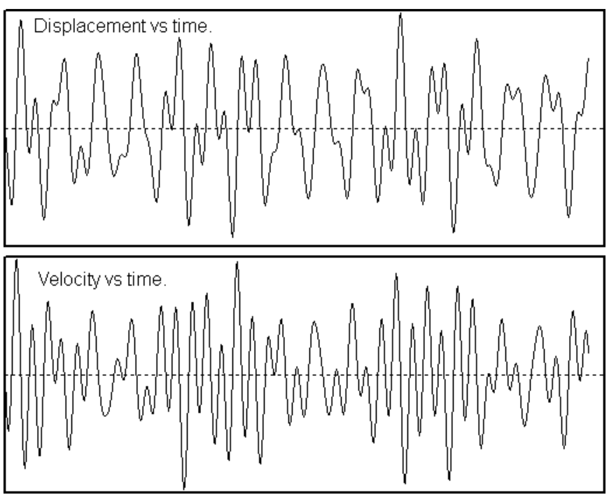

Figure 1: Fragment of a very simple chaotic event. Top: Displacement (Amplitude) vs time. Bottom: Velocity vs time. Obviously not only the amplitudes of oscillation are messy, the velocities are too.

When this simulator showed the evolution of a chaotic oscillator, it was observed that the velocity behaves in an unpredictable way. When the little ball forms part of a pendulum, it was also observed that the chaotic pendulum eventually completes several turns in a given direction around its central point, and not necessarily goes back completing the same number of turns in the opposite direction.

In a few words, it was observed that the speed of a chaotic oscillator has an irregular behavior, impossible to foresee and which deserves further study. In this research and with the aim on detecting any structure -if there is one- in the velocity of a chaotic oscillator, a computer program that generates plottings of the velocity as the simulation takes place, was developed.

The Poincare Map

A methodology to study the evolution of a dynamical system is to analyze -its tridimensional- State Space [2, 3] (Figure 2) and, one way of achieving this is by means of bi-dimensional Poincare Maps [2–4].

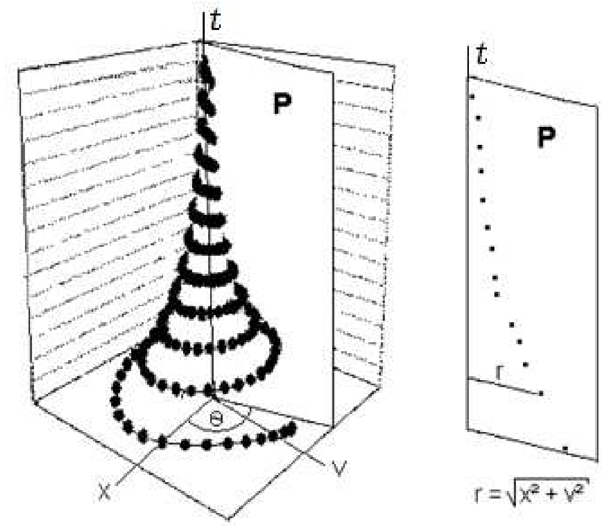

Figure 2: The State Space is the 3D plotting of displacement and velocity versus time (time along the vertical axis). In this case the depicted curve is that of a damped oscillator, for this reason the curve shrinks with time until the oscillator eventually stops. The sketched plane **P** is the Poincaré section (or plane) at an angle θ with the x-axis. The curve intersects the Poincaré plane at some points which constitute the Poincaré Map at angle θ. Notice that theoretically there are infinite Poincaré Maps. The sequence of (x, v, t) points on State Space is known as "The Flow" of the system.

The Poincare Plane **P**, may be seen as a tomographic cut along time of the State Space, see Figure 2. This plane **P**, is defined at some angle with the x-axis. The Poincare Map is the set of all the intersections of the (x, v, t) curve -the flow of the system- with the plane **P**, at a predefined angle.

In this way the Poincare Map contains the structure of the State Space at the angle it is extracted. Obviously, (see Figure 3) the structure of displacement is obtained at angle $\theta = 0$, the structure of velocity is obtained at $\theta = \pi/2$, and so on.

From the experience of this researcher, the Poincaré approach works very well as long of the oscillator maintains constant or almost constant its center of oscillation, like the case of the damped oscillator shown in figures 2 and 3.

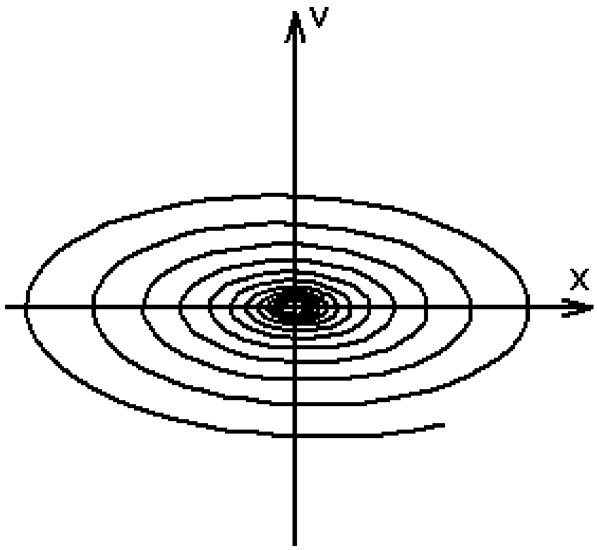

Figure 3: Projection of tridimensional State Space (x, v, t) points over the XV plane for a damped oscillator which maintains constant its oscillation center. Extreme values of amplitude (dislacement) are on positive and negative sides of x-axis, while extremes velocities are on extremes of v-axis.

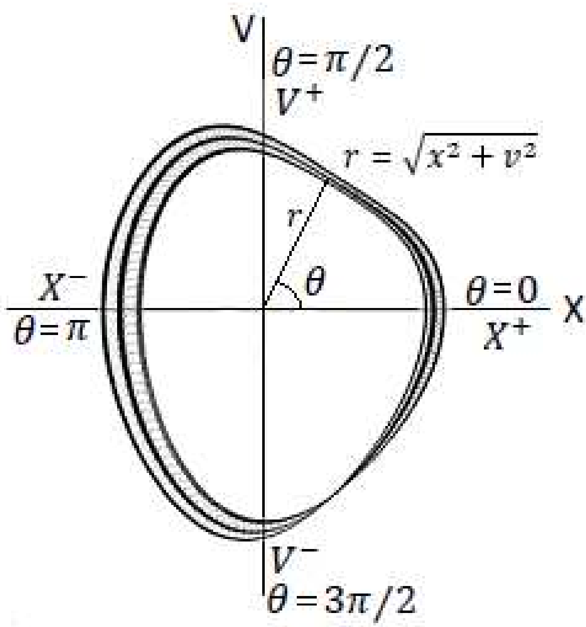

Figure 4: Projection of State Space (x, v, t) points over XV-plane. The sketch shows the connection of X-V axes with different angles for Poincaré Maps. Planes at $\theta = 0$ and $\theta = \pi$ collect the extreme values (positive and negative) of the oscillation amplitude, while planes at $\theta = \pi/2$ and $\theta = 3\pi/2$, capture the extreme values of oscillation velocity.

In the case of an irregular oscillator, which constantly displaces its rotation (equilibrium) center, sophisticated computer programming must be made to extract not-so-obvious information from the State Space. An example of this situation is depicted in Fig. 5, which displays the projection of the tri-dimensional State Space on the VX plane. In this case the oscillator initiates its motion oscillating on the negative side of its resting position and, little by little displaces its rotation center towards the positive side of its equilibrium position. As it results evident in Fig. 5, the Poincare map at 180° will be rather rich in this case, while that at 0° will result very poor.

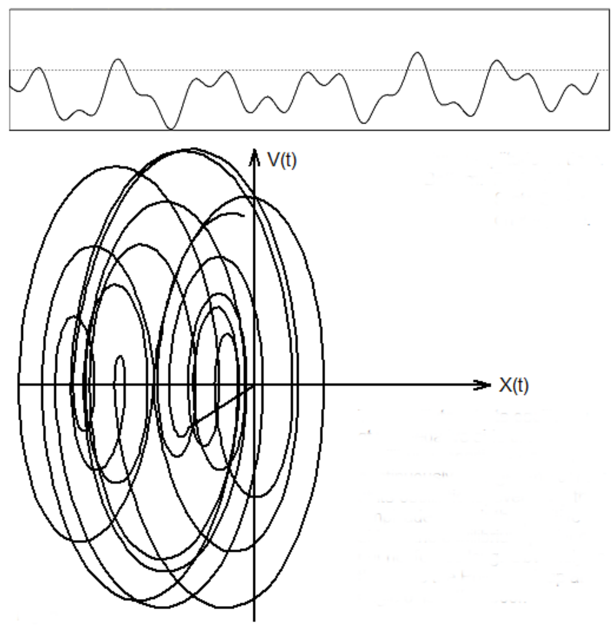

Figure 5: Top: Amplitude vs time. Bottom: Projection of State Space on the X-V plane. The oscillator starts oscillating at the negative side of its equilibrium position and then continuously changes the center of its oscillations, eventually the amplitudes reach the positive side of the equilibrium position, but not for too long. Obviously, in this case the Poincaré Map at angle 0 is rather poor.

State Space of a Chaotic Oscillator

In the case a regular non-chaotic oscillator, the State Space is neat and understandable, to the point that the motion of the oscillator is immediately understood. In the case of a chaotic oscillator (see Fig. 6) the State Space has a highly disordered appearance; it looks literally chaotic in the common sense of the word.

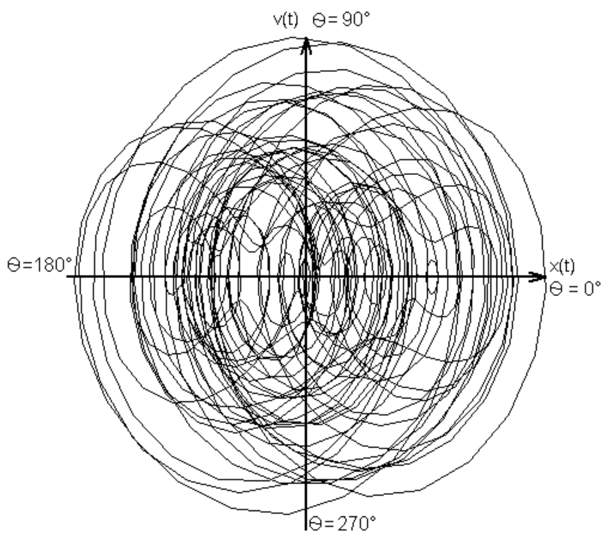

Figure 6: Projection of the tridimensional State Space of a chaotic oscillator on the XV plane. As it can be seen the plotting has a very messy appearance, it looks literally chaotic.

The mathematical model used in this research

The model used in this investigation is that of the nonlinear, damped and forced oscillator, whose differential equation of motion is

$$\frac{d^2\theta}{dt^2} + \left(\frac{b}{mL}\right)\frac{d\theta}{dt} + \omega_0^2 \sin x = \left(\frac{F_0}{mL}\right)\sin \Omega t \quad (1)$$

and where the author of this report has previously detected several chaotic events [5–8].

This second-order differential equation is numerically solved by means of the Runge-Kutta method in a VirtualLab [9] completely developed from scratch by the author. In the Runge-Kutta solution process values of displacement and velocity are generated for every time step, these values are used to depict a virtual and tridimensional State Space in computer memory.

Results of the investigation

The results of this investigation are condensed on figures 7 and 8, which show the extreme values of oscillation amplitude (Fig. 7) and those of velocity (Fig. 8).

In a previous research with the mathematical model used in this investigation, this researcher found that chaotic events in this system were finite -they have a beginning and an end- and it was also encountered that multiple chaotic events are also possible [7, 8]. In this research it has been discovered (see figures 7 and 8) that the velocity has in general a structure similar to that of the corresponding oscillation amplitudes.

Figures 7 and 8, display a common panorama observed in the multiple [7, 8] chaotic events detected in the system under investigation. In all cases the oscillation amplitudes and corresponding maximum velocities have similar structures. Amplitudes and velocities show similar bifurcation cascades, at the same time.

In figure 7, the plottings of the extreme values of amplitude indicate that at first there is a single amplitude of oscillation, which implies a unique oscillation period, then there are two interchanging extreme values of oscillation, which means that the system oscillates switching between two periods, this is, the initial period has bifurcated. Next each of these two periods splits in other two each. The period-bifurcation process continues until it becomes impossible to foresee what will be the following value of the maximum amplitude, this means that the period has become impossible to predict and, that chaos has set in.

In figure 8, the plotting of the extreme values of velocity versus time indicates that initially the system oscillates smoothly accelerating until it reaches a maximum and almost constant value of velocity. After sometime, the system keeps oscillating and accelerating until two maximum and alternating values of velocity are reached, the velocity has undergone bifurcation. Afterwards each maximum velocity splits again, and the process continues, until it becomes impossible to foresee what will be the next extreme value of velocity.

Though impossible to foresee before the investigation, the similitude in structure (similar bifurcation cascades) detected between displacement and velocity for the chaotic oscillator is not so surprising after all, because amplitudes and velocities are 90° apart in State Space, and the flow of the system cannot vary too much through such a short distance.

In figures 7 and 8, it results evident that during a chaotic event, the extreme values of amplitudes as well as those of velocities are not necessarily symmetrical at both sides of the (main) equilibrium position of the oscillator.

Once the system abandons chaos the oscillation amplitudes as well as the corresponding velocities, if not maintained constant, they vary slowly and gradually, with no additional bifurcations.

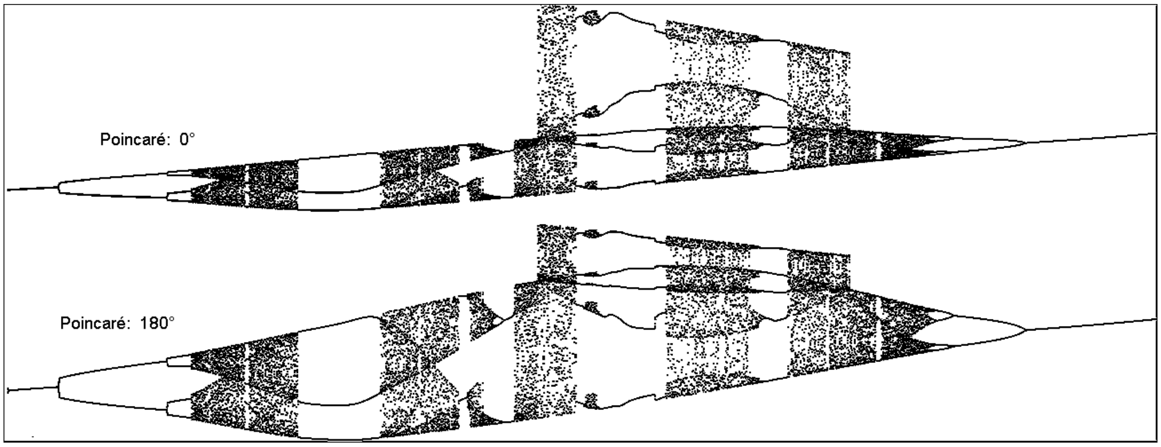

Figure 7: Shown the extreme values of oscillation amplitude versus time.

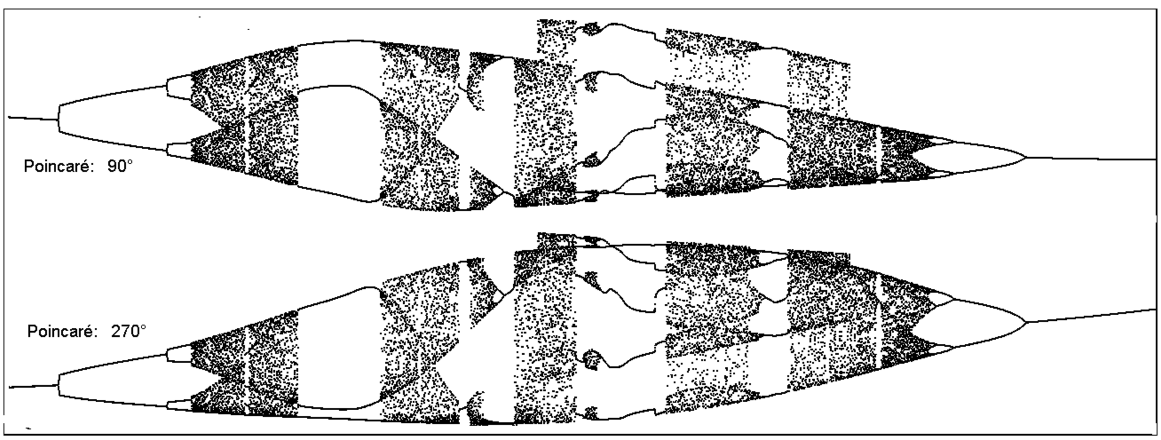

Figure 8: Shown the extreme values of oscillation velocity versus time.

Conclusions

An investigation to detect structure in the velocity during a chaotic event has been carried out for the nonlinear damped and forced oscillator. It has been detected that indeed, there is a structure in the velocity while the system undergoes a chaotic event. Many chaotic events were studied in the above mentioned system and, as in the case of oscillation amplitudes, there are bifurcation cascades in the velocity. In general, it has been found that the velocity structure during a chaotic event in the nonlinear damped and forced oscillator is very similar to the structure of the corresponding amplitudes. This behavior suggests what to expect concerning displacement and velocity in other chaotic systems.

It has been encountered that the extreme values of oscillation amplitudes as well as those of velocity are not symmetrical with respect to the equilibrium position of a chaotic oscillator.

As soon as the system terminates its chaotic stage there are no further bifurcations, neither in amplitude nor in velocity.

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