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Use of the logarithmic decrement to assess the damping in oscillations

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A Virtual Lab to experimentally detect the frequency of the oscillations of a damped oscillator has been created. Once input data such as mass, elastic constant and damping of the medium are entered, the simulation module depicts the oscillations on computer screen, automatically detects the extreme displacements of oscillation and applies the logarithmic decrement algorithm, which leads to the exponent characterizing the gradual shrinking of displacements. From this exponent the frequency of the damped oscillations is calculated, this –as it is expected- is equal to its theoretically calculated value.
Keywords: Damped oscillations, logarithmic decrement, computer simulation, Virtual Lab.

El uso del decremento logarítmico para evaluar el amortiguamiento en las oscilaciones

Se ha creado un Laboratorio Virtual para detectar experimentalmente la frecuencia de oscilación de un oscilador amortiguado. Una vez que se introducen los datos de entrada como masa, constante elástica y amortiguación del medio, el módulo muestra las oscilaciones en la pantalla de la computadora, detecta automáticamente los desplazamientos extremos de la oscilación y aplica el algoritmo de Decremento Logarítmico, que conduce al exponente que caracteriza la contracción gradual de los desplazamientos. De este exponente se calcula la frecuencia de las oscilaciones amortiguadas oscilador, este resultado – tal como se espera- es igual al valor calculado teóricamente.
Palabras claves: Oscilaciones amortiguadas, decremento logarítmico, simulación en computadora, Laboratorio Virtual.

In a damped oscillating system, where the damping is not known, the Logarithmic Decrement may be used to find the damping of the system [1, 2]. The logarithmic decrement is defined as the natural logarithm of the ratio of any two successive maximum displacements in a damped oscillation. Obviously these two maximum amplitudes x_{n+1} and x_n are separated by a certain time t,

$$\frac{x_{n+1}}{x_n} = \exp[-\lambda t], \qquad t = t_{n+1} - t_n.$$

The exponent is negative because in a damped system the amplitudes of the oscillations shrink. In the case with no damping, if the amplitudes of the oscillations would increase, the exponent would be positive and, if the amplitudes were constant, the exponent would be zero.

The expression above is valid provided the oscillations are uniform; this is, as long as the distance between orbits in State Space keeps constant. In Chaotic oscillators the displacements are far from being uniform and the State Space is literally chaotic, in the most common sense of the word [3].

Damped oscillations

Damped oscillations are characterized by the fact that the amplitudes of oscillation tend to reduce as time goes by. Obviously, the higher the damping, the quicker the oscillations shrinkage. Depending on the relationship between the natural frequency of the oscillator and the applied damping, damped oscillations are classified as Critical, Subcritical and Supercritical. In this paper the subcritical case is dealt with, this case is also known as underdamped oscillations.

The differential equation of the underdamped oscillator

From elementary university physics it is known that the differential equation of motion of a system oscillating in presence of a damping [2, 4] is

$$m\frac{d^2\vec{x}}{dt^2} = -k\vec{x} - b\vec{v} \qquad (1)$$

The first term on the right side is the reacting force of the spring (Hooke's Law) and the second term is the viscous damping, indicating that the faster the spring oscillates, the higher the resistance (minus sign) due to the viscosity b of the medium.

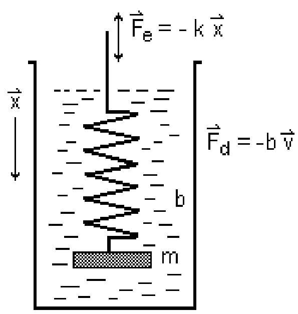

Figure 1: The oscillatory motion of the spring in the liquid is attenuated by the viscosity of the medium whose viscosity constant is b. The elastic force of the spring is F_e and the damping force of the medium is F_d.

Bearing in mind that the velocity is the first temporal derivative of the displacement, the scalar version of equation (1) is written as

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \ .$$

Here, two coefficients are identified

$$\omega_0^2 = \frac{k}{m} \qquad \text{and} \qquad G = \frac{b}{2m}, \tag{2}$$

where ω_0 is the natural frequency of the oscillator, this is the frequency of the free oscillator, and G is the damping, which depends on the viscosity b of the medium.

After inserting ω_0 and G, the differential equation becomes

$$\frac{d^2x}{dt^2} + 2G\frac{dx}{dt} + \omega_0^2 x = 0. \tag{3}$$

In the underdamped case, $\omega_0 > G$, Eq. (3) may be solved by means of any of three methods: (1) Elementary methods of solution of ordinary differential equations (2). The Laplace transform [1]. (3) Physical intuition. This author prefers the latest method, the result is

$$x(t) = A\exp[-\lambda t]\sin(\omega t + \alpha). \tag{4}$$

Replacing Eq. (4) in Eq. (3), we obtain

$$\omega^2 = \omega_0^2 - G^2 \ , \tag{5}$$

in this equation ω is the theoretical angular frequency of the damped oscillator, and the period of the damped oscillations is $T = 2\pi/\omega$.

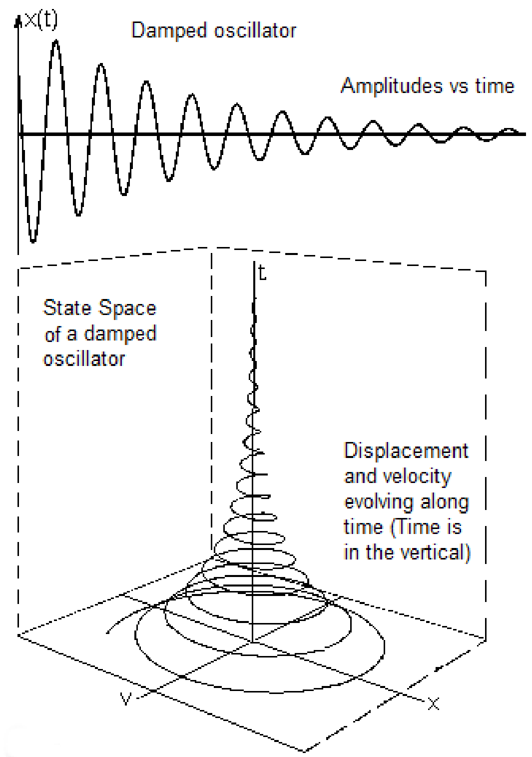

Figure 2: Top: time evolution x(t) of the oscillation for a damped oscillator. Bottom: the corresponding State Space, this is the 3D-plotting of displacement and velocity versus time. It can be seen in both graphs that the amplitudes of oscillation decrease as time elapses, until finally they stop.

The Logarithmic Decrement

The logarithmic decrement [1,2] is based on the assumption that the shrinking in the maximum amplitude in an underdamped oscillation for any two successive oscillations, is given by

$$\frac{x_{n+1}}{x_n} = \exp[-\lambda t], \tag{6}$$

where it is assumed that the shrinking of the orbit in state space is constant, in this expression the values of x_{n+1} and x_n are the maximum amplitudes of two successive oscillations, these are the peaks (or the valleys) in the x versus t plotting, Fig. 2, or the points where the (x, v) curve cuts the x-axis in the 2D version of the state space, Fig. 3.

From elementary university physics, the elapsed time between any two successive oscillation (displacement) peaks is a period T, hence eq. (6) must be rewritten and then applying logarithms to both sides,

$$\lambda = -\frac{1}{T}\ln\left[\frac{x_{n+1}}{x_n}\right] \ . \tag{7}$$

The period T in this equation is the period of the damped oscillations $T = 2\pi/\omega$. The value of λ in Eq.(7) is obtained experimentally by averaging many cases of Eq. (7).

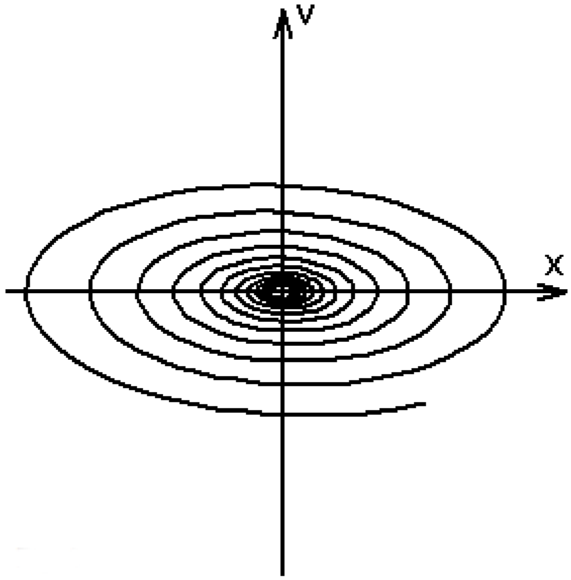

Figure 3: Projection of the (x,v,t) points of State Space over the XV-plane. This is a 2D version of the 3D State Space. In this research the values of maximum displacement, this is the points intersecting the x-axis in this plotting are used to feed eq. (6). These points are the peaks of the X versus t curve.

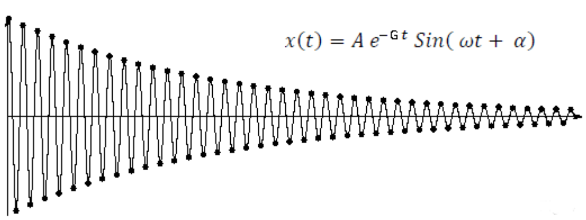

Figure 4: Plotting of displacements versus time x(t) for the damped oscillator. This plotting has been generated by the Virtual Lab being reported. Peaks and valleys have automatically been highlighted.

On the other hand, applying Eq.(4) to two successive amplitude peaks (separated in time by a period T):

$$x_n = A\exp[-\lambda t]\sin(\omega t + \alpha) \tag{8}$$

$$x_{n+1} = A\exp[-G(t+T)]\sin(\omega(t+T) + \alpha)\ . \tag{9}$$

From Eq.(9), $x_{n+1} = A\exp[-G(t+T)]\sin(\omega t + \alpha + \omega T)$, but $\omega = 2\pi/T$, then $x_{n+1} = A\exp[-G(t+T)]\sin(\omega t + \alpha + 2\pi)$, rewritten as

$$x_{n+1} = A\exp[-G(t+T)]\sin(\omega t + \alpha). \tag{10}$$

In this way from (8) and (10),

$$\frac{x_{n+1}}{x_n} = \exp[-GT]. \tag{11}$$

From equations (11), (7) and (5),

$$\lambda = G = \sqrt{\omega_0^2 - \omega^2}\ . \tag{12}$$

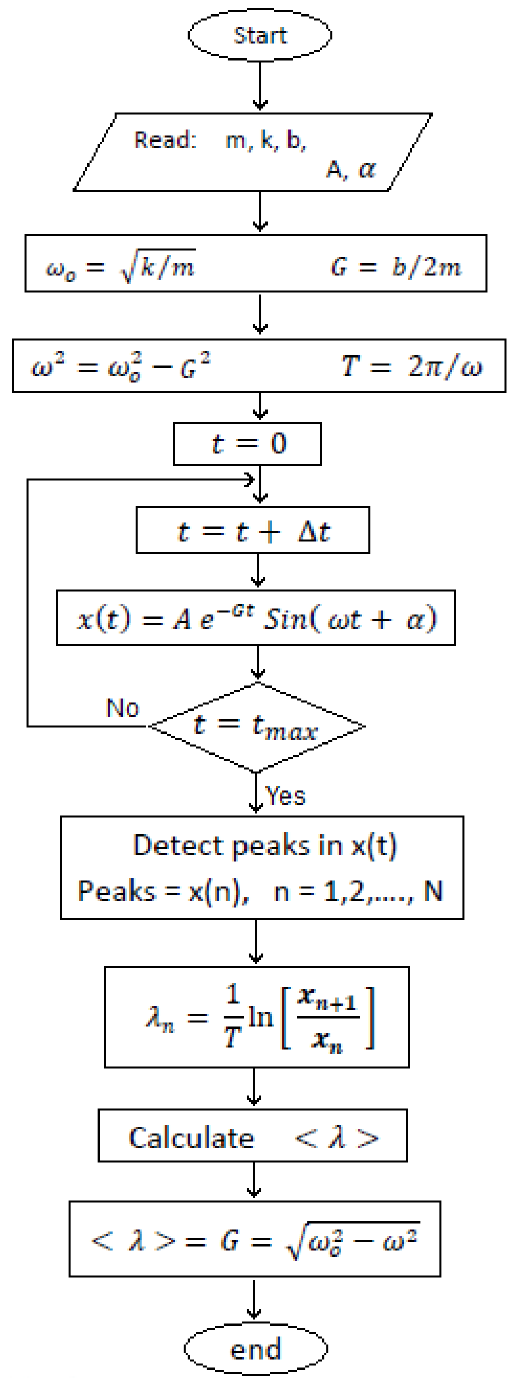

Figure 5: Flowchart of the process in the simulator being reported.

Executing the experiment

The developed module receives the mass m of the oscillator, its elastic constant k as well as the damping b of the medium as input data. Additionally the module reads the maximum amplitude "A" of the oscillations as well as the initial phase. Also the time-steps of the simulation must be entered as part of the input data. See the flowchart in figure (5).

The maximum amplitude of oscillation, the initial phase and the time-steps are included so that the user of the Virtual Lab is enabled to appreciate the effects of changing them, entering these data is not expressly fundamental for execution of the experiment, because they have been included by default, however, larger values of "A" generate large amplitudes of oscillation, and these are more visible.

Once the input data is entered the module computes from Eqs.(2) the theoretical angular frequency of the oscillator and the theoretical damping of the medium. Additionally from Eq.(5) the theoretical value of the angular frequency of the damped oscillations is computed, as well as the period of the damped oscillations.

Next the simulation of Eq.(4) is executed during enough time for the oscillations collapse to zero. Notice that this is not indispensable at all, it would be enough running the experiment during a time equivalent to a few periods so that some values of x_n are available to evaluate Eq.(7).

Once the simulation starts, the curve of displacements versus time $x(t)$ is depicted on computer screen, see fig.(4), and its maximum displacements are automatically detected. Then equation (7) is evaluated for every two successive maximum amplitudes of oscillation.

In Chaos Theory [3] the above mentioned computational detection of peaks and valleys is tantamount to extracting the Poincaré Maps at 0° and at 180° respectively.

Finally the average is calculated and used together with Eq.(12) to obtain the experimental value of the frequency ω of the damped oscillations.

A final checking verifies that the experimental value of ω is equal to its theoretical value.

The Virtual Lab being reported automatically detects peaks and valleys, however only some peaks or valleys are necessary to this experiment. In practical applications the maximum displacements of the oscillations may be detected with the naked eye by simply watching the oscillations and measuring their extreme values.

Demonstrative execution of the Virtual Lab

The following is a typical execution of the simulator being reported, which has been fed with random input data:

```
<><><>   Experiment N = 1      <><><>
 Input data .....

 Mass    m = 4 kg
 Elastic constant    k = 70 N/m
 Viscosity     b =   0.8400
 Amplitude = 68
 Initial phase =   0.5061 rad
 Delta time =   0.0250 s

 Theoretical results:

 Damping of the oscillator    G =  0.1050
 Natural Angular freq.    Wo = 4.1833
 Damped ang. freq.      W =  4.1820
 Damped Oscillations period  T = 1.5024
 N time-steps = 2500
 Experiment duration =  62.5000 s

 Mean Lambda = < Lambda > =  0.1050

 Experimental results:

 < Lambda > = G = Sqrt[ Wo² - W² ]

 Experimental Damping:  G =   0.1050
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It can be seen that the experimental value of the damping G is in agreement with its theoretically calculated value.

Conclusions

A virtual lab to execute experiments dealing with the Logarithmic Decrement of the oscillation amplitudes for an underdamped oscillator has been developed.

Giving the mass of the oscillator as well as its elastic constant and the viscosity of the medium, as input data, the developed module experimentally finds the damping of the oscillator by means of the logarithmic decrement method.

The simulator (Virtual Lab) reported in this paper sheds light on the fact that in real life situations, the damping of the oscillator and the viscosity of the medium in which the oscillator is vibrating, may be found by measuring some successive extremes of the oscillation amplitudes and using the logarithm decrement. The developed module has been included in the Physics Virtual Lab (PVL) [5].

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