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A relativistic theory of the field

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Abstract
As gravitation and electromagnetism are closely analogous long-range interactions, and the current formulation of gravitation is given in terms of geometry, we expect the latter also to appear through the geometry. This unification has however, remained an unfulfilled goal. Thence emerges a relativistic theory of the asymmetric field by generalization of the general relativity. It will demonstrate in a new way that the field-equations chosen for the non-symmetric fields are really the natural ones.

Keywords: Infinitesimal parallel translation, curvature tensor, field equations, Maxwell's equations.

Una teoría relativista del campo

Resumen
Como la gravitación y el electromagnetismo son interacciones de largo alcance muy análogas, y la formulación actual de la gravitación se da en términos de geometría, esperamos que esta última también aparezca a través de la geometría. Sin embargo, esta unificación sigue siendo un objetivo incumplido. De esto surge una teoría relativista del campo asimétrico mediante la generalización de la relatividad general. Se mostrará que una nueva forma de las ecuaciones de campo elegidas para los campos no simétricos son realmente las naturales.

Palabras clave: Transporte paralelo infinitesimal, tensor de curvatura, ecuaciones de campo, ecuaciones de Maxwell.

Introduction

Gravitation is currently explained through the theory of general relativity (GR). As we know, there are at least two major difficulties with GR [1] [2] [3] [4] [5] [6] [7] [8].Firstly, it shows an intrinsic difficulty in its unification with the rest of physics, as it is a nonrenormalizable theory [9] [10] [11] [12]. Asumptions of dark matter and dark energy in order to explain the observations [13] [14]. Let us note that these 'dark entities' could not have been tested so far by any direct detection experiment, besides lacking any convincing theoretical motivation. Anyway, if a theory requires more than 95 % of the content of the Universe in the form of dark entities, it is an alarming signal to turn back to the very foundations of the theory.

Quantum field theory, on the other hand, is plagued with the divergence difficulties. Though the process of renormalization renders the theory in agreement with experiments (but not solving the problem itself), nevertheless this indicates that we cannot ignore contributions from gravitation at very high energies. It has long been speculated that if gravitation is included, certain infinite sets of divergent Feynman diagrams can give finite results [15].

Thus, we see pressing reasons to have a theory of

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gravitation compatible with other interactions-perhaps the electromagnetic. Expectedly, the properties of the new theory may be beyond the conventional paradigms. In the following, we sketch a geometric scenario wherein gravitation and electrodynamics appear naturally unified [16] [17] [18] [19]. However, it should be mentioned that there are two other interactions, the weak (also electroweak [20] [21] [22] [23] and the strong nuclear [24] [25] [26].

Our main task now is to find out whether there is a sufficiently convincing method of finding a unique set of field-equations for the non-symmetric fields [27] [28] [29] [30] [31] [32] [33]. The only reason why this derivation may seem not completely satisfactory is that we subject the field a priori to two conditions:

$$\Gamma^{\lambda}_{\underset{\vee}{\mu\lambda}} = \frac{1}{2}\left(\Gamma^{\lambda}_{\mu\lambda} - \Gamma^{\lambda}_{\lambda\mu}\right) = 0, \tag{1}$$

$$\frac{\partial(\sqrt{-g}g^{\overset{\mu\lambda}{\vee}})}{\partial x^{\lambda}} = \frac{1}{2}\frac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\mu\lambda} - \sqrt{-g}g^{\lambda\mu}\right) = 0. \tag{2}$$

To the convariant tensor $g_{\mu\nu}$ we can associate a contravariant one $g^{\mu\nu}$ uniquely by the condition:

$$g_{\nu\mu}g^{\nu\lambda} = g_{\mu\nu}g^{\lambda\nu} = \delta^{\lambda}_{\mu} \tag{3}$$

where δ^{λ}_{μ} is the Kronecker tensor.

In this article we shall show that an analogous argument can be used for the justification of the field-equations also in our case. It will demonstrate in a new way that the field-equations chosen for the non-symmetric fields are really the natural ones.

The infinitesimal parallel translation

We now introduce a quantity $\Gamma^{\lambda}_{\mu\nu}$ which transforms like the corresponding quantities in Riemannian geometry [34]. In analogy to the quantities of Riemannian geometry, the $\Gamma^{\lambda}_{\mu\nu}$ shall be Hermitian simmetric with respect to the lower indices

$$\Gamma^{\lambda}_{\mu\nu} = \widetilde{\Gamma}^{\lambda}_{\nu\mu}. \tag{4}$$

By contraction of the tensor $\frac{1}{2}\left(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}\right)$ we get the vector

$$\Gamma_{\mu} = \frac{1}{2}\left(\Gamma^{\lambda}_{\mu\lambda} - \Gamma^{\lambda}_{\lambda\mu}\right) = 0. \tag{5}$$

From the fundamental laws it follows that here the parallel translation of a vector is not unique operation for Γ. We therefore introduce the following symbols:

$$\begin{aligned} \delta A^{\overset{\lambda}{+}} &= -\Gamma^{\lambda}_{\mu\nu}A^{\mu}dx^{\nu}, & \delta A_{\underset{+}{\lambda}} &= \Gamma^{\alpha}_{\lambda\eta}A_{\alpha}dx^{\eta}, \\ \delta A^{\overset{\lambda}{-}} &= -\Gamma^{\lambda}_{\nu\mu}A^{\mu}dx^{\nu}, & \delta A_{\underset{-}{\lambda}} &= \Gamma^{\alpha}_{\eta\lambda}A_{\alpha}dx^{\eta}, \\ \delta A^{\overset{\lambda}{0}} &= -\tfrac{1}{2}\left(\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\nu\mu}\right)A^{\mu}dx^{\nu}, & \delta A_{\underset{0}{\lambda}} &= \tfrac{1}{2}\left(\Gamma^{\alpha}_{\lambda\eta} + \Gamma^{\alpha}_{\eta\lambda}\right)A_{\alpha}dx^{\eta}. \end{aligned} \tag{6}$$

Corresponding symbols are introduced for the infinitesimal parallel translation of covariant tensors as well as for covariant differentiation, e. g.:

$$\nabla_{\eta}A^{\overset{\lambda}{+}} = \frac{\partial A^{\lambda}}{\partial x^{\eta}} + \Gamma^{\lambda}_{\mu\eta}A^{\mu} \tag{7}$$

$$\nabla_{\eta}A^{\overset{\lambda}{-}} = \frac{\partial A^{\lambda}}{\partial x^{\eta}} + \Gamma^{\lambda}_{\eta\mu}A^{\mu} \tag{8}$$

$$\nabla_{\eta}A^{\overset{\lambda}{0}} = \frac{\partial A^{\lambda}}{\partial x^{\eta}} + \frac{1}{2}\left(\Gamma^{\lambda}_{\mu\eta} + \Gamma^{\lambda}_{\eta\mu}\right)A^{\mu} \tag{9}$$

$$\nabla_{\eta}A_{\underset{+}{\lambda}} = \frac{\partial A_{\lambda}}{\partial x^{\eta}} - \Gamma^{\alpha}_{\lambda\eta}A_{\alpha} \tag{10}$$

$$\nabla_{\eta}A_{\underset{-}{\lambda}} = \frac{\partial A_{\lambda}}{\partial x^{\eta}} - \Gamma^{\alpha}_{\eta\lambda}A_{\alpha} \tag{11}$$

$$\nabla_{\eta}A_{\underset{0}{\lambda}} = \frac{\partial A_{\lambda}}{\partial x^{\eta}} - \frac{1}{2}\left(\Gamma^{\alpha}_{\lambda\eta} + \Gamma^{\alpha}_{\eta\lambda}\right)A_{\alpha}. \tag{12}$$

In the case of the gravitational theory it is essential that besides the $g_{\mu\nu}$ tensor we also have the symmetric infinitesimal displacement $\Gamma^{\lambda}_{\mu\nu}$. This displacement is connected with $g_{\mu\nu}$ by the equation

$$\frac{\partial g_{\mu\nu}}{\partial x^{\rho}} - g_{\alpha\nu}\Gamma^{\alpha}_{\mu\rho} - g_{\mu\alpha}\Gamma^{\alpha}_{\rho\nu} = 0.$$

If the differentiation index ρ is to be on the right in a certain term, we put $+$ under the corresponding tensor-index; if on the left, put $-$ under the index. As an illustration we give a new form to the preceding equation [35] [36]:

$$\left(\nabla_{\eta}g_{\underset{+}{\mu}\underset{-}{\nu}} \equiv\right)\frac{\partial g_{\mu\nu}}{\partial x^{\eta}} - g_{\alpha\nu}\Gamma^{\alpha}_{\mu\eta} - g_{\mu\alpha}\Gamma^{\alpha}_{\eta\nu} = 0. \tag{13}$$

For the following it is esencial to realize that the right side of equation (13) has tensor character even if Eq. (13) is not satisfied. For the Kronecker tensor we get

$$\nabla_{\eta}\delta^{\overset{\mu}{+}}_{\underset{+}{\nu}} = \delta^{\alpha}_{\nu}\Gamma^{\mu}_{\alpha\eta} - \delta^{\mu}_{\alpha}\Gamma^{\alpha}_{\nu\eta} = 0 = \nabla_{\eta}\delta^{\overset{\mu}{-}}_{\underset{-}{\nu}} = \nabla_{\eta}\delta^{\overset{\mu}{0}}_{\underset{0}{\nu}}.$$

On the other hand:

$$\nabla_{\eta}\delta^{\overset{\mu}{+}}_{\underset{-}{\nu}} = \Gamma^{\mu}_{\nu\eta} - \Gamma^{\mu}_{\eta\nu} + 2\Gamma^{\lambda}_{\underset{\vee}{\eta\alpha}}g^{\mu\alpha}g_{\lambda\nu}$$

$$\nabla_\eta \delta^{\underset{-}{\mu}}_{\underset{+}{\nu}} = \Gamma^\mu_{\eta\nu} - \Gamma^\mu_{\nu\eta} + 2\Gamma^\lambda_{\underset{\vee}{\alpha\eta}} g^{\alpha\mu} g_{\nu\lambda}.$$

Therefore, in constructing tensors under the symbol of differentiation one has to watch the character of the indices carefully. Only for indices of the same character are the operations of contraction and of absolute differentiation interchangeable.

In the equation (13), if we multiply by $\frac{1}{2}g^{\mu\nu}$ we get [36]

$$\frac{1}{2}g^{\mu\nu}\nabla_\eta g_{\underset{+-}{\mu\nu}} = \frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^\eta} - \frac{1}{2}\left(\Gamma^\lambda_{\eta\lambda} + \Gamma^\lambda_{\lambda\eta}\right); \quad (14)$$

and by $\sqrt{-g}$ we compute the vector density

$$\frac{1}{2}\sqrt{-g}g^{\mu\nu}\nabla_\eta g_{\underset{+-}{\mu\nu}} = \frac{\partial\sqrt{-g}}{\partial x^\eta} - \frac{1}{2}\sqrt{-g}\left(\Gamma^\lambda_{\eta\lambda} + \Gamma^\lambda_{\lambda\eta}\right).$$

Thus we define the absolute derivative in terms of scalar density [30]

$$\left(\nabla_\eta\sqrt{-g} \equiv\right)\frac{\partial\sqrt{-g}}{\partial x^\eta} - \frac{1}{2}\sqrt{-g}\left(\Gamma^\lambda_{\eta\lambda} + \Gamma^\lambda_{\lambda\eta}\right) = 0. \quad (15)$$

If we multiply Eq. (6) by $-g^{\mu\rho}g^{\sigma\nu}$ and sum with respect to μ and ν, then, because of (3):

$$g^{\mu\rho}\frac{\partial g_{\mu\nu}}{\partial x^\eta} + \frac{\partial g^{\mu\rho}}{\partial x^\eta}g_{\mu\nu} = 0$$

we get

$$\left(\nabla_\eta g^{\overset{\mu\nu}{+-}} \equiv\right)\frac{\partial g^{\mu\nu}}{\partial x^\eta} + g^{\alpha\nu}\Gamma^\mu_{\alpha\eta} + g^{\mu\alpha}\Gamma^\nu_{\eta\alpha} = 0. \quad (16)$$

The main difference of the theory of the field as compared to the pure theory of gravitation, with regard to the equations determining Γ, lies in the fact that the equations which determine Γ in terms of the $g-$field cannot be solved in a simple manner.

If we multiply (7) by $\sqrt{-g}$ we get that

$$\nabla_\eta\left(\sqrt{-g}A^{\overset{\lambda}{+}}\right) = \frac{\partial\left(\sqrt{-g}A^\lambda\right)}{\partial x^\eta} + \sqrt{-g}A^\mu\Gamma^\lambda_{\mu\eta} - \\ -\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^\eta}\sqrt{-g}A^\lambda,$$

or according to (15)

$$\nabla_\eta\left(\sqrt{-g}A^{\overset{\lambda}{+}}\right) = \frac{\partial\left(\sqrt{-g}A^\lambda\right)}{\partial x^\eta} + \sqrt{-g}A^\mu\Gamma^\lambda_{\mu\eta} - \\ -\frac{1}{2}\sqrt{-g}A^\lambda\left(\Gamma^\rho_{\rho\eta} + \Gamma^\rho_{\eta\rho}\right). \quad (17)$$

In an analogous manner, if we multiplied the right side of (8) by a scalar density $\sqrt{-g}$, then we get the tensor density

$$\nabla_\eta\left(\sqrt{-g}A^{\overset{\lambda}{-}}\right) = \frac{\partial\left(\sqrt{-g}A^\lambda\right)}{\partial x^\eta} + \sqrt{-g}A^\mu\Gamma^\lambda_{\eta\mu} - \\ -\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^\eta}\sqrt{-g}A^\lambda,$$

or according to (15)

$$\nabla_\eta\left(\sqrt{-g}A^{\overset{\lambda}{-}}\right) = \frac{\partial\left(\sqrt{-g}A^\lambda\right)}{\partial x^\eta} + \sqrt{-g}A^\mu\Gamma^\lambda_{\eta\mu} - \\ -\frac{1}{2}\sqrt{-g}A^\lambda\left(\Gamma^\rho_{\rho\eta} + \Gamma^\rho_{\eta\rho}\right) \quad (18)$$

and for the divergence

$$\nabla_\lambda\left(\sqrt{-g}A^{\overset{\lambda}{+}}\right) = \frac{\partial\left(\sqrt{-g}A^\lambda\right)}{\partial x^\lambda} + \sqrt{-g}A^\mu\Gamma^\lambda_{\underset{\vee}{\mu\lambda}}, \quad (19)$$

also

$$\nabla_\lambda\left(\sqrt{-g}A^{\overset{\lambda}{-}}\right) = \frac{\partial\left(\sqrt{-g}A^\lambda\right)}{\partial x^\lambda} - \sqrt{-g}A^\lambda\Gamma^\rho_{\underset{\vee}{\lambda\rho}}. \quad (20)$$

We can now calculate the covariant derivative of a tensor density from the rule for differentiating a product. For example:

$$\nabla_\eta\mathfrak{g}^{\mu\nu} = \nabla_\eta\left(\sqrt{-g}g^{\mu\nu}\right) = \left(\nabla_\eta\sqrt{-g}\right)g^{\mu\nu} + \sqrt{-g}\nabla_\eta g^{\mu\nu}.$$

This vanishes, if (16) is satisfied. More explicitly:

$$\nabla_\eta\left(\sqrt{-g}g^{\mu\nu}\right) = \left(\frac{\partial\sqrt{-g}}{\partial x^\eta} - \sqrt{-g}\Gamma^\lambda_{\underline{\eta\lambda}}\right)g^{\mu\nu} + \\ +\sqrt{-g}\left(\frac{\partial g^{\mu\nu}}{\partial x^\eta} + g^{\alpha\nu}\Gamma^\mu_{\alpha\eta} + g^{\mu\alpha}\Gamma^\nu_{\eta\alpha}\right).$$

Therefore we have:

$$\nabla_\eta\mathfrak{g}^{\overset{\mu\nu}{+-}} = \nabla_\eta\left(\sqrt{-g}g^{\overset{\mu\nu}{+-}}\right) = 0$$

or

$$\nabla_\eta\left(\sqrt{-g}g^{\overset{\mu\nu}{+-}}\right) = \frac{\partial}{\partial x^\eta}\left(\sqrt{-g}g^{\mu\nu}\right) + \sqrt{-g}g^{\alpha\nu}\Gamma^\mu_{\alpha\eta} + \\ +\sqrt{-g}g^{\mu\alpha}\Gamma^\nu_{\eta\alpha} - \sqrt{-g}g^{\mu\nu}\Gamma^\lambda_{\underline{\eta\lambda}} = 0. \quad (21)$$

On the other hand:

$$\nabla_\eta\mathfrak{g}_{\mu\nu} = \nabla_\eta\left(\sqrt{-g}g_{\mu\nu}\right) = \left(\nabla_\eta\sqrt{-g}\right)g_{\mu\nu} + \sqrt{-g}\nabla_\eta g_{\mu\nu}.$$

This vanishes, if (13) is satisfied. More explicitly:

$$\nabla_\eta\left(\sqrt{-g}g_{\mu\nu}\right) = \left(\frac{\partial\sqrt{-g}}{\partial x^\eta} - \sqrt{-g}\Gamma^\lambda_{\underline{\eta\lambda}}\right)g_{\mu\nu} + \\ +\sqrt{-g}\left(\frac{\partial g_{\mu\nu}}{\partial x^\eta} - g_{\alpha\nu}\Gamma^\alpha_{\mu\eta} - g_{\mu\alpha}\Gamma^\alpha_{\eta\nu}\right).$$

Therefore we have:

$$\nabla_\eta\mathfrak{g}_{\underset{+-}{\mu\nu}} = \nabla_\eta\left(\sqrt{-g}g_{\underset{+-}{\mu\nu}}\right) = 0$$

or

$$\nabla_\eta\left(\sqrt{-g}g_{\underset{+-}{\mu\nu}}\right) = \frac{\partial}{\partial x^\eta}\left(\sqrt{-g}g_{\mu\nu}\right) - \sqrt{-g}g_{\alpha\nu}\Gamma^\alpha_{\mu\eta} - \\ -\sqrt{-g}g_{\mu\alpha}\Gamma^\alpha_{\eta\nu} - \sqrt{-g}g_{\mu\nu}\Gamma^\lambda_{\underline{\eta\lambda}} = 0. \quad (22)$$

Curvature

We start from the expression for parallel translation, e. g. according to the first of the equations (6). By translation of a vector along the boundary of an infinitesimal surface-element, it is obtained a tensor of curvature just as in the relativistic theory of gravitational field.

Thus, it is obtained the curvature tensor

$$R^{\lambda}_{\mu\nu\eta}=\frac{\partial\Gamma^{\lambda}_{\mu\nu}}{\partial x^{\eta}}+\Gamma^{\lambda}_{\alpha\eta}\Gamma^{\alpha}_{\mu\nu}-\frac{\partial\Gamma^{\lambda}_{\mu\eta}}{\partial x^{\nu}}-\Gamma^{\lambda}_{\alpha\nu}\Gamma^{\alpha}_{\mu\eta}. \quad (23)$$

Contracting with respect to λ and η it is obtained the contracted curvature tensor

$$R_{\mu\nu}=R^{\lambda}_{\mu\nu\lambda}=\frac{\partial\Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}}+\Gamma^{\lambda}_{\alpha\lambda}\Gamma^{\alpha}_{\mu\nu}-\frac{\partial\Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}}-\Gamma^{\lambda}_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda}. \quad (24)$$

The tensor $R_{\mu\nu}$ is not Hermitian. We form the Hermitian tensor $R'_{\mu\nu}=\frac{1}{2}\left(R_{\mu\nu}+\widetilde{R}_{\nu\mu}\right)$. We thus get

$$R'_{\mu\nu}=\frac{\partial\Gamma^{\rho}_{\mu\nu}}{\partial x^{\rho}}-\frac{1}{2}\left(\frac{\partial\Gamma^{\rho}_{\mu\rho}}{\partial x^{\nu}}+\frac{\partial\Gamma^{\rho}_{\rho\nu}}{\partial x^{\mu}}\right)-\Gamma^{\lambda}_{\mu\rho}\Gamma^{\rho}_{\lambda\nu}+ +\frac{1}{2}\Gamma^{\lambda}_{\mu\nu}\left(\Gamma^{\rho}_{\lambda\rho}+\Gamma^{\rho}_{\rho\lambda}\right). \quad (25)$$

There also exist a non-vanishing contraction with respect to λ and μ

$$R^{\rho}_{\rho\nu\eta}=\frac{\partial}{\partial x^{\eta}}\Gamma^{\rho}_{\rho\nu}-\frac{\partial}{\partial x^{\nu}}\Gamma^{\rho}_{\rho\eta}$$

or

$$R^{\rho}_{\rho\nu\eta}=\frac{1}{2}\frac{\partial}{\partial x^{\eta}}\left(\Gamma^{\rho}_{\rho\nu}+\Gamma^{\rho}_{\nu\rho}\right)-\frac{1}{2}\frac{\partial}{\partial x^{\nu}}\left(\Gamma^{\rho}_{\rho\eta}+\Gamma^{\rho}_{\eta\rho}\right)- -\frac{\partial}{\partial x^{\eta}}\Gamma^{\rho}_{\underset{\vee}{\nu\rho}}+\frac{\partial}{\partial x^{\nu}}\Gamma^{\rho}_{\underset{\vee}{\eta\rho}}$$

which in general does not vanish even if (13) is satisfied. Namely, if we transform the right-hand side using the equation following from (14)

$$\frac{\partial}{\partial x^{\eta}}\left(\Gamma^{\lambda}_{\lambda\nu}+\Gamma^{\lambda}_{\nu\lambda}\right)-\frac{\partial}{\partial x^{\nu}}\left(\Gamma^{\lambda}_{\lambda\eta}+\Gamma^{\lambda}_{\eta\lambda}\right)\equiv 0 \quad (26)$$

we get

$$R^{\rho}_{\rho\nu\eta}=-\frac{\partial}{\partial x^{\eta}}\Gamma^{\rho}_{\underset{\vee}{\nu\rho}}+\frac{\partial}{\partial x^{\nu}}\Gamma^{\rho}_{\underset{\vee}{\eta\rho}}.$$

This will not vanish in general, but, it will vanish when the field satisfies equation (1).

For the anti-Hermitian part, we get:

$$P_{\mu\nu}=\frac{1}{2}\left(R_{\mu\nu}-\widetilde{R}_{\nu\mu}\right)=\frac{1}{2}\left(-\frac{\partial\Gamma^{\rho}_{\mu\rho}}{\partial x^{\nu}}+\frac{\partial\Gamma^{\rho}_{\rho\nu}}{\partial x^{\mu}}\right)+\Gamma^{\lambda}_{\mu\nu}\Gamma^{\rho}_{\underset{\vee}{\lambda\rho}}; \quad (27)$$

considering (26) this becomes

$$P_{\mu\nu}=-\frac{1}{2}\left(\nabla_{\nu}\Gamma^{\rho}_{\underset{\underset{\vee}{+}}{\mu\rho}}+\nabla_{\mu}\Gamma^{\rho}_{\underset{\underset{\vee}{+}}{\nu\rho}}\right) \quad (28)$$

Hence the anti-Hermitian part of $P_{\mu\nu}$ vanishes when (1) and (13) are satisfied.

First we want to make another formal remark, which serves to prepare the derivation of the field equations. If in (21) we contract to form $\nabla_{\nu}\left(\sqrt{-g}g^{\overset{\mu\nu}{+-}}\right)$ and $\nabla_{\mu}\left(\sqrt{-g}g^{\overset{\mu\nu}{+-}}\right)$, then by subtraction we get

$$\frac{1}{2}\left[\nabla_{\nu}\left(\sqrt{-g}g^{\overset{\mu\nu}{+-}}\right)-\nabla_{\nu}\left(\sqrt{-g}g^{\overset{\nu\mu}{+-}}\right)\right]= =\frac{\partial}{\partial x^{\nu}}\left(\sqrt{-g}g^{\overset{\mu\nu}{\vee}}\right)-\sqrt{-g}g^{\underline{\mu\sigma}}\Gamma^{\nu}_{\underset{\vee}{\sigma\nu}} \quad (29)$$

where $\sqrt{-g}g^{\underline{\mu\nu}}$ is the symmetric, $\sqrt{-g}g^{\overset{\mu\nu}{\vee}}$ is the part anti-symmetric of the $\sqrt{-g}g^{\mu\nu}$. Hence, if (13) is satisfied we have identically

$$\frac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\underline{\lambda\sigma}}\Gamma^{\nu}_{\underset{\vee}{\sigma\nu}}\right)\equiv 0. \quad (30)$$

From equation (29) we see that equations (1) and (13) imply

$$\nabla_{\lambda}\left(\sqrt{-g}g^{\overset{\mu\lambda}{\vee}}\right)=0. \quad (31)$$

Field equations

It is now our aim to determine field equations which are compatible with our definitions (13). This we achieve through the application of a method which is already known from the theory of gravitation [38, 39]. From $g_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$ and their derivates we construct a Lagrangian density-function $\mathscr{L}$ whose integral we vary independiently with respect to g and Γ. $\mathscr{L}$ is to be chosen so that the variation with respect to the Γ yields the equation (13) and with respect to g will then yield the proper field equations.

We first construct a new tensor by substracting a certain tensor $S_{\mu\nu}$ from $R'_{\mu\nu}$. According to (14) we have that

$$S_{\eta}=\frac{\partial\left(\log\sqrt{-g}\right)}{\partial x^{\eta}}-\frac{1}{2}\left(\Gamma^{\lambda}_{\eta\lambda}+\Gamma^{\lambda}_{\lambda\eta}\right) \quad (32)$$

is a vector. From it, we construct the tensor $\nabla_{\nu}S_{\underset{+}{\mu}}\ (=S_{\mu\nu})$ getting

$$S_{\mu\nu}=\frac{\partial^{2}\left(\log\sqrt{-g}\right)}{\partial x^{\nu}\partial x^{\mu}}-\frac{\partial\left(\log\sqrt{-g}\right)}{\partial x^{\alpha}}\Gamma^{\alpha}_{\mu\nu}- -\frac{1}{2}\left[\frac{\partial}{\partial x^{\nu}}\left(\Gamma^{\lambda}_{\lambda\mu}+\Gamma^{\lambda}_{\mu\lambda}\right)-\left(\Gamma^{\lambda}_{\lambda\alpha}+\Gamma^{\lambda}_{\alpha\lambda}\right)\Gamma^{\alpha}_{\mu\nu}\right]. \quad (33)$$

We get

$$R^{*}_{\mu\nu}=R'_{\mu\nu}-S_{\mu\nu}=\frac{\partial}{\partial x^{\sigma}}\Gamma^{\sigma}_{\mu\nu}-\Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\lambda\nu}- -\frac{\partial^{2}\left(\log\sqrt{-g}\right)}{\partial x^{\mu}\partial x^{\nu}}+\frac{\partial\left(\log\sqrt{-g}\right)}{\partial x^{\lambda}}\Gamma^{\lambda}_{\mu\nu}. \quad (34)$$

From this we construct with the help of the tensor-density $\sqrt{-g}g^{\mu\nu}$ the Lagrangian density-function

$$\mathscr{L}=\sqrt{-g}g^{\mu\nu}R^{*}_{\mu\nu}. \quad (35)$$

The variation of the integral of $\mathscr{L}$ with respect to $\Gamma^{\lambda}_{\mu\nu}$ and $\sqrt{-g}g^{\mu\nu}$ yields:

$$\begin{aligned}\delta\int \mathscr{L} d\tau = \int d\tau\delta\left(\sqrt{-g}g^{\mu\nu}\right)R^*_{\mu\nu} - \int d\tau\left[\tfrac{\partial^2}{\partial x^\nu\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\right) + \tfrac{\partial}{\partial x^\lambda}\left(\sqrt{-g}g^{\mu\nu}\Gamma^\lambda_{\mu\nu}\right)\right]\delta\left(\ln\sqrt{-g}\right) \\ - \int d\tau\left[\tfrac{\partial}{\partial x^\sigma}\left(\sqrt{-g}g^{\mu\nu}\right) + \sqrt{-g}g^{\lambda\nu}\Gamma^\mu_{\lambda\sigma} + \sqrt{-g}g^{\mu\lambda}\Gamma^\nu_{\sigma\lambda} - \sqrt{-g}g^{\mu\nu}\tfrac{1}{2g}\tfrac{\partial g}{\partial x^\sigma}\right]\delta\Gamma^\sigma_{\mu\nu}.\end{aligned} \qquad (36)$$

From (36) we get, because of (14) and (21):

$$\nabla_\eta\left(\sqrt{-g}g^{\underset{+\,-}{\mu\nu}}\right) = \frac{\partial\left(\sqrt{-g}g^{\mu\nu}\right)}{\partial x^\eta} + \sqrt{-g}g^{\alpha\nu}\Gamma^\mu_{\alpha\eta} + +\sqrt{-g}g^{\mu\alpha}\Gamma^\nu_{\eta\alpha} - \frac{1}{2}\sqrt{-g}g^{\mu\nu}\frac{1}{g}\frac{\partial g}{\partial x^\eta} = 0. \qquad (37)$$

The variation with respect to $\sqrt{-g}g^{\mu\nu}$ yields

$$G_{\mu\nu}\delta\left(\sqrt{-g}g^{\mu\nu}\right) = \delta\left(\sqrt{-g}g^{\mu\nu}\right)R^*_{\mu\nu} - \left[\frac{\partial^2\left(\sqrt{-g}g^{\mu\nu}\right)}{\partial x^\nu\partial x^\mu} + +\frac{\partial}{\partial x^\lambda}\left(\sqrt{-g}g^{\mu\nu}\Gamma^\lambda_{\mu\nu}\right)\delta\left(\log\sqrt{-g}\right)\right.$$

where

$$\delta\left(\log\sqrt{-g}\right) = \frac{1}{2\sqrt{-g}}g_{\mu\nu}\delta\left(\sqrt{-g}g^{\mu\nu}\right).$$

Substituting this expression, we get as the result of variation with respect to $\sqrt{-g}g^{\mu\nu}$

$$G_{\mu\nu} = R^*_{\mu\nu} - \frac{1}{2\sqrt{-g}}\left[\frac{\partial^2\left(\sqrt{-g}g^{\rho\sigma}\right)}{\partial x^\rho\partial x^\sigma} + +\frac{\partial}{\partial x^\lambda}\left(\sqrt{-g}g^{\rho\sigma}\Gamma^\lambda_{\rho\sigma}\right)g_{\mu\nu}.\right. \qquad (38)$$

The field equations following from our variational principle are then

$$\nabla_\eta\left(\sqrt{-g}g^{\underset{+\,-}{\mu\nu}}\right) = 0 \qquad (39)$$

$$G_{\mu\nu} = 0. \qquad (40)$$

The first system is equivalent to (13). The second system can be transformed, using the first. Namely from (21), because of (14):

$$\frac{\partial}{\partial x^\eta}\left(\sqrt{-g}g^{\mu\eta}\right) + \sqrt{-g}g^{\alpha\eta}\Gamma^\mu_{\alpha\eta} - \sqrt{-g}g^{\mu\eta}\Gamma^\lambda_{\underset{\vee}{\eta\lambda}} = 0.$$

In the same way, from (21) and (14) follows:

$$\frac{\partial}{\partial x^\eta}\left(\sqrt{-g}g^{\eta\nu}\right) + \sqrt{-g}g^{\eta\lambda}\Gamma^\mu_{\eta\lambda} + \sqrt{-g}g^{\eta\nu}\Gamma^\lambda_{\underset{\vee}{\eta\lambda}} = 0.$$

We have therefore,

$$\frac{\partial^2\left(\sqrt{-g}g^{\eta\nu}\right)}{\partial x^\nu\partial x^\eta} + \frac{\partial}{\partial x^\nu}\left(\sqrt{-g}g^{\eta\lambda}\Gamma^\nu_{\eta\lambda}\right) = = -\frac{\partial}{\partial x^\nu}\left(\sqrt{-g}g^{\eta\nu}\Gamma^\lambda_{\underset{\vee}{\eta\lambda}}\right) = \frac{\partial}{\partial x^\nu}\left(\sqrt{-g}g^{\nu\eta}\Gamma^\lambda_{\underset{\vee}{\lambda\eta}}\right). \qquad (41)$$

We can therefore write

$$G_{\mu\nu} = R^*_{\mu\nu} - \frac{1}{2\sqrt{-g}}\frac{\partial}{\partial x^\lambda}\left(\sqrt{-g}g^{\lambda\eta}\Gamma^\rho_{\underset{\vee}{\rho\eta}}\right)g_{\mu\nu} \qquad (42)$$

$$G_{\mu\nu} = R^*_{\mu\nu} + \frac{1}{2\sqrt{-g}}\frac{\partial}{\partial x^\lambda}\left(\sqrt{-g}g^{\eta\lambda}\Gamma^\rho_{\underset{\vee}{\eta\rho}}\right)g_{\mu\nu}. \qquad (43)$$

The last term of $G_{\mu\nu}$ vanishes if we consider the equations (1), (29) and (31). The remainder is then identical with the once contracted curvature tensor. Therefore, the field equations are therefore written explicity

$$\nabla_\eta g_{\underset{+\,-}{\mu\nu}} = \frac{\partial g_{\mu\nu}}{\partial x^\eta} - g_{\alpha\nu}\Gamma^\alpha_{\mu\eta} - g_{\mu\alpha}\Gamma^\alpha_{\eta\nu} = 0 \qquad (44)$$

$$\Gamma^\lambda_{\underset{\vee}{\eta\lambda}} = 0 \qquad (45)$$

$$G_{\mu\nu} = \frac{\partial}{\partial x^\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\lambda\nu} - \frac{\partial^2\left(\log\sqrt{-g}\right)}{\partial x^\mu\partial x^\nu} + +\frac{\partial\left(\log\sqrt{-g}\right)}{\partial x^\lambda}\Gamma^\lambda_{\mu\nu} = 0. \qquad (46)$$

The foregoing derivation shows how naturally we can extend general relativity theory to a non-symmetric field, and that the field-equations are really the natural generalizations of the gravitational equations.

The relativistic theory of gravitation

First, we consider $g_{\mu\nu}$ symmetric. By changing μ to ν in (13) and subtraction we obtain:

$$\nabla_\rho g_{\underset{+\,-}{\mu\nu}} - \nabla_\rho g_{\underset{+\,-}{\nu\mu}} = g_{\alpha\nu}\left(\Gamma^\alpha_{\mu\rho} - \Gamma^\alpha_{\rho\mu}\right) + g_{\mu\alpha}\left(\Gamma^\alpha_{\rho\nu} - \Gamma^\alpha_{\nu\rho}\right) = 0. \qquad (47)$$

Therefore, the Γ are symmetric in the last two indices

$$\Gamma^\alpha_{\mu\rho} = \Gamma^\alpha_{\rho\mu}$$
$$\Gamma^\alpha_{\rho\nu} = \Gamma^\alpha_{\nu\rho}.$$

The equations (13) can be resolved and we obtains

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^\nu} + \frac{\partial g_{\nu\rho}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right). \qquad (48)$$

Equation (48), together with $G_{\underline{\mu\nu}} = 0$ are the gravitational equations in the relativistic theory of gravitation. This equations seems to be the most simple and coherent derivation of the gravitational equations for the vacuum to me.

Maxwell's equations

If there is an electromagnetic field, then $g_{\mu\nu}$ or $\sqrt{-g}g^{\mu\nu}$ do contain a anti-symmetric part and cannot solve the system (13) to the $\Gamma^{\alpha}_{\mu\nu}$. We succeed in resolving the problem however, if we restrict ourselves to the first approximation. We shall do this and once again postulate $\Gamma^{\lambda}_{\underset{\vee}{\mu\lambda}} = 0$. Thus we have

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu} \tag{49}$$

where $\gamma_{\mu\nu}$ should be infinitely small in first order. We neglect quantities of second and higher orders. Then the $\Gamma^{\alpha}_{\mu\nu}$ are infinitely small in first order as well.

Therefore the system (44) takes the form

$$\frac{\partial \gamma_{\mu\nu}}{\partial x^{\rho}} - \Gamma^{\nu}_{\mu\rho} - \Gamma^{\mu}_{\rho\nu} = 0. \tag{50}$$

After applying two cyclic permutations of the indices μ, ν and ρ two further equations appear. Then, out of the three equations we may calculate the Γ:

$$\Gamma^{\rho}_{\nu\mu} = \frac{1}{2}\left(\frac{\partial \gamma_{\rho\mu}}{\partial x^{\nu}} + \frac{\partial \gamma_{\nu\rho}}{\partial x^{\mu}} - \frac{\partial \gamma_{\mu\nu}}{\partial x^{\rho}}\right). \tag{51}$$

From equation (51) we deduce that the equation (46) is reduced to the first and third term. If we put the expression (51) into (46), then we obtains

$$-\frac{\partial^2 \gamma_{\nu\mu}}{\partial x^{\rho}\partial x^{\rho}} + \frac{\partial^2 \gamma_{\rho\nu}}{\partial x^{\rho}\partial x^{\mu}} + \frac{\partial^2 \gamma_{\rho\mu}}{\partial x^{\nu}\partial x^{\rho}} - \frac{\partial^2 \gamma_{\rho\rho}}{\partial x^{\nu}\partial x^{\mu}} = 0. \tag{52}$$

Equation (45) then gives

$$\frac{\partial \gamma_{\underset{\vee}{\lambda\alpha}}}{\partial x^{\alpha}} = 0. \tag{53}$$

Now we put the expressions given by $\gamma_{\mu\nu} = \gamma_{\underline{\mu\nu}} + \gamma_{\underset{\vee}{\mu\nu}}$ into (52) and obtain with respect to (53)

$$-\frac{\partial^2 \gamma_{\underline{\nu\mu}}}{\partial x^{\rho}\partial x^{\rho}} + \frac{\partial^2 \gamma_{\underline{\rho\nu}}}{\partial x^{\rho}\partial x^{\mu}} + \frac{\partial^2 \gamma_{\underline{\rho\mu}}}{\partial x^{\nu}\partial x^{\rho}} - \frac{\partial^2 \gamma_{\underline{\rho\rho}}}{\partial x^{\nu}\partial x^{\mu}} = 0 \tag{54}$$

$$\frac{\partial^2 \gamma_{\underset{\vee}{\nu\mu}}}{\partial x^{\rho}\partial x^{\rho}} = 0. \tag{55}$$

The equations (53) and (55) for the electromagnetic field do not contain the quantities $\gamma_{\underline{\mu\nu}}$ which refer to the gravitational field. Thus both fields are independent in first approximation. The equation (53) is one Maxwellian system [40–42].

Concluding remarks

We have constructed a theory of the field, starting from the field of infinitesimal displacement. Also, we calculate the curvature tensor and define the contracted curvature tensor. With this contracted curvature tensor and the variational principle, we deduce the field equations. If we were sure that a non-symmetric tensor $g_{\mu\nu}$ is the right means for describing the structure of the field, then we could hardly doubt that the above field equations are the correct ones. In addition, we have assumed the symmetry of $g_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$ to obtain the law of the pure gravitational field.

Finally, the expressions

$$\frac{\partial \gamma_{\underset{\vee}{\mu\nu}}}{\partial x^{\rho}} + \frac{\partial \gamma_{\underset{\vee}{\rho\mu}}}{\partial x^{\nu}} + \frac{\partial \gamma_{\underset{\vee}{\nu\rho}}}{\partial x^{\mu}},$$

do not vanish necessarily due to (53) and (55), but their divergences of the form

$$\frac{\partial}{\partial x^{\rho}}\left(\frac{\partial \gamma_{\underset{\vee}{\mu\nu}}}{\partial x^{\rho}} + \frac{\partial \gamma_{\underset{\vee}{\rho\mu}}}{\partial x^{\nu}} + \frac{\partial \gamma_{\underset{\vee}{\nu\rho}}}{\partial x^{\mu}}\right) = 0$$

however do. Thus (53) and (55) are Maxwell's equations of empty space.

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