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A relativistic theory of the field II: Hamilton's principle and Bianchi's identities

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Abstract
As gravitation and electromagnetism are closely analogous long-range interactions, and the current formulation of gravitation is given in terms of geometry. Thence emerges a relativistic theory of the field by generalization of the general relativity. The derivation presented shows how naturally we can extend general relativity theory to a non-symmetric field, and that the field-equations are really the generalizations of the gravitational equations. With curvature tensor and the variational principle, we will deduce the field equations and Bianchi's identities. In consecuense, the field equations will find from Bianchi's identities.

Keywords: Infinitesimal parallel translation, Curvature Tensor, field equations, Maxwell's equations..

Una teoría relativista del campo II: Principio de Hamilton e identidades de Bianchi

Resumen
Dado que la gravitación y el electromagnetismo son interacciones de largo alcance muy análogas, la formulación actual de la gravitación se da en términos de geometría. Por lo tanto, surge una teoría relativista del campo mediante la generalización de la relatividad general. La derivación presentada muestra cuán naturalmente podemos extender la teoría de la relatividad general al campo no simétrico, y que las ecuaciones de campo son las generalizaciones de las ecuaciones gravitacionales. Con el tensor de curvatura y el principio variacional, deduciremos las ecuaciones de campo y las identidades de Bianchi. En consecuencia, las ecuaciones de campo se encontrarán a partir de las identidades de Bianchi.

Palabras clave: Tranporte paralelo infinitesimal, tensor de curvatura, ecuaciones de campo, ecuaciones de Maxwell..

Introduction

Einstein tried to treat gravity and electromagnetism unifiedly by means of a unified field theory [1–15]. Previously Maxwell had published in 1873 [16, 17] what we would call the first unified theory, when formulating a field theory that integrated electricity and magnetism, and currently there are attempts to unify gravitation and electromagnetism [18–23]. Since the first attempts of Einstein [24, 25], Kaluza [26] and Klein [27], other types of interactions different from gravity and electromagnetism, such as weak interaction [28] and strong interaction [29, 30], have been the subject of various attempts at unification, and by the end of the 1960s the electroweak theory was formulated [31–34]. In fact, it is a unified field theory of electromagnetism and weak interaction. Attempts to unify the theory of strong interaction with the electroweak model and with gravity (a treatise on quantum gravity [35–38]) have since remained one of the still pending challenges of physicists, a theory that would explain the nature and behavior of all matter.

In the beginning of 20th century, the mathematical theories essential for the creation of the general relativity, they were based on the Riemann metric, which was

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considered as the fundamental concept of general relativity [39–48]. Although, it was later pointed out, correctly, that the element of the theory that allows to avoid the inertial system, it is rather the field of infinitesimal displacement. It replaces the inertial system to the extent that the comparison of vectors at infinitesimally close points becomes possible.

The heuristic strength of the general principle of relativity lies in the fact that it considerably reduces the number of imaginable sets of field equations; the field equations must be covariant with respect to all continuous transformations of the four coordinates. But the problem becomes mathematically well-defined only if we have postulated the dependent variables which are to occur in the equations, and their transformation properties (field-structure). But even if we have chosen the field-structure [49] (in such a way that there exist sufficiently strong relativistic field-equations), the principle of relativity does not determine the field-equations uniquely. The principle of "logical simplicity" must be added (which, however, cannot be formulated in a non-arbitrary way). Only then do we have a definite theory whose physical validity can be tested a posteriori.

For the general theory of gravitation and electromagnetism it is essential that we can associate with the covariant tensor $g_{\mu\nu}$ a contravariant $g^{\mu\nu}$, through the relation $g_{\sigma\lambda}g^{\rho\lambda} = \delta^\rho_\sigma$. This association can be carried over to the non-symmetric case directly. So it is natural to try to extend the theory of gravitation to non-symmetric $g_{\underset{\vee}{\mu\nu}}$ fields.

The main difficulty in this attempt lies in the fact that we can build many more covariant equations from a non-symmetric tensor than from a symmetric one. This is due to the fact that the symmetric part, $g_{\underline{\mu\nu}}$, and the antisymmetric part, $g_{\underset{\vee}{\mu\nu}}$, are tensors independently. Is there a formal point of view which makes one of the many possibilities seem most natural? It seems to me that there is. In the case of the gravitational theory it is essential that besides the $g_{\mu\nu}$ tensor we also have the symmetric infinitesimal displacement $\Gamma^\lambda_{\mu\nu}$. This displacement is connected with $g_{\mu\nu}$ by the equation

$$\frac{\partial g_{\mu\nu}}{\partial x^\rho} - g_{\alpha\nu}\Gamma^\alpha_{\mu\rho} - g_{\mu\alpha}\Gamma^\alpha_{\rho\nu} = 0. \tag{1}$$

But in the symmetric case the order of indices does not matter. How shall we generalize (1) to our case? We make use of the following postulate: there is a tensor $g_{\mu\nu}$ the "conjugate" of $g_{\mu\nu}$, and a "conjugate" $\Gamma^\lambda_{\mu\nu}$ of $\Gamma^\lambda_{\mu\nu}$. It seems reasonable that conjugates should play equivalent roles in the field-equations. So we require that if in any field-equation we replace g and Γ by their conjugates, we should get an equivalent equation. This requirement replaces symmetry in our system.

Our main task now is to find out whether there is a sufficiently convincing method of finding a unique set of field-equations for the non-symmetric fields with the above structure. This problem in a previous publication [50] was solved by forming a variational principle in close analogy to the symmetric case. This way we make sure that the resulting equations will be compatible. The only reason why this derivation may seem not completely satisfactory is that we subject the field a priori to two conditions:

$$\Gamma^\lambda_{\underset{\vee}{\mu\lambda}} = \frac{1}{2}\left(\Gamma^\lambda_{\mu\lambda} - \Gamma^\lambda_{\lambda\mu}\right) = 0, \tag{2}$$

$$\frac{\partial(\sqrt{-g}g^{\underset{\vee}{\mu\lambda}})}{\partial x^\lambda} = \frac{1}{2}\frac{\partial}{\partial x^\lambda}\left(\sqrt{-g}g^{\mu\lambda} - \sqrt{-g}g^{\lambda\mu}\right) = 0. \tag{3}$$

These side-conditions make the derivation more complex than in the gravitational theory.

In the theory of symmetric fields there is a second method of ensuring the compatibility of the field-equations ($R_{\mu\nu} = 0$). We must have four identities connecting the equations. These can be derived by contracting the Bianchi- identities which hold for the curvature tensor:

$$\nabla_{\underset{\bullet}{\sigma}} R_{\mu\nu\underset{\bullet\bullet}{\lambda\rho}} = \nabla_\sigma R_{\mu\nu\lambda\rho} + \nabla_\lambda R_{\mu\nu\rho\sigma} + \nabla_\rho R_{\mu\nu\sigma\lambda} = 0.$$

In this article we have shown that an analogous argument can be used for the justification of the field-equations also in our case. This will give a deeper insight into the structure of non-symmetric fields, and it will demonstrate in a new way that the field-equations chosen for the non-symmetric fields are really the natural ones.

Asymmetric tensors

Given any tensor $A_{\mu\nu}$, it can be written as the sum of a symmetric tensor $A_{\underline{\mu\nu}}$ and an antisymmetric $A_{\underset{\vee}{\mu\nu}}$. These are uniquely determined by the relations:

$$A_{\underline{\mu\nu}} = \frac{1}{2}\left(A_{\mu\nu} + A_{\nu\mu}\right), \tag{4}$$

$$A_{\underset{\vee}{\mu\nu}} = \frac{1}{2}\left(A_{\mu\nu} - A_{\nu\mu}\right). \tag{5}$$

A complication is introduced into this theory by the fact that besides the fundamental tensor $g_{\mu\nu}$ we also have its conjugate

$$\widetilde{g}_{\mu\nu} = g_{\nu\mu}, \tag{6}$$

The other tensors of our theory are defined in terms of $g_{\mu\nu}$. Given a tensor $A_{\mu\nu}$, by its conjugate $\widetilde{A}_{\mu\nu}$ we mean the tensor we get by replacing $g_{\mu\nu}$ in the definition of $A_{\mu\nu}$ by $g_{\mu\nu}$. We shall be particularly interested in tensors in whose definition g and $\widetilde{g}$ play analogous roles; more precisely those tensors for which replacing $g_{\mu\nu}$ by $g_{\nu\mu}$ merely changes $A_{\mu\nu}$ into $A_{\nu\mu}$, or for which

$$\widetilde{A}_{\mu\nu} = A_{\nu\mu}. \tag{7}$$

A tensor having the property (7) is called Hermitian. In analogy to (4) and (5), we can decompose any tensor uniquely into

$$A_{\mu\nu} = \frac{1}{2}\left(A_{\mu\nu} + \widetilde{A}_{\nu\mu}\right) + \frac{1}{2}\left(A_{\mu\nu} - \widetilde{A}_{\nu\mu}\right). \qquad (8)$$

The first term is the Hermitian, the second the anti-Hermitian part of $A_{\mu\nu}$. If the differentiation index ρ is to be on the right in a certain term, we put $+$ under the corresponding tensor-index; if on the left, put $-$ under the index. As an illustration we give a new form of (1):

$$\nabla_\rho g_{\underset{+}{\mu}\underset{-}{\nu}} = \frac{\partial g_{\mu\nu}}{\partial x^\rho} - g_{\alpha\nu}\Gamma^\alpha_{\mu\rho} - g_{\mu\alpha}\Gamma^\alpha_{\rho\nu} = 0. \qquad (9)$$

The theorems about covariant differentiation can be taken over from the symmetric theory, if we are careful to distinguish the two kinds of derivatives. By raising the indices μ and ν in (9) we have:

$$\nabla_\rho g^{\underset{+}{\mu}\underset{-}{\nu}} = \frac{\partial g^{\mu\nu}}{\partial x^\rho} + g^{\alpha\nu}\Gamma^\mu_{\alpha\rho} + g^{\mu\alpha}\Gamma^\nu_{\rho\alpha} = 0. \qquad (10)$$

If we let $\sqrt{-g}$ stand for the square-root of the negative determinant of $g_{\mu\nu}$, then $\sqrt{-g}$ is a scalar density. We can describe a tensor density as a product of $\sqrt{-g}$ and a tensor. Let us study these densities. Multiply (1) by $g^{\mu\nu}$ and sum:

$$\frac{1}{g}\frac{\partial g}{\partial x^\rho} - \Gamma^\lambda_{\lambda\rho} - \Gamma^\lambda_{\rho\lambda} = 0$$

or

$$\frac{\partial\sqrt{-g}}{\partial x^\lambda} - \sqrt{-g}\,\Gamma^\lambda_{\underline{\rho\lambda}} = 0. \qquad (11)$$

It is, therefore, natural to define $\nabla_\rho\left(\sqrt{-g}\right) = \frac{\partial\sqrt{-g}}{\partial x^\lambda} - \sqrt{-g}\,\Gamma^\lambda_{\underline{\rho\lambda}}$. If (9) is satisfied, then $\nabla_\rho\left(\sqrt{-g}\right) = 0$. If we do not assume (1), then $\nabla_\rho g_{\underset{+}{\mu}\underset{-}{\nu}}$ and $\nabla_\rho g^{\underset{+}{\mu}\underset{-}{\nu}}$ do not vanish but they have tensorial character. Also $\nabla_\rho\left(\sqrt{-g}\right)$ has the character of a vector density. This permits us to introduce absolute differentiation of tensor densities. For example: if we multiplied the right side of the equation

$$\nabla_\eta A^{\underset{+}{\lambda}} = \frac{\partial A^\lambda}{\partial x^\eta} + \Gamma^\lambda_{\mu\eta}A^\mu$$

by a scalar density $\sqrt{-g}$, then we get the tensor density

$$\frac{\partial\left(\sqrt{-g}A^\lambda\right)}{\partial x^\eta} + \sqrt{-g}A^\mu\Gamma^\lambda_{\mu\eta} - \frac{\partial\sqrt{-g}}{\partial x^\eta}A^\lambda,$$

or according to (11)

$$\nabla_\eta\left(\sqrt{-g}A^{\underset{+}{\lambda}}\right) = \frac{\partial\left(\sqrt{-g}A^\lambda\right)}{\partial x^\eta} + \sqrt{-g}A^\mu\Gamma^\lambda_{\mu\eta} - \frac{1}{2}\sqrt{-g}A^\lambda\left(\Gamma^\rho_{\rho\eta} + \Gamma^\rho_{\eta\rho}\right). \qquad (12)$$

In an analogous manner, if we multiplied the right side of the equation

$$\nabla_\eta A^{\underset{-}{\lambda}} = \frac{\partial A^\lambda}{\partial x^\eta} + \Gamma^\lambda_{\eta\mu}A^\mu$$

by a scalar density $\sqrt{-g}$, then we get the tensor density

$$\frac{\partial\left(\sqrt{-g}A^\lambda\right)}{\partial x^\eta} + \sqrt{-g}A^\mu\Gamma^\lambda_{\eta\mu} - \frac{\partial\sqrt{-g}}{\partial x^\eta}A^\lambda,$$

or according to (11)

$$\nabla_\eta\left(\sqrt{-g}A^{\underset{-}{\lambda}}\right) = \frac{\partial\left(\sqrt{-g}A^\lambda\right)}{\partial x^\eta} + \sqrt{-g}A^\mu\Gamma^\lambda_{\eta\mu} - \frac{1}{2}\sqrt{-g}A^\lambda\left(\Gamma^\rho_{\rho\eta} + \Gamma^\rho_{\eta\rho}\right) \qquad (13)$$

and for the divergence; by a contraction with respect to the indices λ and η in the equations (12) and (13):

$$\nabla_\lambda\left(\sqrt{-g}A^{\underset{+}{\lambda}}\right) = \frac{\partial\left(\sqrt{-g}A^\lambda\right)}{\partial x^\lambda} + \sqrt{-g}A^\mu\Gamma^\lambda_{\underset{\vee}{\mu\lambda}}, \qquad (14)$$

also

$$\nabla_\lambda\left(\sqrt{-g}A^{\underset{-}{\lambda}}\right) = \frac{\partial\left(\sqrt{-g}A^\lambda\right)}{\partial x^\lambda} - \sqrt{-g}A^\lambda\Gamma^\rho_{\underset{\vee}{\lambda\rho}}. \qquad (15)$$

We can now calculate the covariant derivative of a tensor density from the rule for differentiating a product. For example:

$$\nabla_\rho\mathfrak{g}^{\mu\nu} = \nabla_\rho\left(\sqrt{-g}g^{\mu\nu}\right) = \left(\nabla_\rho\sqrt{-g}\right)g^{\mu\nu} + \sqrt{-g}\nabla_\rho g^{\mu\nu}.$$

This vanishes, if (1) is satisfied. More explicitly:

$$\nabla_\rho\left(\sqrt{-g}g^{\mu\nu}\right) = \left(\frac{\partial\sqrt{-g}}{\partial x^\rho} - \sqrt{-g}\,\Gamma^\lambda_{\underline{\rho\lambda}}\right)g^{\mu\nu} + \sqrt{-g}\left(\frac{\partial g^{\mu\nu}}{\partial x^\rho} + g^{\alpha\nu}\Gamma^\mu_{\alpha\rho} + g^{\mu\alpha}\Gamma^\nu_{\rho\alpha}\right).$$

Therefore we have:

$$\nabla_\rho\mathfrak{g}^{\underset{+}{\mu}\underset{-}{\nu}} = \nabla_\rho\left(\sqrt{-g}g^{\underset{+}{\mu}\underset{-}{\nu}}\right) = 0$$

or

$$\nabla_\rho\left(\sqrt{-g}g^{\underset{+}{\mu}\underset{-}{\nu}}\right) = \frac{\partial}{\partial x^\rho}\left(\sqrt{-g}g^{\mu\nu}\right) + \sqrt{-g}g^{\alpha\nu}\Gamma^\mu_{\alpha\rho} + \sqrt{-g}g^{\mu\alpha}\Gamma^\nu_{\rho\alpha} - \sqrt{-g}g^{\mu\nu}\Gamma^\lambda_{\underline{\rho\lambda}} = 0. \qquad (16)$$

In an analogous manner we may calculate the equation

$$\nabla_\rho\left(\sqrt{-g}g_{\underset{+}{\mu}\underset{-}{\nu}}\right) = \frac{\partial}{\partial x^\rho}\left(\sqrt{-g}g_{\mu\nu}\right) - \sqrt{-g}g_{\alpha\nu}\Gamma^\alpha_{\mu\rho} - \sqrt{-g}g_{\mu\alpha}\Gamma^\alpha_{\rho\nu} - \sqrt{-g}g^{\mu\nu}\Gamma^\lambda_{\underline{\rho\lambda}} = 0. \qquad (17)$$

For completeness we include the following abbreviation

$$A_{\underset{\bullet\bullet\bullet}{\mu\nu\lambda}} = A_{\mu\nu\lambda} + A_{\nu\lambda\mu} + A_{\lambda\mu\nu}.$$

Curvature

Although the G-field does not itself have tensor character, it implies the existence of a tensor [51,52]. The latter is most easily obtained by displacing a vector A^λ according to $\delta A^\lambda = -\Gamma^\lambda_{\rho\sigma} A^\rho dx^\sigma$ along the circumference of an infinitesimal two-dimensional surface element and computing its change in one circuit. This change has vector character.

Let $\underset{0}{x}^\rho$ be the co-ordinates of a fixed point and x^τ those of another point on the circumference. Then $\xi^\tau = x^\tau - \underset{0}{x}^\tau$ is small for all points of the circumference and can be used as a basis for the definition of orders of magnitude. The integral $\oint \delta A^\lambda$ to be computed is then in more explicit notation

$$-\oint \underline{\Gamma}^\lambda_{\sigma\tau} \underline{A}^\sigma dx^\tau$$

or

$$-\oint \underline{\Gamma}^\lambda_{\sigma\tau} \underline{A}^\sigma d\xi^\tau .$$

Underlining of the quantities in the integrand indicates that they are to be taken for successive points of the circumference (and not for the initial point, $\xi^\tau = 0$).

We first compute in the lowest approximation the value of A^λ for an arbitrary point ξ^τ of the circumference. This lowest approximation is obtained by replacing in the integral, extended now over an open path, $\underline{\Gamma}^\lambda_{\sigma\tau}$ and $\underline{A}^\sigma$ by the values $\Gamma^\lambda_{\sigma\tau}$ and A^σ for the initial point of integration ($\xi^\tau = 0$). The integration gives then

$$\underline{A}^\lambda = A^\lambda - \Gamma^\lambda_{\sigma\tau} A^\sigma \int d\xi^\tau = A^\lambda - \Gamma^\lambda_{\sigma\tau} A^\sigma \xi^\tau .$$

What are neglected here, are terms of second or higher order in ξ. With the same approximation one obtains immediately

$$\underline{\Gamma}^\lambda_{\sigma\tau} = \Gamma^\lambda_{\sigma\tau} + \frac{\partial \Gamma^\lambda_{\sigma\tau}}{\partial x^\eta}\xi^\eta .$$

Inserting these expressions in the integral above one obtains first, with an appropriate choice of the summation indices,

$$-\oint \left(\Gamma^\lambda_{\sigma\tau} + \frac{\partial \Gamma^\lambda_{\sigma\tau}}{\partial x^\eta}\xi^\eta\right)\left(A^\lambda - \Gamma^\sigma_{\mu\nu} A^\mu \xi^\nu\right) d\xi^\tau$$

where all quantities, with the exception of ξ, have to be taken for the initial point of integration. We then find

$$-\Gamma^\lambda_{\sigma\tau} A^\sigma \oint d\xi^\tau - \frac{\partial \Gamma^\lambda_{\sigma\tau}}{\partial x^\eta} A^\sigma \oint \xi^\eta d\xi^\tau + \Gamma^\lambda_{\sigma\tau}\Gamma^\sigma_{\mu\nu} A^\mu \oint \xi^\nu d\xi^\tau$$

where the integrals are extended over the closed circumference. (The first term vanishes because its integral vanishes.) The term proportional to $(\xi)^2$ is omitted since it is of higher order. The two other terms may be combined into

$$\left[-\frac{\partial \Gamma^\lambda_{\eta\nu}}{\partial x^\mu} + \Gamma^\lambda_{\alpha\nu}\Gamma^\alpha_{\mu\eta}\right] A^\eta \oint \xi^\mu d\xi^\nu$$

This is the change ΔA^λ of the vector A^λ after displacement along the circumference. We have

$$\oint \xi^\mu d\xi^\nu = \oint d\left(\xi^\mu \xi^\nu\right) - \oint \xi^\nu d\xi^\mu = -\oint \xi^\nu d\xi^\mu .$$

This integral is thus antisymmetric in μ and ν, and in addition it has tensor character. We denote it by $f^{\underset{\vee}{\mu\nu}}$. If $f^{\underset{\vee}{\mu\nu}}$ were an arbitrary tensor, then the vector character of ΔA^λ would imply the tensor character of the bracketed expression in the last but one formula. As it is, we can only infer the tensor character of the bracketed expression if antisymmetrized with respect to μ and ν. This is the curvature tensor

$$R^\lambda_{\mu\nu\eta} = \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\eta} + \Gamma^\lambda_{\alpha\eta}\Gamma^\alpha_{\mu\nu} - \frac{\partial \Gamma^\lambda_{\mu\eta}}{\partial x^\nu} - \Gamma^\lambda_{\alpha\nu}\Gamma^\alpha_{\mu\eta} . \qquad (18)$$

The position of all indices is fixed. There also exist a non-vanishing contraction with respect to λ and μ

$$R^\rho_{\rho\nu\eta} = \frac{\partial}{\partial x^\eta}\Gamma^\rho_{\rho\nu} - \frac{\partial}{\partial x^\nu}\Gamma^\rho_{\rho\eta} \qquad (19)$$

which in general does not vanish even if (9) is satisfied. Namely, if we transform the right-hand side using the equation following from (11)

$$\frac{\partial}{\partial x^\eta}\left(\Gamma^\lambda_{\lambda\nu} + \Gamma^\lambda_{\nu\lambda}\right) - \frac{\partial}{\partial x^\nu}\left(\Gamma^\lambda_{\lambda\eta} + \Gamma^\lambda_{\eta\lambda}\right) \equiv 0 \qquad (20)$$

we get

$$R^\rho_{\rho\nu\eta} = -\frac{\partial}{\partial x^\eta}\Gamma^\rho_{\underset{\vee}{\nu\rho}} + \frac{\partial}{\partial x^\nu}\Gamma^\rho_{\underset{\vee}{\eta\rho}} .$$

This will not vanish in general, but, it will vanish when the field satisfies equation (2).

Contracting with respect to λ and η we obtain the contracted curvature tensor

$$R_{\mu\nu} = R^\lambda_{\mu\nu\lambda} = \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} + \Gamma^\lambda_{\alpha\lambda}\Gamma^\alpha_{\mu\nu} - \frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\nu} - \Gamma^\lambda_{\alpha\nu}\Gamma^\alpha_{\mu\lambda} . \quad (21)$$

The tensor $R_{\mu\nu}$ is not Hermitian. We form the Hermitian tensor $R^*_{\mu\nu} = \frac{1}{2}\left(R_{\mu\nu} + \widetilde{R}_{\nu\mu}\right)$. We thus get

$$R^*_{\mu\nu} = \frac{\partial \Gamma^\rho_{\mu\nu}}{\partial x^\rho} - \frac{1}{2}\left(\frac{\partial \Gamma^\rho_{\mu\rho}}{\partial x^\nu} + \frac{\partial \Gamma^\rho_{\rho\nu}}{\partial x^\mu}\right) - \Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\lambda\nu} + \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\underline{\lambda\rho}} . \qquad (22)$$

For the anti-Hermitian part, we get:

$$P_{\mu\nu} = \frac{1}{2}\left(R_{\mu\nu} - \widetilde{R}_{\nu\mu}\right) = \frac{1}{2}\left(-\frac{\partial \Gamma^\rho_{\mu\rho}}{\partial x^\nu} + \frac{\partial \Gamma^\rho_{\rho\nu}}{\partial x^\mu}\right) + \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\underset{\vee}{\lambda\rho}} ; \qquad (23)$$

considering (20) this becomes

$$P_{\mu\nu} = -\frac{1}{2}\left(\nabla_\nu \Gamma^\rho_{\underset{\vee}{\underset{+}{\mu\rho}}} + \nabla_\mu \Gamma^\rho_{\underset{\vee}{\underset{+}{\nu\rho}}}\right) \qquad (24)$$

hence the anti-Hermitian part of $P_{\mu\nu}$ vanishes when (2) and (9) are satisfied. It is now our task to find compatible field equations (on the basis of a variational principle) so that equations (2) and (9) are part of the field equations.

First we want to make another formal remark, which serves to prepare the derivation of the field equations. If in equation (16) we contract to form $\nabla_\nu\left(\sqrt{-g}g^{\overset{\mu\nu}{+-}}\right)$ and $\nabla_\mu\left(\sqrt{-g}g^{\overset{\mu\nu}{+-}}\right)$ then by subtraction we get

$$\frac{1}{2}\left[\nabla_\nu\left(\sqrt{-g}g^{\overset{\mu\nu}{+-}}\right)-\nabla_\nu\left(\sqrt{-g}g^{\overset{\nu\mu}{+-}}\right)\right]=\frac{\partial}{\partial x^\nu}\left(\sqrt{-g}g^{\underset{\vee}{\mu\nu}}\right)-\sqrt{-g}g^{\underline{\mu\sigma}}\Gamma^{\nu}_{\underset{\vee}{\sigma\nu}} \tag{25}$$

where $\sqrt{-g}g^{\underline{\mu\nu}}$ is the symmetric, $\sqrt{-g}g^{\underset{\vee}{\mu\nu}}$ is the anti-symmetric part of the $\sqrt{-g}g^{\mu\nu}$. Hence, if (9) is satisfied we have identically

$$\frac{\partial}{\partial x^\lambda}\left(\sqrt{-g}g^{\underline{\lambda\sigma}}\Gamma^{\nu}_{\underset{\vee}{\sigma\nu}}\right)\equiv 0. \tag{26}$$

The equations (2) satisfy, therefore, a scalar identity as a result of (9). From equation (25) we see that equations (2) and (9) imply

$$\nabla_\lambda\left(\sqrt{-g}g^{\underset{\vee}{\mu\lambda}}\right)=0. \tag{27}$$

Hamiltonian principle. Field equations

In the case of the real symmetric field we obtain the fields eqations most simply in the following manner. We use as Lagrangian function the scalar density

$$\mathscr{L}_G=\sqrt{-g}g^{\mu\nu}R_{\mu\nu} \tag{28}$$

where

$$R_{\mu\nu}=\frac{\partial}{\partial x^\rho}\Gamma^{\rho}_{\mu\nu}+\Gamma^{\lambda}_{\mu\nu}\Gamma^{\rho}_{\lambda\rho}-\frac{\partial}{\partial x^\nu}\Gamma^{\sigma}_{\mu\rho}-\Gamma^{\lambda}_{\mu\rho}\Gamma^{\sigma}_{\lambda\nu}$$

is the curvature tensor in the relativistic theory of gravitation. If we vary the volume integral of $\mathscr{L}$, i. e.

$$\begin{aligned}\delta\int\mathscr{L}_G d\tau=&-\int d\tau\left[\tfrac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\right)+\sqrt{-g}g^{\lambda\nu}\Gamma^{\mu}_{\lambda\rho}+\sqrt{-g}g^{\mu\lambda}\Gamma^{\nu}_{\rho\lambda}-\sqrt{-g}g^{\mu\nu}\Gamma^{\lambda}_{\rho\lambda}\right]\delta\Gamma^{\rho}_{\mu\nu}\\&+\int d\tau\delta^{\nu}_{\rho}\left[\tfrac{\partial}{\partial x^\lambda}\left(\sqrt{-g}g^{\mu\nu}\right)+\sqrt{-g}g^{\lambda\sigma}\Gamma^{\lambda}_{\lambda\sigma}\right]\delta\Gamma^{\rho}_{\mu\nu}-\int d\tau\sqrt{-g}\left(R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R\right)\delta g^{\mu\nu}\end{aligned}$$

independiently with respect to Γ and g, then variation with respect to Γ yields

$$-\left[\tfrac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\right)+\sqrt{-g}g^{\lambda\nu}\Gamma^{\mu}_{\lambda\rho}+\sqrt{-g}g^{\mu\lambda}\Gamma^{\nu}_{\rho\lambda}-\sqrt{-g}g^{\mu\nu}\Gamma^{\lambda}_{\rho\lambda}\right]+\delta^{\nu}_{\rho}\left[\tfrac{\partial}{\partial x^\lambda}\left(\sqrt{-g}g^{\mu\nu}\right)+\sqrt{-g}g^{\lambda\sigma}\Gamma^{\lambda}_{\lambda\sigma}\right]=0$$

or Eq. (1), and variation with respect to g yields the equations $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=0$, or $R_{\mu\nu}=0$. If we apply the same method in the relativistic theory of the field

$$\begin{aligned}\delta\int\mathscr{L}d\tau=&-\int d\tau\left[\tfrac{\partial}{\partial x^\rho}\left(\sqrt{-g}g^{\mu\nu}\right)+\sqrt{-g}g^{\alpha\nu}\Gamma^{\mu}_{\alpha\rho}+\sqrt{-g}g^{\mu\alpha}\Gamma^{\nu}_{\rho\alpha}-\tfrac{1}{2}\sqrt{-g}g^{\mu\nu}\left(\Gamma^{\lambda}_{\rho\lambda}+\Gamma^{\lambda}_{\lambda\rho}\right)\right]\delta\Gamma^{\rho}_{\mu\nu}\\&+\tfrac{1}{2}\int d\tau\left[\tfrac{\partial}{\partial x^\lambda}\left(\sqrt{-g}g^{\mu\lambda}\right)+\sqrt{-g}g^{\alpha\lambda}\Gamma^{\mu}_{\alpha\lambda}-\sqrt{-g}g^{\alpha\nu}\Gamma^{\sigma}_{\underset{\vee}{\alpha\sigma}}\right]\delta^{\nu}_{\rho}\delta\Gamma^{\rho}_{\mu\nu}\\&+\tfrac{1}{2}\int d\tau\left[\tfrac{\partial}{\partial x^\lambda}\left(\sqrt{-g}g^{\lambda\nu}\right)+\sqrt{-g}g^{\lambda\alpha}\Gamma^{\nu}_{\lambda\alpha}+\sqrt{-g}g^{\alpha\nu}\Gamma^{\sigma}_{\underset{\vee}{\alpha\sigma}}\right]\delta^{\mu}_{\rho}\delta\Gamma^{\rho}_{\mu\nu}\\&+\tfrac{1}{2}\int d\tau\left(\sqrt{-g}g^{\mu\alpha}\Gamma^{\sigma}_{\underset{\vee}{\alpha\sigma}}\delta^{\nu}_{\rho}-\sqrt{-g}g^{\alpha\nu}\Gamma^{\sigma}_{\underset{\vee}{\alpha\sigma}}\delta^{\mu}_{\rho}\right)\delta\Gamma^{\rho}_{\mu\nu}+\int d\tau\delta\left(\sqrt{-g}g^{\mu\nu}\right)R_{\mu\nu}\end{aligned}$$

with $\mathscr{L}=\sqrt{-g}g^{\mu\nu}R_{\mu\nu}$. Then we see a complication, since the variation with respect to Γ does not inmediately yield the equation (9), which we wish to keep in any case. The variation with respect to Γ yields

$$\begin{aligned}&-\left[\tfrac{\partial}{\partial x^\rho}\left(\sqrt{-g}g^{\mu\nu}\right)+\sqrt{-g}g^{\mu\alpha}\Gamma^{\nu}_{\rho\alpha}+\sqrt{-g}g^{\alpha\nu}\Gamma^{\mu}_{\alpha\rho}-\tfrac{1}{2}\sqrt{-g}g^{\mu\nu}\left(\Gamma^{\lambda}_{\rho\lambda}+\Gamma^{\lambda}_{\lambda\mu}\right)\right]\\&+\tfrac{1}{2}\delta^{\nu}_{\rho}\left[\tfrac{\partial}{\partial x^\lambda}\left(\sqrt{-g}g^{\mu\lambda}\right)+\sqrt{-g}g^{\alpha\lambda}\Gamma^{\mu}_{\alpha\lambda}-\sqrt{-g}g^{\mu\alpha}\Gamma^{\sigma}_{\underset{\vee}{\alpha\sigma}}\right]\\&+\tfrac{1}{2}\delta^{\mu}_{\rho}\left[\tfrac{\partial}{\partial x^\lambda}\left(\sqrt{-g}g^{\lambda\nu}\right)+\sqrt{-g}g^{\lambda\alpha}\Gamma^{\nu}_{\lambda\alpha}+\sqrt{-g}g^{\alpha\nu}\Gamma^{\sigma}_{\underset{\vee}{\alpha\sigma}}\right]+\tfrac{1}{2}\left(\sqrt{-g}g^{\mu\alpha}\Gamma^{\sigma}_{\underset{\vee}{\alpha\sigma}}\delta^{\nu}_{\rho}-\sqrt{-g}g^{\alpha\nu}\Gamma^{\sigma}_{\underset{\vee}{\alpha\sigma}}\delta^{\mu}_{\rho}\right)=0.\end{aligned} \tag{29}$$

The first bracket is $\nabla_\rho\left(\sqrt{-g}g^{\overset{\mu\nu}{+-}}\right)$; the second and third brackets are contractions of this quantity, i. e.

$$-\nabla_\rho\left(\sqrt{-g}g^{\overset{\mu\nu}{+-}}\right)+\tfrac{1}{2}\delta^{\nu}_{\rho}\nabla_\lambda\left(\sqrt{-g}g^{\overset{\mu\lambda}{+-}}\right)+\tfrac{1}{2}\delta^{\mu}_{\rho}\nabla_\lambda\left(\sqrt{-g}g^{\overset{\lambda\nu}{+-}}\right)+\tfrac{1}{2}\left(\sqrt{-g}g^{\mu\alpha}\Gamma^{\sigma}_{\underset{\vee}{\alpha\sigma}}\delta^{\nu}_{\rho}-\sqrt{-g}g^{\alpha\nu}\Gamma^{\sigma}_{\underset{\vee}{\alpha\sigma}}\delta^{\mu}_{\rho}\right)=0. \tag{30}$$

If there were no fourth bracket in (29) would imply the vanishing of $\nabla_\rho\left(\sqrt{-g}g^{\overset{\mu\nu}{+-}}\right)$, that is, (10). However, this would require the vanishing of $\Gamma^{\sigma}_{\underset{\vee}{\alpha\sigma}}$ to which demand we have no right for the time being.

We can resolve this difficulty in the following manner. We can compute the equations of (29)

$$-\left[\tfrac{\partial}{\partial x^{\rho}}\left(\sqrt{-g}g^{\underline{\mu\nu}}\right)-\sqrt{-g}g^{\underline{\mu\nu}}\Gamma^{\alpha}_{\underline{\rho\alpha}}+\sqrt{-g}g^{\underline{\mu\alpha}}\Gamma^{\nu}_{\underline{\rho\alpha}}+\sqrt{-g}g^{\underset{\vee}{\mu\alpha}}\Gamma^{\nu}_{\underset{\vee}{\rho\alpha}}\right]+\tfrac{1}{2}\delta^{\nu}_{\rho}\left[\tfrac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\underline{\mu\lambda}}\right)+\sqrt{-g}g^{\underline{\alpha\lambda}}\Gamma^{\mu}_{\underline{\alpha\lambda}}+\sqrt{-g}g^{\underset{\vee}{\alpha\lambda}}\Gamma^{\mu}_{\underset{\vee}{\alpha\lambda}}\right]$$
$$+\tfrac{1}{2}\delta^{\mu}_{\rho}\left[\tfrac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\underline{\lambda\nu}}\right)+\sqrt{-g}g^{\underline{\lambda\alpha}}\Gamma^{\nu}_{\underline{\lambda\alpha}}+\sqrt{-g}g^{\underset{\vee}{\lambda\alpha}}\Gamma^{\sigma}_{\underset{\vee}{\lambda\alpha}}\right]-\left(\sqrt{-g}g^{\underset{\vee}{\alpha\mu}}\Gamma^{\nu}_{\underset{\vee}{\alpha\rho}}+\sqrt{-g}g^{\underline{\alpha\nu}}\Gamma^{\mu}_{\underline{\alpha\rho}}\right)=0$$

and

$$-\left[\tfrac{\partial}{\partial x^{\rho}}\left(\sqrt{-g}g^{\underset{\vee}{\mu\nu}}\right)-\sqrt{-g}g^{\underset{\vee}{\mu\nu}}\Gamma^{\alpha}_{\underline{\rho\alpha}}+\sqrt{-g}g^{\underline{\mu\alpha}}\Gamma^{\nu}_{\underset{\vee}{\rho\alpha}}+\sqrt{-g}g^{\underset{\vee}{\mu\alpha}}\Gamma^{\nu}_{\underline{\rho\alpha}}\right]+\tfrac{1}{2}\delta^{\nu}_{\rho}\left[\tfrac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\underset{\vee}{\mu\lambda}}\right)+\sqrt{-g}g^{\underset{\vee}{\alpha\lambda}}\Gamma^{\mu}_{\underline{\alpha\lambda}}+\sqrt{-g}g^{\underline{\alpha\lambda}}\Gamma^{\mu}_{\underset{\vee}{\alpha\lambda}}\right]$$
$$+\tfrac{1}{2}\delta^{\mu}_{\rho}\left[\tfrac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\underset{\vee}{\lambda\nu}}\right)+\sqrt{-g}g^{\underset{\vee}{\lambda\alpha}}\Gamma^{\nu}_{\underline{\lambda\alpha}}+\sqrt{-g}g^{\underline{\lambda\alpha}}\Gamma^{\sigma}_{\underset{\vee}{\lambda\alpha}}\right]-\left(\sqrt{-g}g^{\underline{\alpha\nu}}\Gamma^{\mu}_{\underset{\vee}{\alpha\rho}}+\sqrt{-g}g^{\underset{\vee}{\alpha\rho}}\Gamma^{\mu}_{\underline{\alpha\rho}}\right)=0.$$

Therefore, we form of the second equation

$$-\tfrac{\partial}{\partial x^{\rho}}\left(\sqrt{-g}g^{\underset{\vee}{\mu\nu}}\right)-\sqrt{-g}g^{\underline{\alpha\nu}}\Gamma^{\mu}_{\underset{\vee}{\alpha\rho}}-\sqrt{-g}g^{\underset{\vee}{\alpha\nu}}\Gamma^{\mu}_{\underline{\alpha\rho}}-\sqrt{-g}g^{\underline{\mu\alpha}}\Gamma^{\nu}_{\underset{\vee}{\rho\alpha}}-\sqrt{-g}g^{\underset{\vee}{\mu\alpha}}\Gamma^{\nu}_{\underline{\rho\alpha}}$$
$$+\sqrt{-g}g^{\underset{\vee}{\mu\nu}}\Gamma^{\alpha}_{\underline{\rho\alpha}}+\tfrac{1}{2}\tfrac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\underset{\vee}{\mu\lambda}}\right)\delta^{\nu}_{\rho}+\tfrac{1}{2}\tfrac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\underset{\vee}{\lambda\nu}}\right)=0.$$

If we contract this equation with respect to ν and ρ

$$\frac{1}{2}\frac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\underset{\vee}{\mu\lambda}}\right)+\sqrt{-g}g^{\underline{\mu\alpha}}\Gamma^{\rho}_{\underset{\vee}{\alpha\rho}}=0. \qquad (31)$$

From this we can deduce that the necessary and sufficient condition for $\Gamma^{\rho}_{\underset{\vee}{\alpha\rho}}=0$ is that $\frac{\partial}{\partial x^{\nu}}\left(\sqrt{-g}g^{\underset{\vee}{\mu\nu}}\right)=0$, i. e. Eqs. (2) and (3). In order to satisfy this identically it suffices to asume

$$\sqrt{-g}g^{\underset{\vee}{\mu\lambda}}=\frac{\partial}{\partial x^{\tau}}\left(\sqrt{-g}g^{\mu\lambda\tau}\right) \qquad (32)$$

where $\sqrt{-g}g^{\mu\lambda\tau}$ is a tensor density which is antisymmetric in all three indices. That is, we require that $\sqrt{-g}g^{\underset{\vee}{\mu\lambda}}$ be derived from a "vector potential". Therefore, we substitute in the Lagrange function

$$\sqrt{-g}g^{\mu\nu}=\sqrt{-g}g^{\underline{\mu\nu}}+\frac{\partial}{\partial x^{\tau}}\left(\sqrt{-g}g^{\mu\lambda\tau}\right) \qquad (33)$$

and vary independiently with respect to the Γ then yields (9), as we have shown. The variation respect to the $\sqrt{-g}g^{\underline{\mu\nu}}$ and $\sqrt{-g}g^{\mu\lambda\tau}$ yields the equations

$$R_{\underline{\mu\nu}}=0, \qquad (34)$$

$$\frac{\partial}{\partial x^{\lambda}}R_{\underset{\vee}{\mu\nu}}+\frac{\partial}{\partial x^{\mu}}R_{\underset{\vee}{\nu\lambda}}+\frac{\partial}{\partial x^{\nu}}R_{\underset{\vee}{\lambda\mu}}=0. \qquad (35)$$

Considering (16), each of the systems $\Gamma^{\rho}_{\underset{\vee}{\alpha\rho}}=0$ and $\frac{\partial}{\partial x^{\nu}}\left(\sqrt{-g}g^{\underset{\vee}{\mu\nu}}\right)=0$ implies the other; this is proven by showing that (9) implies the equation

$$\frac{\partial}{\partial x^{\rho}}\left(\sqrt{-g}g^{\underset{\vee}{\mu\rho}}\right)-\sqrt{-g}g^{\underline{\mu\alpha}}\Gamma^{\rho}_{\underset{\vee}{\alpha\rho}}=0.$$

The second line of (30) vanishes because of $\Gamma^{\sigma}_{\underset{\vee}{\alpha\sigma}}=0$. If we contract (30) first according to ρ and ν, then according to μ and ρ we get the two equations

$$\begin{aligned}\nabla_{\rho}\left(\sqrt{-g}g^{\underset{+-}{\mu\rho}}\right)+\tfrac{1}{2}\nabla_{\rho}\left(\sqrt{-g}g^{\underset{+-}{\rho\mu}}\right)&=0\\ \nabla_{\rho}\left(\sqrt{-g}g^{\underset{+-}{\rho\nu}}\right)+\tfrac{1}{2}\nabla_{\rho}\left(\sqrt{-g}g^{\underset{+-}{\nu\rho}}\right)&=0.\end{aligned} \qquad (36)$$

Adding these two equations we get

$$\nabla_{\rho}\left(\sqrt{-g}g^{\underset{+-}{\mu\rho}}\right)+\nabla_{\rho}\left(\sqrt{-g}g^{\underset{+-}{\rho\mu}}\right)=0. \qquad (37)$$

Equation (25) which was based on the definition of absolute differentiation yields considering $\Gamma^{\rho}_{\underset{\vee}{\alpha\rho}}=0$ and $\frac{\partial}{\partial x^{\nu}}\left(\sqrt{-g}g^{\underset{\vee}{\mu\nu}}\right)=0$

$$\nabla_{\rho}\left(\sqrt{-g}g^{\underset{+-}{\mu\rho}}\right)-\nabla_{\rho}\left(\sqrt{-g}g^{\underset{+-}{\rho\mu}}\right)=0. \qquad (38)$$

Hence $\nabla_{\rho}\left(\sqrt{-g}g^{\underset{+-}{\mu\rho}}\right)=0$ and $\nabla_{\rho}\left(\sqrt{-g}g^{\underset{+-}{\rho\mu}}\right)=0$. Equation (30) reduces therefore to

$$\nabla_{\rho}\left(\sqrt{-g}g^{\underset{+-}{\mu\nu}}\right)=0. \qquad (39)$$

If we omit $\frac{\partial}{\partial x^{\nu}}\left(\sqrt{-g}g^{\underset{\vee}{\mu\nu}}\right)=0$, then he system of field equations not weakened is therefore:

$$\nabla_{\rho}g_{\underset{+-}{\mu\nu}}=\frac{\partial g_{\mu\nu}}{\partial x^{\rho}}-g_{\alpha\nu}\Gamma^{\alpha}_{\mu\rho}-g_{\mu\alpha}\Gamma^{\alpha}_{\rho\nu}=0,$$

$$\Gamma^{\lambda}_{\underset{\vee}{\mu\lambda}}=0,$$

$$R_{\underline{\mu\nu}}=0,$$

$$\frac{\partial}{\partial x^{\lambda}}R_{\underset{\vee}{\mu\nu}}+\frac{\partial}{\partial x^{\mu}}R_{\underset{\vee}{\nu\lambda}}+\frac{\partial}{\partial x^{\nu}}R_{\underset{\vee}{\lambda\mu}}=0.$$

On the other hand, we choose the Lagrangian function

$$\mathscr{L} = \sqrt{-g}g^{\mu\nu}R_{\mu\nu} = \delta^{\sigma}_{\alpha}\sqrt{-g}g^{\mu\nu}R^{\alpha}_{\mu\nu\sigma} \tag{40}$$

in the Hamilton's principle

$$\delta\int d\tau\mathscr{L} = \int d\tau\delta\left(\sqrt{-g}g^{\mu\nu}\right)\delta^{\sigma}_{\alpha}R^{\alpha}_{\mu\nu\sigma} + \int d\tau\delta^{\sigma}_{\alpha}\sqrt{-g}g^{\mu\nu}\delta R^{\alpha}_{\mu\nu\sigma} = 0. \tag{41}$$

We vary (40) relative to the $\Gamma's$:

$$\int d\tau\delta\left(\sqrt{-g}g^{\mu\nu}\right)R_{\mu\nu} + \int d\tau\delta^{\sigma}_{\alpha}\sqrt{-g}g^{\mu\nu} \left[\nabla_{\sigma}\left(\delta\Gamma^{\overset{\alpha}{+}}_{\underset{+-}{\mu\,\nu}}\right) - \nabla_{\nu}\left(\delta\Gamma^{\overset{\alpha}{+}}_{\underset{++}{\mu\,\sigma}}\right)\right]. \tag{42}$$

Then we can write (42) as:

$$\int d\tau\delta\left(\sqrt{-g}g^{\mu\nu}\right)R_{\mu\nu} + \int d\tau\nabla_{\sigma}\left(\sqrt{-g}g^{\underset{+-}{\mu\,\nu}}\delta\Gamma^{\overset{\sigma}{+}}_{\underset{+-}{\mu\,\nu}}\right) - \int d\tau\nabla_{\sigma}\left(\sqrt{-g}g^{\underset{+-}{\mu\,\nu}}\delta^{\overset{\sigma}{+}}_{\underset{+}{\alpha}}\right)\delta\Gamma^{\alpha}_{\mu\nu}$$

$$-\int d\tau\nabla_{\nu}\left(\sqrt{-g}g^{\underset{+-}{\mu\,\nu}}\delta\Gamma^{\overset{\sigma}{+}}_{\underset{++}{\mu\,\sigma}}\right) + \int d\tau\nabla_{\nu}\left(\sqrt{-g}g^{\underset{+-}{\mu\,\nu}}\delta^{\overset{\sigma}{+}}_{\underset{+}{\alpha}}\right)\delta\Gamma^{\alpha}_{\mu\sigma}. \tag{43}$$

Let us see what $\nabla_{\sigma}\left(\sqrt{-g}g^{\underset{+-}{\mu\,\nu}}\delta\Gamma^{\overset{\sigma}{+}}_{\underset{+-}{\mu\,\nu}}\right)$ contributes to the integral. That is,

$$\nabla_{\lambda}\left(\sqrt{-g}g^{\underset{+-}{\mu\,\nu}}\delta\Gamma^{\overset{\sigma}{+}}_{\underset{+-}{\mu\,\nu}}\right) = \frac{\partial\left(\sqrt{-g}g^{\mu\nu}\delta\Gamma^{\sigma}_{\mu\nu}\right)}{\partial x^{\lambda}} + \left(\sqrt{-g}g^{\mu\nu}\delta\Gamma^{\sigma}_{\mu\nu}\right)\Gamma^{\lambda}_{\underset{\vee}{\mu\lambda}}. \tag{44}$$

The first term is an ordinary divergence, and hence contributes nothing to the integral. We see that we need (2) to make the second term vanish. By subjecting the field to equation (2) we make sure that $\nabla_{\sigma}\left(\sqrt{-g}g^{\underset{+-}{\mu\,\nu}}\delta\Gamma^{\overset{\sigma}{+}}_{\underset{+-}{\mu\,\nu}}\right)$ (and similarly $\nabla_{\nu}\left(\sqrt{-g}g^{\underset{+-}{\mu\,\nu}}\delta\Gamma^{\overset{\sigma}{+}}_{\underset{++}{\mu\,\sigma}}\right)$) contributes nothing to the integral. So we may omit these from (43) and write:

$$\int d\tau\delta\left(\sqrt{-g}g^{\mu\nu}\right)R_{\mu\nu} + \int d\tau\left[-\nabla_{\sigma}\left(\sqrt{-g}g^{\underset{+-}{\mu\,\nu}}\delta^{\overset{\sigma}{+}}_{\underset{+}{\alpha}}\right) + \nabla_{\sigma}\left(\sqrt{-g}g^{\underset{+-}{\mu\,\sigma}}\delta^{\overset{\nu}{+}}_{\underset{+}{\alpha}}\right)\right]\delta\Gamma^{\alpha}_{\mu\nu} \tag{45}$$

Or since $\nabla_{\sigma}\left(\delta^{\overset{\nu}{+}}_{\underset{+}{\alpha}}\right)$ vanishes:

$$\int d\tau\delta\left(\sqrt{-g}g^{\mu\nu}\right)R_{\mu\nu} + \int d\tau\left[-\nabla_{\alpha}\left(\sqrt{-g}g^{\underset{+-}{\mu\,\nu}}\right) + \nabla_{\sigma}\left(\sqrt{-g}g^{\underset{+-}{\mu\,\sigma}}\right)\delta^{\overset{\nu}{+}}_{\underset{+}{\alpha}}\right]\delta\Gamma^{\alpha}_{\mu\nu} \tag{46}$$

We cannot conclude yet that the quantity in brackets vanishes, because the $\Gamma^{\lambda}_{\mu\nu}$ are not independent but satisfy (2). But we could conclude the vanishing of these quantities if they depended on only 60 parameters instead of the 64 $\nabla_{\underset{+}{\alpha}}\left(\sqrt{-g}g^{\underset{+-}{\mu\,\nu}}\right)$. This is actually so, for the following reason: we have the equation (25). By subjecting the field to (2) and (3), we make sure that these four quantities vanish. Hence only 60 of the $\nabla_{\alpha}\left(\sqrt{-g}g^{\underset{+-}{\mu\,\nu}}\right)$ are independent. The same must be true of the square bracketed quantities in (46). Thus we can conclude from (46) that all these vanish:

$$-\nabla_{\alpha}\left(\sqrt{-g}g^{\underset{+-}{\mu\,\nu}}\right) + \nabla_{\sigma}\left(\sqrt{-g}g^{\underset{+-}{\mu\,\sigma}}\right)\delta^{\overset{\nu}{+}}_{\underset{+}{\alpha}} = 0. \tag{47}$$

Contracting with respect to ν and α we have $\nabla_{\sigma}\left(\sqrt{-g}g^{\underset{+-}{\mu\,\sigma}}\right) = 0$. Hence all the $\nabla_{\alpha}\left(\sqrt{-g}g^{\underset{+-}{\mu\,\sigma}}\right)$ vanish. Therefore also the $\nabla_{\alpha}g_{\underset{+-}{\mu\,\sigma}}$. Thus we have derived that

$$\nabla_{\rho}g_{\underset{+-}{\mu\,\nu}} = 0. \tag{48}$$

But we must remember that the $\sqrt{-g}g^{\mu\nu}$ satisfy (3). This can be done most easily by setting equation (33) and varying with respect to $\sqrt{-g}g^{\underline{\mu\nu}}$ and $\sqrt{-g}g^{\mu\lambda\tau}$, which are independent. We get the equations

$$R_{\underline{\mu\nu}} = 0,$$

$$\frac{\partial}{\partial x^{\dot{\lambda}}}R_{\underset{\vee}{\dot{\mu}\dot{\nu}}} = 0.$$

This completes the derivation of the field-equations:

$$\Gamma^{\lambda}_{\underset{\vee}{\mu\lambda}} = 0,$$

$$\nabla_{\rho}g_{\underset{+-}{\mu\,\nu}} = 0,$$

$$R_{\underline{\mu\nu}} = 0,$$

$$\frac{\partial}{\partial x^{\lambda}}R_{\underset{\vee}{\mu\nu}} + \frac{\partial}{\partial x^{\mu}}R_{\underset{\vee}{\nu\lambda}} + \frac{\partial}{\partial x^{\nu}}R_{\underset{\vee}{\lambda\mu}} = \frac{\partial}{\partial x^{\dot{\lambda}}}R_{\underset{\vee}{\dot{\mu}\dot{\nu}}} = 0.$$

We can further justify the a priori assumption of (2) by the fact that this equation is necessary and sufficient to make $R_{\mu\nu}$ a Hermitian tensor.

The Bianchi's identities

A direct computation shows that the covariant derivate of curvature tensor

$$\nabla_{\underset{\bullet}{\lambda}} R^{\overset{\rho}{+}}_{\underset{+\bullet\bullet}{\mu\nu\sigma}} = \nabla_\lambda R^{\overset{\rho}{+}}_{\underset{+}{\mu\nu\sigma}} + \nabla_\nu R^{\overset{\rho}{+}}_{\underset{+}{\mu\sigma\lambda}} + \nabla_\sigma R^{\overset{\rho}{+}}_{\underset{+}{\mu\lambda\nu}}$$

satisfies the identities:

$$\nabla_{\underset{\bullet}{\lambda}} R^{\overset{\rho}{+}}_{\underset{+\bullet\bullet}{\mu\nu\sigma}} = \frac{\partial}{\partial x^{\underset{\bullet}{\lambda}}} R^{\rho}_{\underset{\bullet\bullet}{\mu\nu\sigma}} + R^{\alpha}_{\underset{\bullet\bullet}{\mu\sigma\lambda}} \Gamma^{\rho}_{\underset{\bullet}{\alpha\nu}} - R^{\rho}_{\underset{\bullet\bullet}{\alpha\sigma\lambda}} \Gamma^{\alpha}_{\underset{\bullet}{\mu\nu}} = 0. \quad (49)$$

From (18) we can form the covariant curvature tensor in analogy to the symmetric case,

$$R_{\mu\nu\lambda\eta} = g_{\alpha\mu} R^{\alpha}_{\nu\lambda\eta}. \quad (50)$$

The choice of $g_{\alpha\mu}$ instead of $g_{\mu\alpha}$ may seem arbitrary, but this is not really so. We have to lower the index ρ in the identities (49). The contravariant index μ has the $+$ differentiation character, so it must be summed with a similar index, i.e. the first index of g. Only this way can we lower the index μ in (49) without introducing additional terms. Thus we get the covariant identities

$$g_{\rho\alpha} \nabla_{\underset{\bullet}{\lambda}} R^{\overset{\rho}{+}}_{\underset{+\bullet\bullet}{\mu\nu\sigma}} = \nabla_{\underset{\bullet}{\lambda}} \left(g_{\underset{+-}{\rho\alpha}} R^{\overset{\rho}{+}}_{\underset{+\bullet\bullet}{\mu\nu\sigma}} \right) = \nabla_{\underset{\bullet}{\lambda}} R_{\underset{-+\bullet\bullet}{\alpha\mu\nu\sigma}} = 0. \quad (51)$$

For what follows we must also find the symmetry properties of $R_{\alpha\mu\nu\sigma}$. From (18) it is clear that $R^{\lambda}_{\mu\nu\sigma}$ is antisymmetric in $(\nu\sigma)$. From (50) we see that $R_{\alpha\mu\nu\sigma}$ has the same property:

$$R_{\lambda\mu\nu\sigma} = -R_{\lambda\mu\sigma\nu}. \quad (52)$$

If we differentiate (1) with respect to σ and antisymmetrize with respect to ρ and σ, we have

$$-\frac{\partial g_{\alpha\nu}}{\partial x^\sigma} \Gamma^{\alpha}_{\mu\rho} - \frac{\partial g_{\mu\alpha}}{\partial x^\sigma} \Gamma^{\alpha}_{\rho\nu} + \frac{\partial g_{\alpha\nu}}{\partial x^\rho} \Gamma^{\alpha}_{\mu\sigma} + \frac{\partial g_{\mu\alpha}}{\partial x^\rho} \Gamma^{\alpha}_{\sigma\nu} +$$

$$g_{\alpha\nu} \left(\frac{\partial \Gamma^{\alpha}_{\mu\sigma}}{\partial x^\rho} - \frac{\partial \Gamma^{\alpha}_{\mu\rho}}{\partial x^\sigma} \right) + g_{\mu\alpha} \left(\frac{\partial \Gamma^{\alpha}_{\sigma\nu}}{\partial x^\rho} - \frac{\partial \Gamma^{\alpha}_{\rho\nu}}{\partial x^\sigma} \right) = 0.$$

Using (1) again on the first four terms and then collecting terms

$$-g_{\alpha\nu} \left(\frac{\partial \Gamma^{\alpha}_{\mu\rho}}{\partial x^\sigma} - \frac{\partial \Gamma^{\alpha}_{\mu\sigma}}{\partial x^\rho} \right) - g_{\mu\alpha} \left(\frac{\partial \Gamma^{\alpha}_{\rho\nu}}{\partial x^\sigma} - \frac{\partial \Gamma^{\alpha}_{\sigma\nu}}{\partial x^\rho} \right) -$$

$$g_{\alpha\nu} \Gamma^{\alpha}_{\eta\sigma} \Gamma^{\eta}_{\mu\rho} - g_{\mu\alpha} \Gamma^{\alpha}_{\sigma\eta} \Gamma^{\eta}_{\rho\nu} + g_{\alpha\nu} \Gamma^{\alpha}_{\eta\rho} \Gamma^{\eta}_{\mu\sigma} + g_{\mu\alpha} \Gamma^{\alpha}_{\rho\eta} \Gamma^{\eta}_{\sigma\nu} = 0$$

or using (18) and (50), we have

$$R_{\mu\nu\rho\sigma} = -\widetilde{R}_{\nu\mu\rho\sigma}. \quad (53)$$

This expresses that $R_{\mu\nu\rho\sigma}$ is anti-Hermitian in $(\mu\nu)$; this is the manner in which the antisymmetry of $R_{\mu\nu\rho\sigma}$(in the gravitational theory) generalizes to our case.

In (51) it is not immediately clear that $\nabla_{\underset{\bullet}{\lambda}} R_{\underset{-+\bullet\bullet}{\alpha\mu\nu\sigma}}$ is a tensor. We are now in a position to give a more useful form for (51) in which this is obvious, i. e.:

$$\nabla_\lambda R_{\underset{-+-+}{\mu\nu\rho\sigma}} + \nabla_\rho R_{\underset{-+++}{\mu\nu\sigma\lambda}} + \nabla_\sigma R_{\underset{-+--}{\mu\nu\lambda\rho}} = \nabla_{\underset{\bullet}{\lambda}} R_{\underset{-+\bullet\bullet}{\mu\nu\rho\sigma}} -$$

$$R_{\mu\nu\alpha\sigma} \Gamma^{\alpha}_{\lambda\rho} - R_{\mu\nu\rho\alpha} \Gamma^{\alpha}_{\sigma\lambda} - R_{\mu\nu\alpha\lambda} \Gamma^{\alpha}_{\sigma\rho} -$$

$$R_{\mu\nu\sigma\alpha} \Gamma^{\alpha}_{\lambda\rho} - R_{\mu\nu\alpha\rho} \Gamma^{\alpha}_{\sigma\lambda} - R_{\mu\nu\lambda\alpha} \Gamma^{\alpha}_{\sigma\rho}. \quad (54)$$

The first term on the right side of the equation vanishes by (51), the last six cancel out due to (52). Therefore,

$$\nabla_\lambda R_{\underset{-+-+}{\mu\nu\rho\sigma}} + \nabla_\rho R_{\underset{-+++}{\mu\nu\sigma\lambda}} + \nabla_\sigma R_{\underset{-+--}{\mu\nu\lambda\rho}} = 0, \quad (55)$$

where this equation is called "Bianchi's identities". We are now in a position to carry out the derivation of the identities for the field equations. In analogy to the gravitational theory, we contract (55) by $g^{\mu\rho} g^{\nu\rho}$. Making use of (52), we get

$$g^{\mu\rho} g^{\nu\rho} \left[\nabla_\lambda R_{\underset{-+-+}{\mu\nu\rho\sigma}} + \nabla_\rho R_{\underset{-+++}{\mu\nu\sigma\lambda}} + \nabla_\sigma R_{\underset{-+--}{\mu\nu\lambda\rho}} \right] = 0$$

or using (9)

$$g^{\nu\rho} \nabla_\lambda \left(g^{\overset{\sigma\mu}{+-}} R_{\underset{-+-+}{\mu\nu\rho\sigma}} \right) - g^{\nu\rho} \nabla_\rho \left(g^{\overset{\sigma\mu}{+-}} R_{\underset{-+++}{\mu\nu\lambda\sigma}} \right) -$$

$$g^{\sigma\mu} \nabla_\sigma \left(g^{\overset{\nu\rho}{+-}} R_{\underset{-+--}{\mu\nu\rho\lambda}} \right) = 0. \quad (56)$$

Let us define

$$R_{\mu\nu} = g^{\sigma\lambda} R_{\lambda\mu\nu\sigma} \quad (57)$$

$$S_{\sigma\lambda} = g^{\mu\nu} R_{\lambda\mu\nu\sigma} \quad (58)$$

where

$$R_{\mu\nu} = g^{\sigma\lambda} g_{\alpha\lambda} R^{\alpha}_{\mu\nu\sigma} = \delta^{\sigma}_{\alpha} R^{\alpha}_{\mu\nu\sigma} = \frac{\partial}{\partial x^\sigma} \Gamma^{\sigma}_{\mu\nu} +$$

$$\Gamma^{\eta}_{\mu\nu} \Gamma^{\sigma}_{\eta\sigma} - \frac{\partial}{\partial x^\nu} \Gamma^{\sigma}_{\mu\sigma} - \Gamma^{\eta}_{\mu\sigma} \Gamma^{\sigma}_{\eta\nu}. \quad (59)$$

Then we have

$$g^{\nu\rho} \left[\nabla_\lambda R_{\underset{+-}{\nu\rho}} - \nabla_\rho R_{\underset{++}{\nu\lambda}} - \nabla_\nu S_{\underset{--}{\lambda\rho}} \right] = 0. \quad (60)$$

We need some connection between R and S. From (52) and (53) we see that

$$R_{\mu\nu\rho\sigma} = -\widetilde{R}_{\nu\mu\rho\sigma} = -R_{\nu\mu\rho\sigma}.$$

Multiply by $g^{\mu\sigma}$ $(= \widetilde{g}^{\sigma\mu})$ and sum:

$$S_{\rho\nu} = \widetilde{R}_{\nu\rho}. \quad (61)$$

If R were Hermitian, R and S would be identical. Hence we have a new reason for requiring that $R_{\mu\nu}$ should be Hermitian. But from (59) we see that $R_{\mu\nu}$ has an anti-Hermitian part (compare with (8)):

$$\frac{1}{2} \left(R_{\mu\nu} - \widetilde{R}_{\nu\mu} \right) = \frac{1}{2} \left(-\frac{\partial \Gamma^{\rho}_{\mu\rho}}{\partial x^\nu} + \frac{\partial \Gamma^{\rho}_{\rho\nu}}{\partial x^\mu} \right) -$$

$$\frac{1}{2} \Gamma^{\lambda}_{\mu\nu} \left(\Gamma^{\sigma}_{\sigma\lambda} - \Gamma^{\sigma}_{\lambda\sigma} \right). \quad (62)$$

If we use (11), then we have

$$\frac{1}{2}\left(R_{\mu\nu}-\widetilde{R}_{\nu\mu}\right)=-\frac{1}{2}\left(\frac{\partial\Gamma^{\rho}_{\underset{\vee}{\mu\rho}}}{\partial x^{\nu}}+\frac{\partial\Gamma^{\rho}_{\underset{\vee}{\nu\sigma}}}{\partial x^{\mu}}-\Gamma^{\lambda}_{\mu\nu}\Gamma^{\rho}_{\underset{\vee}{\lambda\rho}}\right). \tag{63}$$

From this we see that $R_{\mu\nu}$ is Hermitian if we subject the field to the four conditions

$$\Gamma^{\lambda}_{\underset{\vee}{\mu\lambda}}=0.$$

It then follows from (61) that

$$S_{\rho\nu}=R_{\rho\nu}, \tag{64}$$

and (60) becomes

$$g^{\nu\rho}\left[\nabla_{\lambda}R_{\underset{+-}{\nu\,\rho}}-\nabla_{\rho}R_{\underset{++}{\nu\,\lambda}}-\nabla_{\nu}R_{\underset{--}{\lambda\,\rho}}\right]=0. \tag{65}$$

These identities hold for all fields where Γ is defined by (1) and is subject to (2). We might jump to the conclusion that the field equations should stipulate the vanishing of all $R_{\mu\nu}$. This set, together with (1) and (2) would, however, be overdetermined. We can get a weaker set of equations by observing how $R_{\underset{\vee}{\mu\nu}}$ enters (65). The contribution of $R_{\underset{\vee}{\mu\nu}}$ to the equations is:

$$g^{\nu\rho}\left(\nabla_{\lambda}R_{\underline{\nu\rho}}-\nabla_{\rho}R_{\underline{\nu\lambda}}-\nabla_{\nu}R_{\underline{\lambda\rho}}\right)+$$

$$g^{\nu\rho}\left[\nabla_{\lambda}R_{\underset{\vee}{\underset{+-}{\nu\,\rho}}}-\nabla_{\rho}R_{\underset{\vee}{\underset{++}{\nu\,\lambda}}}-\nabla_{\nu}R_{\underset{\vee}{\underset{--}{\lambda\,\rho}}}\right]=0$$

which can be written as

$$\nabla_{\lambda}\left(g^{\nu\rho}R_{\underline{\nu\rho}}\right)-\nabla_{\rho}\left(g^{\nu\rho}R_{\underline{\nu\lambda}}\right)-\nabla_{\nu}\left(g^{\nu\rho}R_{\underline{\lambda\rho}}\right)+$$

$$g^{\nu\rho}\left[\frac{\partial}{\partial x^{\lambda}}R_{\underset{\vee}{\nu\rho}}-R_{\underset{\vee}{\alpha\rho}}\Gamma^{\alpha}_{\nu\lambda}-R_{\underset{\vee}{\nu\alpha}}\Gamma^{\alpha}_{\lambda\rho}\right]+$$

$$g^{\nu\rho}\left[-\frac{\partial}{\partial x^{\rho}}R_{\underset{\vee}{\nu\lambda}}+R_{\underset{\vee}{\alpha\lambda}}\Gamma^{\alpha}_{\nu\rho}+R_{\underset{\vee}{\nu\alpha}}\Gamma^{\alpha}_{\lambda\rho}\right]-$$

$$g^{\nu\rho}\left[\frac{\partial}{\partial x^{\nu}}R_{\underset{\vee}{\rho\lambda}}+R_{\underset{\vee}{\lambda\alpha}}\Gamma^{\alpha}_{\nu\rho}+R_{\underset{\vee}{\alpha\rho}}\Gamma^{\alpha}_{\nu\lambda}\right]=0$$

this equation is equivalent to

$$\nabla_{\lambda}R-\nabla_{\rho}R^{\rho}_{\lambda}-\nabla_{\nu}R^{\nu}_{\lambda}+$$

$$g^{\nu\rho}\left[\frac{\partial}{\partial x^{\lambda}}R_{\underset{\vee}{\nu\rho}}-\frac{\partial}{\partial x^{\rho}}R_{\underset{\vee}{\nu\lambda}}-\frac{\partial}{\partial x^{\nu}}R_{\underset{\vee}{\rho\lambda}}\right]=0$$

or

$$\nabla_{\rho}\left(R^{\rho}_{\lambda}-\frac{1}{2}\delta^{\rho}_{\lambda}R\right)+$$

$$g^{\nu\rho}\left[\frac{\partial}{\partial x^{\lambda}}R_{\underset{\vee}{\nu\rho}}+\frac{\partial}{\partial x^{\rho}}R_{\underset{\vee}{\lambda\nu}}+\frac{\partial}{\partial x^{\nu}}R_{\underset{\vee}{\lambda\rho}}\right]=0,$$

that is

$$-g^{\underline{\nu\rho}}\nabla_{\rho}\left(R_{\underline{\lambda\nu}}-\frac{1}{2}g_{\underline{\lambda\nu}}R\right)+g^{\nu\rho}\frac{\partial}{\partial x^{\underset{\bullet}{\lambda}}}R_{\underset{\vee}{\underset{\bullet\bullet}{\nu\rho}}}=0.$$

Since we see that $R_{\mu\nu}$ enters the equations only in the combination $\frac{\partial}{\partial x^{\underset{\bullet}{\lambda}}}R_{\underset{\vee}{\underset{\bullet\bullet}{\nu\rho}}}$ it is natural to choose the field equations for $R_{\underset{\vee}{\mu\nu}}$ as

$$\frac{\partial}{\partial x^{\underset{\bullet}{\lambda}}}R_{\underset{\vee}{\underset{\bullet\bullet}{\nu\rho}}}=0$$

instead of $R_{\underline{\mu\nu}}=0$. So we get the field equations:

$$\Gamma^{\lambda}_{\underset{\vee}{\mu\lambda}}=0,$$

$$R_{\underline{\mu\nu}}=0,$$

$$\frac{\partial}{\partial x^{\underset{\bullet}{\lambda}}}R_{\underset{\vee}{\underset{\bullet\bullet}{\mu\nu}}}=0,$$

where the $\Gamma^{\lambda}_{\mu\nu}$ are defined by:

$$\nabla_{\rho}g_{\underset{+-}{\mu\,\nu}}=0.$$

The foregoing derivation shows how naturally we can extend general relativity theory to a non-symmetric field, and that the field-equations are really the natural generalizations of the gravitational equations.

The theory of gravitational field as special case

Let the $g_{\mu\nu}$ be symmetric. By changing μ to ν in (9) and subtraction we obtain in understandable notation:

$$\nabla_{\rho}g_{\underset{+-}{\mu\,\nu}}-\nabla_{\rho}g_{\underset{+-}{\nu\,\mu}}=g_{\alpha\nu}\left(\Gamma^{\alpha}_{\mu\rho}-\Gamma^{\alpha}_{\rho\mu}\right)+g_{\mu\alpha}\left(\Gamma^{\alpha}_{\rho\nu}-\Gamma^{\alpha}_{\nu\rho}\right)=0. \tag{66}$$

Therefore, the Γ are symmetric in the last two indices as in Riemannian geometry and the theory of general relativity, that is

$$\Gamma^{\alpha}_{\mu\rho}=\Gamma^{\alpha}_{\rho\mu} \tag{67}$$

$$\Gamma^{\alpha}_{\rho\nu}=\Gamma^{\alpha}_{\nu\rho}. \tag{68}$$

The equations (9) can be resolved in a well-known manner, and we obtain

$$\Gamma^{\lambda}_{\mu\nu}=\frac{1}{2}g^{\lambda\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^{\nu}}+\frac{\partial g_{\nu\rho}}{\partial x^{\mu}}-\frac{\partial g_{\mu\nu}}{\partial x^{\rho}}\right). \tag{69}$$

Equation (69), together with (34) is the well-known law of gravitation. If we have presumed the symmetry of the $g_{\mu\nu}$ at the beginning, we would have arrived at (69) and (34) directly. This seems to be the most simple and coherent derivation of the gravitational equations for the vacuum. Therefore it should be seen as a natural attempt to encompass the law of electromagnetism by generalizing these considerations rightly.

Relations to Maxwell's electromagnetic theory

If there is an electromagnetic field, that means the $g_{\mu\nu}$ or the $\sqrt{-g}g^{\mu\nu}$ do contain a skew-symmetric part, we cannot solve the equations (9) any more with respect to the $\Gamma^{\alpha}_{\mu\nu}$, which significantly complicates the clearness of the whole system. We succeed in resolving the problem however, if we restrict ourselves to the first approximation. We shall do this and once again postulate the vanishing of $\Gamma^{\lambda}_{\underset{\vee}{\mu\lambda}}$. Thus we start with the ansatz

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\underline{\mu\nu}} + \gamma_{\underset{\vee}{\mu\nu}} \tag{70}$$

where by the $\gamma_{\underline{\mu\nu}}$ should be symmetric, and $\gamma_{\underset{\vee}{\mu\nu}}$ the skew-symmetric, both should be infinitely small in first order. We neglect quantities of second and higher orders. Then the $\Gamma^{\alpha}_{\mu\nu}$ are infinitely small in first order as well.

Under these circumstances the systems (9) and (10) takes the more simple form

$$\frac{\partial g_{\mu\nu}}{\partial x^{\rho}} - \Gamma^{\nu}_{\mu\rho} - \Gamma^{\mu}_{\rho\nu} = 0 \tag{71}$$

$$\frac{\partial g^{\mu\nu}}{\partial x^{\rho}} + \Gamma^{\mu}_{\nu\rho} + \Gamma^{\nu}_{\rho\mu} = 0. \tag{72}$$

After applying two cyclic permutations of the indices μ, ν and ρ two further equations appear. Then, out of the three equations we may calculate the Γ in a similar manner as in the symmetric case. One obtains

$$\Gamma^{\rho}_{\nu\mu} = \frac{1}{2}\left(\frac{\partial g_{\rho\mu}}{\partial x^{\nu}} + \frac{\partial g_{\nu\rho}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}}\right). \tag{73}$$

Equation (21) is reduced to the first and third term. If we put the expression $\Gamma^{\rho}_{\nu\mu}$ from (73) therein, we obtain

$$-\frac{\partial^2 g_{\nu\mu}}{\partial x^{\rho}\partial x^{\rho}} + \frac{\partial^2 g_{\rho\nu}}{\partial x^{\rho}\partial x^{\mu}} + \frac{\partial^2 g_{\rho\mu}}{\partial x^{\nu}\partial x^{\rho}} - \frac{\partial^2 g_{\rho\rho}}{\partial x^{\nu}\partial x^{\mu}} = 0. \tag{74}$$

Before further consideration of (74), we develop the series from equation (2). Equation (2) then gives

$$(G_{\lambda} \equiv)\, \frac{\partial \gamma_{\underset{\vee}{\lambda\alpha}}}{\partial x^{\alpha}} = 0. \tag{75}$$

Now we put the expressions given by (70) into (74) and obtain with respect to (75)

$$-\frac{\partial^2 \gamma_{\underline{\nu\mu}}}{\partial x^{\rho}\partial x^{\rho}} + \frac{\partial^2 \gamma_{\underline{\rho\nu}}}{\partial x^{\rho}\partial x^{\mu}} + \frac{\partial^2 \gamma_{\underline{\rho\mu}}}{\partial x^{\nu}\partial x^{\rho}} - \frac{\partial^2 \gamma_{\underline{\rho\rho}}}{\partial x^{\nu}\partial x^{\mu}} = 0 \tag{76}$$

$$\frac{\partial^2 \gamma_{\underset{\vee}{\nu\mu}}}{\partial x^{\rho}\partial x^{\rho}} = 0. \tag{77}$$

The expressions (76), which may be simplified as usual by proper choice of coordinates, are the same as in the absence of an electromagnetic field. In the same manner, the equations (75) and (77) for the electromagnetic field do not contain the quantities $\gamma_{\underline{\mu\nu}}$ which refer to the gravitational field. Thus both fields are (in accordance with experience) independent in first approximation.

The equations (75), (77) are nearly equivalent to Maxwell's equations of empty space. Equation (75) is one Maxwellian system. The equation (75) can be replaced considering $G_{\lambda} = 0$ by

$$\left(G_{\underset{\vee}{\mu\nu}} \equiv\right) \frac{\partial^2 \gamma_{\underset{\vee}{\mu\nu}}}{\partial x^{\alpha}\partial x^{\alpha}} = 0. \tag{78}$$

We now have the identity

$$\frac{\partial G_{\underset{\vee}{\mu\nu}}}{\partial x^{\nu}} - \frac{\partial^3 \gamma_{\underset{\vee}{\mu\nu}}}{\partial x^{\nu}\partial x^{\alpha}\partial x^{\alpha}} \equiv 0$$

or

$$\frac{\partial G_{\underset{\vee}{\mu\nu}}}{\partial x^{\nu}} - \frac{\partial^2 G_{\mu}}{\partial x^{\alpha}\partial x^{\alpha}} \equiv 0. \tag{79}$$

After differenciate equation (78) with to respect to ρ, we found the next expression

$$\frac{\partial G_{\underset{\vee}{\mu\nu}}}{\partial x^{\rho}} - \frac{\partial^2}{\partial x^{\alpha}\partial x^{\alpha}}\left(\frac{\partial \gamma_{\underset{\vee}{\mu\nu}}}{\partial x^{\rho}}\right) = 0. \tag{80}$$

After applying two cyclic permutations of the indices μ, ν and ρ two further equations appear. Then, we obtain

$$\frac{\partial G_{\underset{\vee}{\mu\nu}}}{\partial x^{\rho}} + \frac{\partial G_{\underset{\vee}{\rho\mu}}}{\partial x^{\nu}} + \frac{\partial G_{\underset{\vee}{\nu\rho}}}{\partial x^{\mu}} - \frac{\partial^2}{\partial x^{\alpha}\partial x^{\alpha}} \left(\frac{\partial \gamma_{\underset{\vee}{\mu\nu}}}{\partial x^{\rho}} + \frac{\partial \gamma_{\underset{\vee}{\rho\mu}}}{\partial x^{\nu}} + \frac{\partial \gamma_{\underset{\vee}{\nu\rho}}}{\partial x^{\mu}}\right) \equiv 0. \tag{81}$$

Therefore, the equations which according to field equations hold for an antisymmetric (electromagnetic) field are

$$\frac{\partial \gamma_{\underset{\vee}{\lambda\alpha}}}{\partial x^{\alpha}} = 0 \tag{82}$$

$$\frac{\partial^2}{\partial x^{\alpha}\partial x^{\alpha}}\left(\frac{\partial \gamma_{\underset{\vee}{\mu\nu}}}{\partial x^{\rho}} + \frac{\partial \gamma_{\underset{\vee}{\rho\mu}}}{\partial x^{\nu}} + \frac{\partial \gamma_{\underset{\vee}{\nu\rho}}}{\partial x^{\mu}}\right) = 0. \tag{83}$$

If, in the equation (83), the expression inside the parentheses would itself vanish, then we would have Maxwell's equations for empty space [53, 54].

Concluding remarks

A relativistic theory of the field has been presented, starting from the field of infinitesimal displacement. Also, we calculated the curvature tensor and define the contracted curvature tensor. With this curvature tensor and the variational principle, we deduced the field equations, and Bianchi's identities. In consecuense, the field equations have been found from Bianchi's identities. In addition, if we were sure that a non-symmetric tensor $g_{\mu\nu}$ is the right means for describing the structure of the field, then we could hardly doubt that the above equations are the correct ones. The foregoing derivation shows how naturally we can extend general relativity theory to a non-symmetric field, and that the field-equations are really the generalizations of the gravitational equations. If there is an electromagnetic field, that means the $g_{\mu\nu}$ or the $\sqrt{-g}g^{\mu\nu}$ do contain a skew-symmetric part, we cannot solve the equations (9) any more with respect to the $\Gamma^{\alpha}_{\mu\nu}$, which significantly complicates the clearness of the whole system. We succeed in resolving the problem however, if we restrict ourselves to the first approximation. Therefore, we obtain equation (75), the first system of Maxwell's equations. If, in the equation (83), the expression inside the parentheses would itself vanish, then we would have Maxwell's equations for empty space, whose solutions therefore satisfy our equations. Maxwell's equations of empty space seem to be too weak, however, is not a (justified) objection to the theory since we do not know to which solutions of the linearized equations there correspond rigorous solutions which are regular in the entire space. It is clear from the start that in a consistent field theory which claims to be complete (in contrast e.g. to the pure theory of gravitation) only those solutions are to be considered which are regular in the entire space. Whether such (non-trivial) solutions exist is as yet unknown.

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