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General relativistic theory of gravitation and electrodynamics

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Abstract
As gravitation and electromagnetism are closely analogous long-range interactions, and the current formulation of gravitation is given in terms of geometry, we expect the latter also to appear throug the geometry. We look for the formally most simple expression for the law of gravitation in the absence of an electromagnetic field, and then the most natural generalization of this law. This theory contain Maxwell's theory in first approximation. In the following we outline the scheme of the general theory and then show in which sense this contains the law of the pure gravitational field and Maxwell's theory.

Keywords: Curvature tensor, field equations, Maxwell's equations.

Teoría relativista general de gravitación y electrodinámica

Resumen
Como la gravitación y el electromagnetismo son interacciones de largo alcance muy análogas, y la formulación actual de la gravitación se da en términos de geometría, esperamos que esta última también aparezca a través de la geometría. Buscamos la expresión formalmente más simple para la ley de la gravitación en ausencia de un campo electromagnético, y luego la generalización más natural de esta ley. Esta teoría contiene la teoría de Maxwell en primera aproximación. A continuación, esbozamos el esquema de la teoría general y luego mostramos en qué sentido contiene la ley del campo gravitacional puro y la teoría de Maxwell.

Palabras clave: Tensor de curvatura, ecuaciones de campo, ecuaciones de Maxwell.

Introduction

The first attempts of unification of Einstein [1] and Kaluza [2],other types of interactions different from gravity and electromagnetism, such as weak interaction and strong interaction, have been the subject of various attempts at unification, and by the end of the 1960s the electroweak theory was formulated [3–6]. In fact, it is a unified field theory of electromagnetism and weak interaction. Attempts to unify the theory of strong interaction [7, 8] with the electroweak model and with gravity (see [9–12]) have since remained one of the still pending challenges of physicists.

Theoretical physicists working in the field of general relativity [13–21] still do not have a consensus on the unification of gravitation and electromagnetic. However, so far we have not found a convincing formalism for this connection [22–28]. Our task now is to find out if exist a set of field equations for the non-symmetric fields. This problem was solved by forming a variational principle in close analogy to the symmetric case [29, 30]. This way we make sure that the resulting equations will be compatible. These variational principle make the derivation more complex than in the gravitational theory.

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The general theory

We consider a 4-dimensional continuum with a field $\Gamma^{\lambda}_{\mu\nu}$ which defines infinitesimal vector shifts according to the relation

$$dA^{\lambda} = -\Gamma^{\lambda}_{\mu\nu} A^{\mu} dx^{\nu}. \tag{1}$$

We start from the expression (1). By translation of a real vector along the boundary of an infinitesimal surface-element, we obtain a tensor of curvature as in general relativity.

Therefore, we have the real curvature-tensor

$$R^{\lambda}_{\mu\nu\eta} = \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\eta}} + \Gamma^{\lambda}_{\alpha\eta}\Gamma^{\alpha}_{\mu\nu} - \frac{\partial \Gamma^{\lambda}_{\mu\eta}}{\partial x^{\nu}} - \Gamma^{\lambda}_{\alpha\nu}\Gamma^{\alpha}_{\mu\eta} \tag{2}$$

and contracting with respect to λ and η one obtains the contracted curvature tensor

$$R_{\mu\nu} = R^{\lambda}_{\mu\nu\lambda} = \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} + \Gamma^{\lambda}_{\alpha\lambda}\Gamma^{\alpha}_{\mu\nu} - \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} - \Gamma^{\lambda}_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda}. \tag{3}$$

Independently from this $\Gamma^{\lambda}_{\mu\nu}$, we introduce a contravariant tensor density $\sqrt{-g}g^{\mu\nu}$. From both quantities we obtain the Lagrangian density

$$\mathscr{L} = \sqrt{-g}g^{\mu\nu}R_{\mu\nu} \tag{4}$$

and the variations of the integral

$$\mathscr{I} = \int \mathscr{L} dx_1 dx_2 dx_3 dx_4 \tag{5}$$

with respect to the $\sqrt{-g}g^{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$ are independent.

Next, we determine the field equations. From $\sqrt{-g}g^{\mu\nu}$ and $R_{\mu\nu}$ construct a Lagrangian density-function $\mathscr{L}$ whose integral we vary independently with respect to $\sqrt{-g}g^{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$.

The variation of the integral of $\mathscr{L}$:

$$\begin{aligned}\delta\int \mathscr{L} d\tau = &-\int d\tau \left[\tfrac{\partial}{\partial x^{\mu}}\left(\sqrt{-g}g^{\mu\nu}\right) + \sqrt{-g}g^{\lambda\nu}\Gamma^{\mu}_{\lambda\rho} + \sqrt{-g}g^{\mu\lambda}\Gamma^{\nu}_{\rho\lambda} - \sqrt{-g}g^{\mu\nu}\Gamma^{\lambda}_{\rho\lambda}\right]\delta\Gamma^{\rho}_{\mu\nu} \\ &+ \int d\tau \delta^{\nu}_{\rho}\left[\tfrac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\mu\nu}\right) + \sqrt{-g}g^{\lambda\sigma}\Gamma^{\lambda}_{\lambda\sigma}\right]\delta\Gamma^{\rho}_{\mu\nu} + \int d\tau \delta\left(\sqrt{-g}g^{\mu\nu}\right)R_{\mu\nu}\end{aligned}$$

with respect to $\sqrt{-g}g^{\mu\nu}$yields

$$R_{\mu\nu} = 0 \tag{6}$$

and with respect to the $\Gamma^{\alpha}_{\mu\nu}$, we obtain

$$\frac{\partial\left(\sqrt{-g}g^{\mu\nu}\right)}{\partial x^{\rho}} + \sqrt{-g}g^{\lambda\nu}\Gamma^{\mu}_{\lambda\rho} + \sqrt{-g}g^{\mu\lambda}\Gamma^{\nu}_{\rho\lambda} - \sqrt{-g}g^{\mu\nu}\Gamma^{\lambda}_{\rho\lambda} - \delta^{\nu}_{\rho}\left[\frac{\partial\left(\sqrt{-g}g^{\mu\sigma}\right)}{\partial x^{\sigma}} + \sqrt{-g}g^{\lambda\sigma}\Gamma^{\mu}_{\lambda\sigma}\right] = 0. \tag{7}$$

Now, if we contract of (7) by ν and α or μ and α, we obtain the equations

$$3\left[\frac{\partial\left(\sqrt{-g}g^{\mu\rho}\right)}{\partial x^{\rho}} + \sqrt{-g}g^{\lambda\rho}\Gamma^{\mu}_{\lambda\rho}\right] + 2\sqrt{-g}g^{\mu\rho}\Gamma^{\lambda}_{\underset{\vee}{\rho\lambda}} = 0, \tag{8}$$

$$\frac{\partial\left(\sqrt{-g}g^{\lambda\nu}\right)}{\partial x^{\lambda}} - \frac{\partial\left(\sqrt{-g}g^{\nu\lambda}\right)}{\partial x^{\lambda}} = 0. \tag{9}$$

To the covariant tensor $g_{\mu\nu}$ we can associate a contravariant one $g^{\mu\nu}$ uniquely by the condition:

$$g_{\mu\alpha}g^{\nu\alpha} = g_{\alpha\mu}g^{\alpha\nu} = \delta^{\nu}_{\mu} \tag{10}$$

and if we now multiply (7) by $\sqrt{-g}g_{\mu\nu}$, we obtain:

$$2\sqrt{-g}g^{\mu\rho}\left\{\frac{\partial}{\partial x^{\rho}}\left[\ln\sqrt{\det\left(\sqrt{-g}g_{\mu\nu}\right)}\right] + \Gamma^{\lambda}_{\rho\lambda}\right\} + 2\sqrt{-g}g^{\mu\rho}\Gamma^{\lambda}_{\underset{\vee}{\rho\lambda}} + \delta^{\rho}_{\rho}\left[\frac{\partial\left(\sqrt{-g}g^{\mu\lambda}\right)}{\partial x^{\lambda}} + \sqrt{-g}g^{\lambda\sigma}\Gamma^{\mu}_{\lambda\sigma}\right] = 0. \tag{11}$$

We write the equations (8) and (11) in the form:

$$\frac{1}{3}\sqrt{-g}g^{\mu\rho}\Gamma^{\lambda}_{\underset{\vee}{\rho\lambda}} = \sqrt{-g}g^{\mu\rho}\left\{\frac{\partial}{\partial x^{\rho}}\left[\ln\sqrt{\det\left(\sqrt{-g}g_{\mu\nu}\right)}\right] + \Gamma^{\lambda}_{\rho\lambda}\right\} \tag{12}$$

or

$$\frac{1}{3}\sqrt{-g}g^{\mu\rho}\Gamma^{\lambda}_{\underset{\vee}{\rho\lambda}} = -\frac{1}{2}\left[\frac{\partial\left(\sqrt{-g}g^{\mu\lambda}\right)}{\partial x^{\lambda}} + \sqrt{-g}g^{\lambda\rho}\Gamma^{\mu}_{\lambda\rho}\right]. \tag{13}$$

If we insert equation (13) into (7), we can write

$$\frac{\partial\left(\sqrt{-g}g^{\mu\nu}\right)}{\partial x^{\rho}}+\sqrt{-g}g^{\lambda\nu}\Gamma^{\mu}_{\lambda\rho}+\sqrt{-g}g^{\mu\lambda}\Gamma^{\mu}_{\lambda\rho}-\sqrt{-g}g^{\mu\nu}\Gamma^{\lambda}_{\rho\lambda}+\frac{2}{3}\delta^{\nu}_{\rho}\sqrt{-g}g^{\mu\alpha}\Gamma^{\lambda}_{\underset{\vee}{\alpha\lambda}}=0 \tag{14}$$

in conjunction with (9). Equation (14), after multiplying by $\sqrt{-g}g_{\mu\beta}$ and $\sqrt{-g}g_{\alpha\nu}$ and manipulating indices, can also take the following form

$$\frac{\partial g_{\mu\nu}}{\partial x^{\rho}}-g_{\mu\alpha}\Gamma^{\alpha}_{\rho\nu}-g_{\alpha\nu}\Gamma^{\alpha}_{\mu\rho}+g_{\mu\nu}\left[\frac{\partial\left(\ln\sqrt{-g}\right)}{\partial x^{\rho}}+\Gamma^{\lambda}_{\rho\lambda}\right]+\frac{2}{3}g_{\mu\rho}\Gamma^{\lambda}_{\underset{\vee}{\lambda\nu}}=0. \tag{15}$$

Therefore, we have the field equations

$$R_{\mu\nu}=\frac{\partial\Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}}+\Gamma^{\lambda}_{\alpha\lambda}\Gamma^{\alpha}_{\mu\nu}-\frac{\partial\Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}}-\Gamma^{\lambda}_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda}=0, \tag{16}$$

$$\frac{\partial\left(\sqrt{-g}g^{\lambda\nu}\right)}{\partial x^{\lambda}}-\frac{\partial\left(\sqrt{-g}g^{\nu\lambda}\right)}{\partial x^{\lambda}}=0,$$

$$\frac{\partial g_{\mu\nu}}{\partial x^{\rho}}-g_{\mu\alpha}\Gamma^{\alpha}_{\rho\nu}-g_{\alpha\nu}\Gamma^{\alpha}_{\mu\rho}+g_{\mu\nu}\left[\frac{\partial\left(\ln\sqrt{-g}\right)}{\partial x^{\rho}}+\Gamma^{\lambda}_{\rho\lambda}\right]+\frac{2}{3}g_{\mu\rho}\Gamma^{\lambda}_{\underset{\vee}{\lambda\nu}}=0.$$

We interpret the symmetric part of $\sqrt{-g}g_{\mu\nu}$ as metric tensor and the skew-symmetric part as electromagnetic field, and we assume

$$\frac{\partial\left(\ln\sqrt{-g}\right)}{\partial x^{\lambda}}+\Gamma^{\lambda}_{\rho\lambda}=0$$

and

$$\Gamma^{\lambda}_{\underset{\vee}{\nu\lambda}}=0. \tag{17}$$

If there were no bracket in (7) would imply the vanishing of

$$\frac{\partial\left(\sqrt{-g}g^{\mu\nu}\right)}{\partial x^{\rho}}+\sqrt{-g}g^{\lambda\nu}\Gamma^{\mu}_{\lambda\rho}+\sqrt{-g}g^{\mu\lambda}\Gamma^{\nu}_{\rho\lambda}-\sqrt{-g}g^{\mu\nu}\Gamma^{\lambda}_{\rho\lambda}, \tag{18}$$

however, this would require the vanishing $\Gamma^{\lambda}_{\underset{\vee}{\nu\lambda}}$. We can resolve this difficulty in the following manner. We can compute the equations of (7)

$$\begin{aligned}&\tfrac{\partial}{\partial x^{\rho}}\left(\sqrt{-g}g^{\underline{\mu\nu}}\right)+\sqrt{-g}g^{\underline{\lambda\nu}}\Gamma^{\mu}_{\underline{\lambda\rho}}+\sqrt{-g}g^{\overset{\lambda\nu}{\vee}}\Gamma^{\mu}_{\underset{\vee}{\lambda\rho}}+\sqrt{-g}g^{\underline{\mu\lambda}}\Gamma^{\nu}_{\underline{\rho\lambda}}+\sqrt{-g}g^{\overset{\mu\lambda}{\vee}}\Gamma^{\nu}_{\underset{\vee}{\rho\lambda}}-\sqrt{-g}g^{\underline{\mu\nu}}\Gamma^{\lambda}_{\underline{\rho\lambda}}\\&-\sqrt{-g}g^{\overset{\mu\nu}{\vee}}\Gamma^{\lambda}_{\underset{\vee}{\rho\lambda}}-\delta^{\nu}_{\rho}\left[\tfrac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\underline{\mu\lambda}}\right)+\sqrt{-g}g^{\underline{\lambda\sigma}}\Gamma^{\mu}_{\underline{\lambda\sigma}}+\sqrt{-g}g^{\overset{\lambda\sigma}{\vee}}\Gamma^{\mu}_{\underset{\vee}{\lambda\sigma}}\right]=0,\end{aligned} \tag{19}$$

$$\begin{aligned}&\tfrac{\partial}{\partial x^{\rho}}\left(\sqrt{-g}g^{\overset{\mu\nu}{\vee}}\right)+\sqrt{-g}g^{\underline{\lambda\nu}}\Gamma^{\mu}_{\underset{\vee}{\lambda\rho}}+\sqrt{-g}g^{\overset{\lambda\nu}{\vee}}\Gamma^{\mu}_{\underline{\lambda\rho}}+\sqrt{-g}g^{\underline{\mu\lambda}}\Gamma^{\nu}_{\underset{\vee}{\rho\lambda}}+\sqrt{-g}g^{\overset{\mu\lambda}{\vee}}\Gamma^{\nu}_{\underline{\rho\lambda}}-\sqrt{-g}g^{\underline{\mu\nu}}\Gamma^{\lambda}_{\underset{\vee}{\rho\lambda}}\\&-\sqrt{-g}g^{\overset{\mu\nu}{\vee}}\Gamma^{\lambda}_{\underline{\rho\lambda}}-\delta^{\nu}_{\rho}\left[\tfrac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\overset{\mu\lambda}{\vee}}\right)+\sqrt{-g}g^{\underline{\lambda\sigma}}\Gamma^{\mu}_{\underset{\vee}{\lambda\sigma}}+\sqrt{-g}g^{\overset{\lambda\sigma}{\vee}}\Gamma^{\mu}_{\underline{\lambda\sigma}}\right]=0.\end{aligned} \tag{20}$$

If we contract the equation (20) with respect to ν and ρ

$$\frac{3}{2}\frac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\overset{\mu\lambda}{\vee}}\right)+\sqrt{-g}g^{\underline{\mu\lambda}}\Gamma^{\rho}_{\underset{\vee}{\lambda\rho}}=0. \tag{21}$$

From this we can deduce that the necessary and sufficient condition for $\Gamma^{\rho}_{\underset{\vee}{\rho\lambda}}=0$ is that $\partial_{\rho}\left(\sqrt{-g}g^{\overset{\mu\rho}{\vee}}\right)=0$. In order to satisfy this identically it suffices to asume

$$\sqrt{-g}g^{\overset{\mu\nu}{\vee}}=\frac{\partial}{\partial x^{\rho}}\left(\sqrt{-g}g^{\mu\nu\rho}\right) \tag{22}$$

where $\sqrt{-g}g^{\overset{\mu\nu}{\vee}}$ is a tensor density which is antisymmetric in all three indices. That is, we require that $\sqrt{-g}g^{\overset{\mu\nu}{\vee}}$ be derived from a "vector potential". We therefore substitute in the Lagrangian density

$$\sqrt{-g}g^{\mu\nu}=\sqrt{-g}g^{\underline{\mu\nu}}+\frac{\partial}{\partial x^{\rho}}\left(\sqrt{-g}g^{\mu\nu\rho}\right) \tag{23}$$

and vary independiently with respect to the Γ then yields

$$\frac{\partial g_{\mu\nu}}{\partial x^{\rho}}-g_{\alpha\nu}\Gamma^{\alpha}_{\mu\rho}-g_{\mu\alpha}\Gamma^{\alpha}_{\rho\nu}=0. \tag{24}$$

The variation respect to the $\sqrt{-g}g^{\underline{\mu\nu}}$ and $\sqrt{-g}g^{\mu\lambda\tau}$

yields the equations

$$R_{\underline{\mu\nu}} = 0, \tag{25}$$

$$\frac{\partial}{\partial x^{\lambda}} R_{\underset{\vee}{\mu\nu}} + \frac{\partial}{\partial x^{\mu}} R_{\underset{\vee}{\nu\lambda}} + \frac{\partial}{\partial x^{\nu}} R_{\underset{\vee}{\lambda\mu}} = 0. \tag{26}$$

If we omit $\frac{\partial}{\partial x^{\nu}}\left(\sqrt{-g}\, g^{\overset{\mu\nu}{\vee}}\right) = 0$, then the system of field equations not weakened is therefore:

$$R_{\underline{\mu\nu}} = 0,$$

$$\frac{\partial}{\partial x^{\lambda}} R_{\underset{\vee}{\mu\nu}} + \frac{\partial}{\partial x^{\mu}} R_{\underset{\vee}{\nu\lambda}} + \frac{\partial}{\partial x^{\nu}} R_{\underset{\vee}{\lambda\mu}} = 0,$$

$$\Gamma^{\lambda}_{\underset{\vee}{\mu\lambda}} = 0,$$

$$\frac{\partial g_{\mu\nu}}{\partial x^{\rho}} - g_{\alpha\nu}\Gamma^{\alpha}_{\mu\rho} - g_{\mu\alpha}\Gamma^{\alpha}_{\rho\nu} = 0.$$

The foregoing derivation shows how naturally we can extend general relativity theory to a general relativistic theory of the asymmetric field.

The Bianchi's identities

In the case of the gravitational theory it is essential that besides the $g_{\mu\nu}$ tensor we also have the symmetric infinitesimal displacement $\Gamma^{\lambda}_{\mu\nu}$. This displacement is connected with $g_{\mu\nu}$ by the equation

$$\nabla_{\rho} g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^{\rho}} - g_{\alpha\nu}\Gamma^{\alpha}_{\mu\rho} - g_{\mu\alpha}\Gamma^{\alpha}_{\rho\nu} = 0, \tag{27}$$

where we have defined the covariant derivate $\nabla_{\rho} g_{\mu\nu}$. If the differentiation index ρ is to be on the right in a certain term, we put + under the corresponding tensor-index; if on the left, put − under the index. As an illustration we give a new form of (27):

$$\nabla_{\rho} g_{\underset{+}{\mu}\underset{-}{\nu}} = \frac{\partial g_{\mu\nu}}{\partial x^{\rho}} - g_{\alpha\nu}\Gamma^{\alpha}_{\mu\rho} - g_{\mu\alpha}\Gamma^{\alpha}_{\rho\nu} = 0. \tag{28}$$

In the theory of symmetric fields there is a second method of ensuring the compatibility of the field-equations ($R_{\mu\nu} = 0$). We must have four identities connecting the equations. These can be derived by contracting the Bianchi-identities which hold for the curvature tensor:

$$\nabla_{\underset{\bullet}{\sigma}} R_{\mu\nu\underset{\bullet}{\lambda}\underset{\bullet}{\rho}} = \nabla_{\sigma} R_{\mu\nu\lambda\rho} + \nabla_{\lambda} R_{\mu\nu\rho\sigma} + \nabla_{\rho} R_{\mu\nu\sigma\lambda} = 0.$$

In this article, we have shown that an analogous argument can be used for the justification of the field-equations also in our case.

A direct computation shows that the covariant derivate of curvature tensor

$$\nabla_{\underset{\bullet}{\lambda}} R^{\overset{\rho}{+}}_{\underset{+}{\mu}\underset{\bullet}{\nu}\underset{\bullet}{\sigma}} = \nabla_{\lambda} R^{\overset{\rho}{+}}_{\underset{+}{\mu}\nu\sigma} + \nabla_{\nu} R^{\overset{\rho}{+}}_{\underset{+}{\mu}\sigma\lambda} + \nabla_{\sigma} R^{\overset{\rho}{+}}_{\underset{+}{\mu}\lambda\nu}$$

satisfies the identities:

$$\nabla_{\underset{\bullet}{\lambda}} R^{\overset{\rho}{+}}_{\underset{+}{\mu}\underset{\bullet}{\nu}\underset{\bullet}{\sigma}} = \frac{\partial}{\partial x^{\underset{\bullet}{\lambda}}} R^{\rho}_{\mu\underset{\bullet}{\nu}\underset{\bullet}{\sigma}} + R^{\alpha}_{\mu\underset{\bullet}{\sigma}\underset{\bullet}{\lambda}}\Gamma^{\rho}_{\alpha\underset{\bullet}{\nu}} - R^{\rho}_{\alpha\underset{\bullet}{\sigma}\underset{\bullet}{\lambda}}\Gamma^{\alpha}_{\mu\underset{\bullet}{\nu}} = 0. \tag{29}$$

From (2) we can form the covariant curvature tensor in analogy to the symmetric case,

$$R_{\mu\nu\lambda\eta} = g_{\alpha\mu} R^{\alpha}_{\nu\lambda\eta}. \tag{30}$$

The choice of $g_{\alpha\mu}$ instead of $g_{\mu\alpha}$ may seem arbitrary, but this is not really so. We have to lower the index ρ in the identities (29). The contravariant index μ has the + differentiation character, so it must be summed with a similar index, i.e. the first index of g. Only this way can we lower the index μ in (29) without introducing additional terms. Thus we get the covariant identities

$$g_{\rho\alpha}\nabla_{\underset{\bullet}{\lambda}} R^{\overset{\rho}{+}}_{\underset{+}{\mu}\underset{\bullet}{\nu}\underset{\bullet}{\sigma}} = \nabla_{\underset{\bullet}{\lambda}}\left(g_{\underset{+}{\rho}\underset{-}{\alpha}} R^{\overset{\rho}{+}}_{\underset{+}{\mu}\underset{\bullet}{\nu}\underset{\bullet}{\sigma}}\right) = \nabla_{\underset{\bullet}{\lambda}} R_{\underset{-}{\alpha}\underset{+}{\mu}\underset{\bullet}{\nu}\underset{\bullet}{\sigma}} = 0. \tag{31}$$

For what follows we must also find the symmetry properties of $R_{\alpha\mu\nu\sigma}$. From (2) it is clear that $R^{\lambda}_{\mu\nu\sigma}$ is antisymmetric in $(\nu\sigma)$. From (30) we see that $R_{\alpha\mu\nu\sigma}$ has the same property:

$$R_{\lambda\mu\nu\sigma} = -R_{\lambda\mu\sigma\nu}. \tag{32}$$

If we differentiate (24) with respect to σ and antisymmetrize with respect to ρ and σ, we have:

$$-\frac{\partial g_{\alpha\nu}}{\partial x^{\sigma}}\Gamma^{\alpha}_{\mu\rho} - \frac{\partial g_{\mu\alpha}}{\partial x^{\sigma}}\Gamma^{\alpha}_{\rho\nu} + \frac{\partial g_{\alpha\nu}}{\partial x^{\rho}}\Gamma^{\alpha}_{\mu\sigma} + \frac{\partial g_{\mu\alpha}}{\partial x^{\rho}}\Gamma^{\alpha}_{\sigma\nu} + g_{\alpha\nu}\left(\frac{\partial \Gamma^{\alpha}_{\mu\sigma}}{\partial x^{\rho}} - \frac{\partial \Gamma^{\alpha}_{\mu\rho}}{\partial x^{\sigma}}\right) + g_{\mu\alpha}\left(\frac{\partial \Gamma^{\alpha}_{\sigma\nu}}{\partial x^{\rho}} - \frac{\partial \Gamma^{\alpha}_{\rho\nu}}{\partial x^{\sigma}}\right) = 0.$$

Using (24) again on the first four terms and then collecting terms

$$-g_{\alpha\nu}\left(\frac{\partial \Gamma^{\alpha}_{\mu\rho}}{\partial x^{\sigma}} - \frac{\partial \Gamma^{\alpha}_{\mu\sigma}}{\partial x^{\rho}}\right) - g_{\mu\alpha}\left(\frac{\partial \Gamma^{\alpha}_{\rho\nu}}{\partial x^{\sigma}} - \frac{\partial \Gamma^{\alpha}_{\sigma\nu}}{\partial x^{\rho}}\right) - g_{\alpha\nu}\Gamma^{\alpha}_{\eta\sigma}\Gamma^{\eta}_{\mu\rho} - g_{\mu\alpha}\Gamma^{\alpha}_{\sigma\eta}\Gamma^{\eta}_{\rho\nu} + g_{\alpha\nu}\Gamma^{\alpha}_{\eta\rho}\Gamma^{\eta}_{\mu\sigma} + g_{\mu\alpha}\Gamma^{\alpha}_{\rho\eta}\Gamma^{\eta}_{\sigma\nu} = 0$$

or using (2) and (30), we have

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}. \tag{33}$$

This expresses that $R_{\mu\nu\rho\sigma}$ is anti-symmetric in $(\mu\nu)$; this is the manner in which the antisymmetry of $R_{\mu\nu\rho\sigma}$ (in the gravitational theory) generalizes to our case.

In (31) it is not immediately clear that $\nabla_{\underset{\bullet}{\lambda}} R_{\underset{-+\bullet\bullet}{\alpha\mu\nu\sigma}}$ is a tensor. We are now in a position to give a more useful form for (31) in which this is obvious, i. e.:

$$\begin{aligned}\nabla_{\lambda} R_{\underset{-+-+}{\mu\nu\rho\sigma}} + \nabla_{\rho} R_{\underset{-+++}{\mu\nu\sigma\lambda}} + \nabla_{\sigma} R_{\underset{-+--}{\mu\nu\lambda\rho}} = \nabla_{\underset{\bullet}{\lambda}} R_{\underset{-+\bullet\bullet}{\mu\nu\rho\sigma}} - R_{\mu\nu\alpha\sigma}\Gamma^{\alpha}_{\lambda\rho} - R_{\mu\nu\rho\alpha}\Gamma^{\alpha}_{\sigma\lambda} - R_{\mu\nu\alpha\lambda}\Gamma^{\alpha}_{\sigma\rho} \\ - R_{\mu\nu\sigma\alpha}\Gamma^{\alpha}_{\lambda\rho} - R_{\mu\nu\alpha\rho}\Gamma^{\alpha}_{\sigma\lambda} - R_{\mu\nu\lambda\alpha}\Gamma^{\alpha}_{\sigma\rho}.\end{aligned} \tag{34}$$

The first term on the right side of the equation vanishes by (31), the last six cancel out due to (32). Therefore,

$$\nabla_{\lambda} R_{\underset{-+-+}{\mu\nu\rho\sigma}} + \nabla_{\rho} R_{\underset{-+++}{\mu\nu\sigma\lambda}} + \nabla_{\sigma} R_{\underset{-+--}{\mu\nu\lambda\rho}} = 0, \tag{35}$$

where this equation is called "Bianchi's identities". We are now in a position to carry out the derivation of the identities for the field equations. In analogy to the gravitational theory, we contract (35) by $g^{\mu\rho}g^{\nu\rho}$. Making use of (32), we get

$$g^{\mu\rho}g^{\nu\rho}\left[\nabla_{\lambda} R_{\underset{-+-+}{\mu\nu\rho\sigma}} + \nabla_{\rho} R_{\underset{-+++}{\mu\nu\sigma\lambda}} + \nabla_{\sigma} R_{\underset{-+--}{\mu\nu\lambda\rho}}\right] = 0$$

or using $\nabla_{\rho} g_{\underset{+-}{\mu\nu}} = \frac{\partial}{\partial x^{\rho}} g_{\mu\nu} - g_{\alpha\nu}\Gamma^{\alpha}_{\mu\rho} - g_{\mu\alpha}\Gamma^{\alpha}_{\rho\nu} = 0$

$$g^{\nu\rho}\nabla_{\lambda}\left(g^{\overset{\sigma\mu}{+-}} R_{\underset{-+-+}{\mu\nu\rho\sigma}}\right) - g^{\nu\rho}\nabla_{\rho}\left(g^{\overset{\sigma\mu}{+-}} R_{\underset{-+++}{\mu\nu\lambda\sigma}}\right) - g^{\sigma\mu}\nabla_{\sigma}\left(g^{\overset{\nu\rho}{+-}} R_{\underset{-+--}{\mu\nu\rho\lambda}}\right) = 0. \tag{36}$$

Let us define

$$R_{\mu\nu} = g^{\sigma\lambda} R_{\lambda\mu\nu\sigma} \tag{37}$$

$$S_{\sigma\lambda} = g^{\mu\nu} R_{\lambda\mu\nu\sigma} \tag{38}$$

where

$$R_{\mu\nu} = g^{\sigma\lambda} g_{\alpha\lambda} R^{\alpha}_{\mu\nu\sigma} = \delta^{\sigma}_{\alpha} R^{\alpha}_{\mu\nu\sigma} = \frac{\partial}{\partial x^{\sigma}}\Gamma^{\sigma}_{\mu\nu} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\sigma}_{\eta\sigma} - \frac{\partial}{\partial x^{\nu}}\Gamma^{\sigma}_{\mu\sigma} - \Gamma^{\eta}_{\mu\sigma}\Gamma^{\sigma}_{\eta\nu}. \tag{39}$$

Then we have

$$g^{\nu\rho}\left[\nabla_{\lambda} R_{\underset{+-}{\nu\rho}} - \nabla_{\rho} R_{\underset{++}{\nu\lambda}} - \nabla_{\nu} S_{\underset{--}{\lambda\rho}}\right] = 0. \tag{40}$$

We need some connection between R and S. From (32) and we see that

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}.$$

Multiply by $g^{\mu\sigma}$ and sum:

$$S_{\rho\nu} = R_{\rho\nu}. \tag{41}$$

It then follows from (41) that

$$g^{\nu\rho}\left[\nabla_{\lambda} R_{\underset{+-}{\nu\rho}} - \nabla_{\rho} R_{\underset{++}{\nu\lambda}} - \nabla_{\nu} R_{\underset{--}{\lambda\rho}}\right] = 0. \tag{42}$$

These identities hold for all fields where Γ is defined by (24) and is subject to (17). We might jump to the conclusion that the field equations should stipulate the vanishing of all $R_{\mu\nu}$. This set, together with (24) and (17) would, however, be overdetermined. We can get a weaker set of equations by observing how $R_{\underset{\vee}{\mu\nu}}$ enters (42). The contribution of $R_{\underset{\vee}{\mu\nu}}$ to the equations is:

$$g^{\nu\rho}\left(\nabla_{\lambda} R_{\underline{\nu\rho}} - \nabla_{\rho} R_{\underline{\nu\lambda}} - \nabla_{\nu} R_{\underline{\lambda\rho}}\right) + g^{\nu\rho}\left[\nabla_{\lambda} R_{\underset{\vee}{\underset{+-}{\nu\rho}}} - \nabla_{\rho} R_{\underset{\vee}{\underset{++}{\nu\lambda}}} - \nabla_{\nu} R_{\underset{\vee}{\underset{--}{\lambda\rho}}}\right] = 0$$

which can be written as

$$\nabla_{\lambda}\left(g^{\nu\rho} R_{\underline{\nu\rho}}\right) - \nabla_{\rho}\left(g^{\nu\rho} R_{\underline{\nu\lambda}}\right) - \nabla_{\nu}\left(g^{\nu\rho} R_{\underline{\lambda\rho}}\right) + g^{\nu\rho}\left[\frac{\partial}{\partial x^{\lambda}} R_{\underset{\vee}{\nu\rho}} - R_{\underset{\vee}{\alpha\rho}}\Gamma^{\alpha}_{\nu\lambda} - R_{\underset{\vee}{\nu\alpha}}\Gamma^{\alpha}_{\lambda\rho}\right]$$

$$+g^{\nu\rho}\left[-\frac{\partial}{\partial x^{\rho}}R_{\underset{\vee}{\nu\lambda}}+R_{\underset{\vee}{\alpha\lambda}}\Gamma^{\alpha}_{\nu\rho}+R_{\underset{\vee}{\nu\alpha}}\Gamma^{\alpha}_{\lambda\rho}-\frac{\partial}{\partial x^{\nu}}R_{\underset{\vee}{\rho\lambda}}+R_{\underset{\vee}{\lambda\alpha}}\Gamma^{\alpha}_{\nu\rho}+R_{\underset{\vee}{\alpha\rho}}\Gamma^{\alpha}_{\nu\lambda}\right]=0$$

this equation is equivalent to

$$\nabla_{\lambda}R-\nabla_{\rho}R^{\rho}_{\lambda}-\nabla_{\nu}R^{\nu}_{\lambda}+g^{\nu\rho}\left[\frac{\partial}{\partial x^{\lambda}}R_{\underset{\vee}{\nu\rho}}-\frac{\partial}{\partial x^{\rho}}R_{\underset{\vee}{\nu\lambda}}-\frac{\partial}{\partial x^{\nu}}R_{\underset{\vee}{\rho\lambda}}\right]=0$$

or

$$\nabla_{\rho}\left(R^{\rho}_{\lambda}-\frac{1}{2}\delta^{\rho}_{\lambda}R\right)+g^{\nu\rho}\left[\frac{\partial}{\partial x^{\lambda}}R_{\underset{\vee}{\nu\rho}}+\frac{\partial}{\partial x^{\rho}}R_{\underset{\vee}{\lambda\nu}}+\frac{\partial}{\partial x^{\nu}}R_{\underset{\vee}{\lambda\rho}}\right]=0,$$

that is

$$-g^{\underline{\nu\rho}}\nabla_{\rho}\left(R_{\underline{\lambda\nu}}-\frac{1}{2}g_{\underline{\lambda\nu}}R\right)+g^{\nu\rho}\frac{\partial}{\partial x^{\underset{\bullet}{\lambda}}}R_{\underset{\vee}{\underset{\bullet\bullet}{\nu\rho}}}=0.$$

Since we see that $R_{\mu\nu}$ enters the equations only in the combination $\frac{\partial}{\partial x^{\underset{\bullet}{\lambda}}}R_{\underset{\vee}{\underset{\bullet\bullet}{\nu\rho}}}$ it is natural to choose the field equations for $R_{\underset{\vee}{\mu\nu}}$ as

$$\frac{\partial}{\partial x^{\underset{\bullet}{\lambda}}}R_{\underset{\vee}{\underset{\bullet\bullet}{\nu\rho}}}=0$$

instead of $R_{\underline{\mu\nu}}=0$. Therefore, we get the field equations:

$$\Gamma^{\lambda}_{\underset{\vee}{\mu\lambda}}=0,$$

$$R_{\underline{\mu\nu}}=0,$$

$$\frac{\partial}{\partial x^{\underset{\bullet}{\lambda}}}R_{\underset{\vee}{\underset{\bullet\bullet}{\mu\nu}}}=0,$$

where the $\Gamma^{\lambda}_{\mu\nu}$ are defined by:

$$\nabla_{\rho}g_{\underset{+-}{\mu\nu}}=0.$$

The foregoing derivation shows how naturally we can extend general relativity theory to a non-symmetric field.

Gravitational theory

Let the $g_{\mu\nu}$ be symmetric. In the case of the symmetric field we obtain the fields equations most simply in the following manner. We use as Lagrangian function the scalar density

$$\mathscr{L}_G=\sqrt{-g}g^{\mu\nu}R_{\mu\nu} \tag{43}$$

where $R_{\mu\nu}$is the curvature tensor in the relativistic theory of gravitation.

If we vary the volume integral of $\mathscr{L}_G$, i. e.

$$\begin{aligned}\delta\int\mathscr{L}_G d^4x=&-\int d^4x\left[\tfrac{\partial}{\partial x^{\mu}}\left(\sqrt{-g}g^{\mu\nu}\right)+\sqrt{-g}g^{\lambda\nu}\Gamma^{\mu}_{\lambda\rho}+\sqrt{-g}g^{\mu\lambda}\Gamma^{\nu}_{\rho\lambda}-\sqrt{-g}g^{\mu\nu}\Gamma^{\lambda}_{\rho\lambda}\right]\delta\Gamma^{\rho}_{\mu\nu}\\&+\int d^4x\delta^{\nu}_{\rho}\left[\tfrac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\mu\nu}\right)+\sqrt{-g}g^{\lambda\sigma}\Gamma^{\lambda}_{\lambda\sigma}\right]\delta\Gamma^{\rho}_{\mu\nu}-\int d^4x\sqrt{-g}\left(R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R\right)\delta g^{\mu\nu}\end{aligned}$$

independiently with respect to Γ and g, then variation with respect to Γ yields

$$\begin{aligned}-&\left[\tfrac{\partial}{\partial x^{\mu}}\left(\sqrt{-g}g^{\mu\nu}\right)+\sqrt{-g}g^{\lambda\nu}\Gamma^{\mu}_{\lambda\rho}+\sqrt{-g}g^{\mu\lambda}\Gamma^{\nu}_{\rho\lambda}-\sqrt{-g}g^{\mu\nu}\Gamma^{\lambda}_{\rho\lambda}\right]\\&+\delta^{\nu}_{\rho}\left[\tfrac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\mu\nu}\right)+\sqrt{-g}g^{\lambda\sigma}\Gamma^{\lambda}_{\lambda\sigma}\right]=0\end{aligned}$$

or

$$\frac{\partial g_{\mu\nu}}{\partial x^{\rho}}-g_{\alpha\nu}\Gamma^{\alpha}_{\mu\rho}-g_{\mu\alpha}\Gamma^{\alpha}_{\rho\nu}=0 \tag{44}$$

and variation with respect to g yields the equations

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=0, \tag{45}$$

or

$$R_{\mu\nu}=0. \tag{46}$$

The variation of the gravitational action, $\int d^4x\mathscr{L}_G$, with respect to $g_{\mu\nu}$ leads to the Einstein's field equations of general relativity, and the variation with respect to the affine connection, $\Gamma^{\lambda}_{\mu\nu}$, reveals that the connection is necessarily the metric connection.

Relations to Maxwell's electromagnetic theory

If there is an electromagnetic field, that means the $g_{\mu\nu}$ or the $\sqrt{-g}g^{\mu\nu}$ do contain a skew-symmetric part, we cannot solve the equations (24) any more with respect to the $\Gamma^{\alpha}_{\mu\nu}$, which significantly complicates the clearness of the whole system. We succeed in resolving the problem however, if we restrict ourselves to the first approximation [29, 30]. We have conducted this and once again postulate the vanishing of $\Gamma^{\lambda}_{\underset{\vee}{\mu\lambda}}$. Equation (17) then gives

$$(G_\lambda \equiv)\, \frac{\partial \gamma_{\underset{\vee}{\lambda\alpha}}}{\partial x^{\alpha}} = 0. \tag{47}$$

The equation (47) can be replaced considering $G_\lambda = 0$ by

$$\left(G_{\underset{\vee}{\mu\nu}} \equiv\right) \frac{\partial^2 \gamma_{\underset{\vee}{\mu\nu}}}{\partial x^{\alpha}\partial x^{\alpha}} = 0. \tag{48}$$

We now have the identity

$$\frac{\partial G_{\underset{\vee}{\mu\nu}}}{\partial x^{\nu}} - \frac{\partial^3 \gamma_{\underset{\vee}{\mu\nu}}}{\partial x^{\nu}\partial x^{\alpha}\partial x^{\alpha}} \equiv 0$$

or

$$\frac{\partial G_{\underset{\vee}{\mu\nu}}}{\partial x^{\nu}} - \frac{\partial^2 G_{\mu}}{\partial x^{\alpha}\partial x^{\alpha}} \equiv 0. \tag{49}$$

The identity (48) implies the density of the electric current $J_\mu = \frac{\partial}{\partial x^{\nu}}\gamma_{\underset{\vee}{\mu\nu}}$. After differenciate equation (48) with to respect to ρ, we found the next expression

$$\frac{\partial G_{\underset{\vee}{\mu\nu}}}{\partial x^{\rho}} - \frac{\partial^2}{\partial x^{\alpha}\partial x^{\alpha}}\left(\frac{\partial \gamma_{\underset{\vee}{\mu\nu}}}{\partial x^{\rho}}\right) = 0. \tag{50}$$

After applying two cyclic permutations of the indices μ, ν and ρ two further equations appear. Then, we obtain

$$\frac{\partial G_{\underset{\vee}{\mu\nu}}}{\partial x^{\rho}} + \frac{\partial G_{\underset{\vee}{\rho\mu}}}{\partial x^{\nu}} + \frac{\partial G_{\underset{\vee}{\nu\rho}}}{\partial x^{\mu}} - \frac{\partial^2}{\partial x^{\alpha}\partial x^{\alpha}}\left(\frac{\partial \gamma_{\underset{\vee}{\mu\nu}}}{\partial x^{\rho}} + \frac{\partial \gamma_{\underset{\vee}{\rho\mu}}}{\partial x^{\nu}} + \frac{\partial \gamma_{\underset{\vee}{\nu\rho}}}{\partial x^{\mu}}\right) \equiv 0. \tag{51}$$

Therefore, the equations which according to field equations hold for an antisymmetric (electromagnetic) field are

$$\frac{\partial \gamma_{\underset{\vee}{\lambda\alpha}}}{\partial x^{\alpha}} = 0, \tag{52}$$

$$\frac{\partial^2}{\partial x^{\alpha}\partial x^{\alpha}}\left(\frac{\partial \gamma_{\underset{\vee}{\mu\nu}}}{\partial x^{\rho}} + \frac{\partial \gamma_{\underset{\vee}{\rho\mu}}}{\partial x^{\nu}} + \frac{\partial \gamma_{\underset{\vee}{\nu\rho}}}{\partial x^{\mu}}\right) = 0. \tag{53}$$

If, in the equation (53), the expression inside the parentheses would itself vanish, then we would have Maxwell's equations for empty space, whose solutions therefore satisfy our equations.

Let ϕ_ν be the components of the electromagnetic potential vector. From them we form the components $\gamma_{\underset{\vee}{\mu\nu}}$ of the electromagnetic field, in acordance with the system of equations

$$\gamma_{\underset{\vee}{\mu\nu}} = \frac{\partial \phi_\mu}{\partial x^{\nu}} - \frac{\partial \phi_\nu}{\partial x^{\mu}}. \tag{54}$$

From equation (54), we have the systems of equations

$$\frac{\partial \gamma_{\underset{\vee}{\mu\nu}}}{\partial x^{\rho}} + \frac{\partial \gamma_{\underset{\vee}{\rho\mu}}}{\partial x^{\nu}} + \frac{\partial \gamma_{\underset{\vee}{\nu\rho}}}{\partial x^{\mu}} = 0 \tag{55}$$

The system (55) thus contains essentially four equations which are written out as follows:

$$\frac{\partial \gamma_{\underset{\vee}{23}}}{\partial x^{0}} + \frac{\partial \gamma_{\underset{\vee}{30}}}{\partial x^{2}} + \frac{\partial \gamma_{\underset{\vee}{02}}}{\partial x^{3}} = 0, \tag{56}$$

$$\frac{\partial \gamma_{\underset{\vee}{30}}}{\partial x^{1}} + \frac{\partial \gamma_{\underset{\vee}{01}}}{\partial x^{3}} + \frac{\partial \gamma_{\underset{\vee}{13}}}{\partial x^{0}} = 0, \tag{57}$$

$$\frac{\partial \gamma_{\underset{\vee}{01}}}{\partial x^{2}} + \frac{\partial \gamma_{\underset{\vee}{12}}}{\partial x^{0}} + \frac{\partial \gamma_{\underset{\vee}{20}}}{\partial x^{1}} = 0, \tag{58}$$

$$\frac{\partial \gamma_{\underset{\vee}{12}}}{\partial x^{3}} + \frac{\partial \gamma_{\underset{\vee}{23}}}{\partial x^{1}} + \frac{\partial \gamma_{\underset{\vee}{31}}}{\partial x^{2}} = 0. \tag{59}$$

This system correspond to the second of Maxwell's system of equations. We recognize this at once by setting

$$\begin{aligned} &\gamma_{\underset{\vee}{23}} = H_x, \quad \gamma_{\underset{\vee}{31}} = H_y, \quad \gamma_{\underset{\vee}{12}} = H_z, \\ &\gamma_{\underset{\vee}{10}} = E_x, \quad \gamma_{\underset{\vee}{20}} = E_y, \quad \gamma_{\underset{\vee}{30}} = E_z. \end{aligned} \tag{60}$$

Then in place of (56), (57), (58) and (59) we may set, in the usual notation of the three-dimensional vector analysis

$$-\frac{1}{c}\frac{\partial \overrightarrow{H}}{\partial t} = \nabla \times \overrightarrow{E}, \tag{61}$$

$$\nabla \cdot \overrightarrow{H} = 0. \tag{62}$$

Now, we take $0 = \frac{\partial}{\partial x^{\nu}}\gamma_{\underset{\vee}{\mu\nu}}$we obtain in place of (52)

$$\nabla \cdot \overrightarrow{E} = 0, \tag{63}$$

$$\nabla \times \overrightarrow{H} = \frac{1}{c}\frac{\partial \overrightarrow{E}}{\partial t}. \tag{64}$$

Therefore, we deduce the Maxwel's first system. Thus (52) and (53) are substantially the Maxwell's equations [31-34] of empty space.

Concluding remarks

A theory of non-symmetric fields has been presented, based on the infinitesimal displacement field. Subsequently, the actual curvature tensor was deduced and we defined the contracted curvature tensor. With this curvature tensor and the variational principle, we derive the field equations and Bianchi's identities. In consecuense, the field equations have been found from Bianchi's identities. Nevertheless, the unification of gravitation and electrodynamics has however remained an unfulfilled goal.

In addition, we have assumed the symmetry of $g_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$ to obtain the law of the pure gravitational field and $\sqrt{-g}g_{\mu\nu}$ do contain a skew-symmetric part if and only if there is an electromagnetic field. Thus (52) and (53) are substantially the Maxwell's equations.

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