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Hamiltonian formalism of the Bianchi's models

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Abstract
Lately the Cosmic Background Radiation (CMB) data have resulted in anomalies or deviations with respect to the standard model of cosmology, which has led several cosmologists to consider alternative models to the standard model (homogeneous and isotropic), such as the Bianchi models, which are homogeneous but anisotropic. Based on these motivations to consider alternative models, we propose to study, in the present work, the algebraic classification of the Bianchi models and each of the Bianchi space-times, applying the ADM formalism of general relativity in its Hamiltonian version and the groups G_3. The dynamic equations are shown with the help of the Hamiltonian density $\mathcal{H}$ and the Poisson parentheses, in other words, the equation of motion are presented for each of the Bianchi space-times. Some theoretical consequences of these equations are discussed when we take the limit $\Omega \to -\infty$ and the fixed parameters β_+ and β_-, consequently, we find that the dependent part of the gravitational potential from the Hamiltonian Density tends to zero and from the equations of motion we find the constant of motion, $p_\Omega = p_{\beta_+} = p_{\beta_-} = \text{constant}$.

Keywords: Cosmology, Bianchi's models, ADM formalism.

Formalismo Hamiltoniano de los modelos de Bianchi

Resumen
Últimamente los datos de la Radiación Cósmica de Fondo (CMB) han dado como resultado anomalías o desviaciones con respecto al modelo estándar de la cosmología, lo cual ha llevado a varios cosmólogos a considerar modelos alternativos al modelo estándar (homogeneo e isotrópico), como los modelos de Bianchi, los cuales son homogéneos pero anisotrópicos. Basándonos en estas motivaciones para considerar modelos alternativos, proponemos estudiar, en el presente trabajo, la clasificación algebraíca de los modelos de Bianchi y cada uno de los espacio-tiempo de Bianchi, aplicando el formalismo ADM de relatividad general en su versión Hamiltoniana y los grupos G_3. Se muestran las ecuaciones dinámicas con ayuda de la densidad Hamiltoniana $\mathcal{H}$ y los paréntesis de Poisson, en otras palabras, se presentan las ecuaciones de movimiento para cada uno de los espacio-tiempo de Bianchi. Se discuten algunas consecuencias de carácter teórico en dichas ecuaciones cuando tomamos el límite $\Omega \to -\infty$ y los parámetros β_+ y β_- fijos, en consecuencia, encontramos que la parte dependiente del potencial gravitacional de la densidad Hamiltoniana tiende a cero y de las ecuaciones de movimiento encontramos la constante de movimiento, $p_\Omega = p_{\beta_+} = p_{\beta_-} = \text{constante}$.

Palabras clave: Cosmología, modelos de Bianchi, formalismo ADM.

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1 Introduction

Cosmology is the branch of physics that studies the origin of the Universe on its largest scale. At first, it was known as mechanics of the celestial and it was the study of the heavens; there were different philosophical currents in ancient Greece, promoted by Aristarchus, Aristotle and Ptolemy, proposing different theories of what was observed. In particular, there was Ptolemy's geocentric theory in which the center of the entire known and unknown universe was the Earth, until Copernicus and many years later in the 16th century Kepler and Galileo Galilei proposed a heliocentric model. Later, in 1687, Newton extended the works of the latter, formulating the 3 laws of motion and the universal law of gravitation [1], with which was born modern cosmology, that is, the analytical cosmology.

In 1915 Albert Einstein, aided by the equivalence principle, the tensor calculus and Mach's law, published the field equations $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=-\kappa T_{\mu\nu}$, which describe the dynamics of the geometry of space-time [2]. Shortly after, various solutions to this equation were published, which are the structure of modern cosmology, where it is found that the dominant force under this assumption is the force of gravity. In addition to the above, modern cosmology assumes that the Universe, on large scales, is homogeneous and isotropic, which helped to more easily solve the field equations proposed by Einstein, because the metric is symmetric. This type of metric was developed by Alexander Friedmann and later worked by Howard Percy Robertson and Arthur Geoffrey Walker among others.

If we apply the general relativity [3–15] to cosmological models, then is investigated the past, present and future of the Universe. In addition, the modern theoretical cosmology, sticks to the so-called cosmological principle. This principle establishes that at large scales the Universe is homogeneous and isotropic, that is, there are no privileged positions or directions in the Universe. The assumption of isotropy and homogeneity of the universe helps to solve Einstein's equations [16, 17] more easily. The hypothesis of isotropy and homogeneity applied to general relativity opened the field of modern cosmology with the construction of models that accept exact solutions, which are known as models of Friedmann-Lemaître-Robertson-Walker (FLRW) [18–23].

This article is focused on Bianchi's models type A and B, which are especially homogeneous and anisotropic [24]; that is, there are privileged positions, but not privileged directions. The classification of this type of models was made by Luigi Bianchi in 1897 [25].

In section 2, we present the Friedmann-Lemaitre-Robertson-Walker model (FLRW). Starting from the FLRW metric and considering the energy-moment tensor for the Universe, when considering a perfect fluid, we can use the field equations of gravitation to find the Friedmann equations; which provide information on the dynamics of the behavior of the Universe. In the present work this section is presented the FLRW model, with the aim of noting that the FLRW models are particular cases of some of the Bianchi models. In section 3, we develop the formalism of the different cases of Bianchi cosmological models type A and B. These cosmological models will be analyzed without matter, cosmological constant and scalar potential. First, a general model for Bianchi's cosmological models will be described; where $\mathcal{L}_G$ is the Lagrangian geometric density. Once the geometric Lagrangian density $\mathcal{L}_G$ is found, we can find the Hamiltonian density (see appendix A). We will use Hamiltonian density $\mathcal{H}$ to develop the dynamics of Bianchi's cosmological models. Finally, we present a table with the structure constants that give an algebraic classification of each Bianchi's models. Therefore, the structure constants are of the utmost importance in this work since they are the ones that provide an algebraic classification of the Bianchi's models in accordance with group theory. From the Hamiltonian density, we study the dynamics of each of the Bianchi's models by calculating each of the Poisson brackets of each canonical variable, and through which it was possible to conclude that in the limit when $\Omega \to -\infty$ each Hamiltonian constraint could be interpreted with a time-dependent gravitational potential and when considering the equations of motion where the temporal derivatives of the canonical moments are found, we can obtain the conservation equation, $p_\Omega^2 = p_{\beta_+}^2 + p_{\beta_-}^2$.

2 FLRW model

The Schwarzschil metric

We will start by studying the Schwarzschild's line element, since it will be useful in FLRW models. Let us consider the Sun as a point mass and the gravitational field around it, we assume it to be statically and spherically symmetric. Consequently, in the coordinate system $x^\mu = \left(x^0, x^1, x^2, x^3\right) = (t, r, \theta, \phi)$ the metric tensor will only be a function of $x^1 = r$, that is, $g_{\mu\nu} = g_{\mu\nu}(r)$. Furthermore, as the radial coordinate tends to infinity, that is, when $r \to \infty$ and the metric tensor reduces to the metric Minkowski tensor $\eta_{\mu\nu}$, in other words, we obtain the Minkowski line element in spherical polar coordinates

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \qquad (1)$$

In general, when spacetime is not flat; that is, when space-time is curved, we consider the square of the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Applying the temporal isotropy in the line element for a curved space-time, in other words, the line element in a curved space-time would not change under the transformation $x^0 = t \to -t$, therefore, we can write the line element as:

$$ds^2 = g_{00}dt^2 - g_{ik}dx^i dx^k, \quad (2)$$

where $g_{01} = g_{02} = g_{03} = 0$ and $i, k = 1, 2, 3$. Too, applying isotropy at $x^2 = \theta$ y $x^3 = \phi$; that is, the line element ds does not change under the transformations $x^2 = \theta \to -\theta$ y $x^3 = \phi \to -\phi$, implying $g_{12} = g_{13} = g_{23} = 0$, therefore, equation (2) becomes the scalar equation:

$$ds^2 = g_{00}dt^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\phi^2. \quad (3)$$

When $r \to \infty$ equation (3) reduces to equation (1), therefore we write equation (3) in the form

$$ds^2 = A(r)\,dt^2 - B(r)\,dr^2 - C(r)\,r^2 d\theta^2 - D(r)\,r^2 \sin^2\theta d\phi^2. \quad (4)$$

Let us consider an angular change of direction by an angle α in two planes:

1. In a vertical plane, a change of direction by an angle $\alpha = d\theta$ of the z axis, is obtained, from equation (4), the result

$$ds_1^2 = -C(r)\,r^2\alpha^2; \quad (5)$$

1. In a horizontal plane (equatorial plane, $\theta = \pi/2$) by the same angle $\alpha = d\phi$ to obtain from the equation

$$ds_2^2 = -D(r)\,r^2\alpha^2. \quad (6)$$

The isotropy in three dimensions requires that the condition $ds_1 = ds_2$ is fulfilled, therefore from equations (5) and (6) we find that $C = D$. From the preceding considerations, equation (4) is transformed to the result

$$ds^2 = A(r)\,dt^2 - B(r)\,dr^2 - C(r)\,r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \quad (7)$$

Introducing a new coordinate by $r' = \sqrt{C(r)}r$. If we differentiate this new coordinate, we obtain

$$dr' = \left(\frac{1}{2\sqrt{C}}\frac{dC}{dr} + \sqrt{C}\right)dr,$$

of this ordinary differential the second term of equation (7) takes the form

$$B(r)\,dr^2 = B(r)\left(\frac{1}{2\sqrt{C}}\frac{dC}{dr} + \sqrt{C}\right)^{-2} dr'^2 = B'(r')\,dr'^2. \quad (8)$$

With the help of equations (7) and (9) we can rewrite the infinitesimal line element as:

$$ds^2 = A'(r')\,dt^2 - B'(r')\,dr'^2 - r'^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

Since $A'(r'), B'(r') > 0$, we can write the above equation as:

$$ds^2 = \exp[\nu(r)]\,dt^2 - \exp[\lambda(r)]\,dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \quad (9)$$

By using equation (9) in the field equations of gravitation in the vacuum and solving the system of differential equations we can rewrite ds^2 as follows

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

this is the famous Schwarzschild's line element [26]. It can be analyzed that this line element is reduced to the Minkowski's line element, that is, equation (1), when $r \to \infty$.

Deduction of the FLRW metric

Instead of the four coordinates for which the spatial isotropy of the universe is most evident, we will now choose different coordinates that are more convenient from the point of view of physical interpretation.

Since the temporal lines with respect to the coordinates x_1, x_2 and x_3 are constant and x_0 variable, we choose the geodesics of the particle that in the form of central symmetry are straight lines that pass through the center, similarly to how the space-time decomposition is done in ADM formalism. Also let x_0 be the metric distance to the center. In such a coordinate system the metric is of the form:

$$ds^2 = (dx_0)^2 - d\sigma^2 = (dx_0)^2 - g_{ik}dx^i dx^k, \quad (10)$$

where $d\sigma^2$ is the metric on one of the hypersurfaces and $i, k = 1, 2, 3$.

The elements of the spatial metric tensor g_{ik} that belong to different hypersurfaces will then be in the same way on all hypersurfaces with the only difference that there will be a positive factor; called scale factor, which depends on x_0:

$$g_{ik} = \gamma_{ik}a^2, \quad (11)$$

where the components of γ_{ik} depend on x_1, x_2 and x_3 only, and a is a function of x_0. Therefore, introducing equation (11) on the right hand side of equation (10) gives

$$d\sigma^2 = a^2\gamma_{ik}dx^i dx^k = a^2\left(d\sigma'\right)^2. \quad (12)$$

Using the Schwarzschild line element; that is, equation (9), it follows that the line element in parentheses on the right side of equation (12) takes the form:

$$d\sigma'^2 = \gamma_{ik}dx^i dx^k = e^{\lambda}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \quad (13)$$

On the other hand, the first non-zero component of the Ricci tensor for the metric of equation (13) is

$$R_{11} = \frac{1}{r}\frac{d\lambda}{dr}, \tag{14}$$

furthermore R_{22} and R_{33} are given by

$$R_{22} = \csc^2\theta R_{33} = 1 + \frac{1}{2} r e^{-\lambda}\frac{d\lambda}{dr} - e^{-\lambda}. \tag{15}$$

Regarding Gaussian curvature [27], mathematically a space of constant curvature is characterized by the equation

$$R_{\lambda\mu\nu\kappa} = k\left(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu}\right). \tag{16}$$

The spaces with constant curvature are qualitatively different depending on whether the curvature is positive, negative, or zero. In the case of a three-dimensional space, equation (16) is written as

$$R_{ijkl} = k\left(g_{ik}g_{jl} - g_{il}g_{jk}\right).$$

Contracting the previous equation with g^{ik}, we obtain

$$R_{jl} = g^{ik}R_{ijkl} = 2kg_{jl}. \tag{17}$$

Using the components of the Ricci tensor; that is, using equations (14) and (15) and the line element of equation (13), from equation (17) we obtain the ordinary differential equations

$$\frac{1}{r}\frac{d\lambda}{dr} = 2k\exp\left(-\lambda\right),$$
$$1 + \frac{1}{2} r\exp\left(-\lambda\right)\frac{d\lambda}{dr} - \exp\left(-\lambda\right) = 2kr^2.$$

The solution of the system of ordinary differential equations above is given by the analytical equation

$$\exp\left(-\lambda\right) = 1 - kr. \tag{18}$$

The homogeneity and isotropy imposed on space-time make admissible the three types of geometries for space described in the FLRW model metric and are classified as open universe if $k = -1$ (ie, hyperbolic space), flat if $k = 0$ (ie, Euclidean space) or closed if $k = 1$ (ie, spherical space). After insert the solution of equation (18) in equation (13), then the resulting equation is introduced in equation (12) and with this result finally substituting it in equation (10), we obtain the FLRW metric:

$$ds^2 = (dt)^2 - [a\left(t\right)]^2\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right], \tag{19}$$

where k describes the curvature and is constant in time and $a\left(t\right)$ is the scale factor; which is time dependent and can be interpreted as the radius or size of the universe. Obviously, once k and $a\left(t\right)$ are specified the spacetime metric is completely determined.

Geometrically, as shown below, $a\left(t\right)$ can be seen as the radius of the universe, since the hypersurfaces considered below represent the three types of possible Universes according to the FLRW metric, consequently, this describes the dynamical properties of the different homogeneous and isotropic universes. Physically, a very useful quantity to define the scale factor is the Hubble parameter (sometimes called the Hubble constant), given by

$$H\left(t\right) = \frac{1}{a}\frac{da}{dt}.$$

The Hubble parameter refers to how fast most distant galaxies are receding from us via Hubble's law [28],$v = Hd$. This is the relationship that was discovered by Edwin Hubble, and has been verified with great accuracy by modern methods of observation.

The FLRW metric can also be determined from the geometry of three-dimensional spaces of constant curvature. Therefore, consider the Cartesian equation of a spherical hypersurface

$$x^2 + y^2 + z^2 + w^2 = a^2.$$

The infinitesimal distance (line element) in this case would be:

$$d\sigma^2 = dx^2 + dy^2 + dz^2 + dw^2. \tag{20}$$

Let us consider the following transformations in a four-dimensional Euclidean space with the coordinates (x, y, z, w):

$$\begin{gathered} w = a\cos\psi, \\ x = a\sin\psi\cos\theta, \\ y = a\sin\psi\sin\theta\cos\phi, \\ z = a\sin\psi\sin\theta\sin\phi. \end{gathered} \tag{21}$$

Differentiating equations (21), substituting the total differentials in equation (20) and after making the necessary simplifications we obtain:

$$d\sigma^2 = a^2\left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]. \tag{22}$$

Taking the radial transformation $\sin\psi = r$; therefore, the total differential is $dr = \cos\psi d\psi$, from which the mathematical expression $d\psi^2 = \left(1-r^2\right)^{-1}dr^2$, is obtained, and consequently the line element of equation (22) is determined by the equation:

$$d\sigma^2 = a^2\left[\frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]. \tag{23}$$

With the equation (23), we write the metric of the three-dimensional homogeneous spherical surface in the form:

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]. \quad (24)$$

Similarly, if we consider a homogeneous surface of negative curvature with the infinitesimal line element $d\sigma = \sqrt{-dw^2 + dx^2 + dy^2 + dz^2}$, we obtain

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1+r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \quad (25)$$

and by considering a homogeneous surface of null curvature with an infinitesimal line element $d\sigma = \sqrt{dx^2 + dy^2 + dz^2}$, we have

$$ds^2 = dt^2 - a^2(t)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]. \quad (26)$$

If we confine equations (24), (25) and (26) we obtain the FLRW metric expressed in equation (19), where evidently $k = -1, 0, 1$.

Friedmann equations

Suppose now that the Universe is filled with an ideal fluid; frictionless adiabatic fluid, that is, fluid characterized by the fact that in a local coordinate system of a fluid element there is only one isotropic pressure. Therefore, the energy-moment tensor for a Universe of this type according to the theory of general relativity can be represented by:

$$T^{\mu\nu} = (\rho + p)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - pg^{\mu\nu}. \quad (27)$$

The tensor of equation (27) can be obtained from the consideration of a frame in free fall, in which the perfect fluid is at rest in a small neighborhood. In this framework, the metric tensor would be $g_{\mu\nu} = \eta_{\mu\nu}$, the four-speed is given by $\frac{dx^\mu}{ds} = (1,0,0,0)$ and the moment energy tensor is determined by

$$T^{\alpha\beta} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}.$$

In some general coordinate frame, the energy-moment tensor is determined by its transformation law, that is, $T^{\mu\nu} = \frac{\partial x^\mu}{\partial \xi^\alpha}\frac{\partial x^\nu}{\partial \xi^\beta}T^{\alpha\beta}$. Using the transformation law of the metric tensor $g^{\mu\nu} = \frac{\partial x^\mu}{\partial \xi^\alpha}\frac{\partial x^\nu}{\partial \xi^\beta}\eta^{\alpha\beta}$ and the quadri-velocity in the frame of free fall; where $\eta^{\alpha\beta}$ is the Minkowski's metric tensor, from the transformation law of the energy-moment tensor, we obtain the equation (27).

Making use of the FLRW metric; that is, making use of equation (19), and introducing equation (27) in the field equations $R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -8\pi G T^{\mu\nu}$, we obtain the Friedmann equations [19]:

$$\begin{aligned} \tfrac{2}{a}\tfrac{d^2a}{dt^2} + \tfrac{1}{a^2}\left(\tfrac{da}{dt}\right)^2 + \tfrac{k}{a^2} &= -8\pi G p, \\ 3\left[\tfrac{1}{a^2}\left(\tfrac{da}{dt}\right)^2 + \tfrac{k}{a^2}\right] &= 8\pi G\rho, \end{aligned} \quad (28)$$

where the first equation of (28) corresponds to $G^{ii} = R^{ii} - \frac{1}{2}g^{ii}R = -8\pi T^{ii}$ with $i = 1,2,3$ and the second of the previous equations corresponds to the 0-0 component; that is, $R^{00} - \frac{1}{2}g^{00}R = -8\pi G T^{00}$. The above equations provide information on how the universe behaves as an ideal fluid.

3 Bianchi's cosmological models

In this section, we develop the formalism of the different cases of the Bianchi's cosmological models, i. e., type A and B. Bianchi's cosmological models will be analyzed without matter, cosmological constant and scalar potential. First, a general model for the Bianchi's models will be described; where is the geometric Lagrangian density $\mathcal{L}_G$. Once the geometric Lagrangian density $\mathcal{L}_G$ is found, the Hamiltonian density is developed. Finally, the Hamiltonian density $\mathcal{H}$ will be used to develop the dynamics of the Bianchi's cosmological models.

The homogeneity and isotropy of the cosmological models are directly related to the intrinsic symmetries of the manifold; which in simple terms and locally looks like a piece of the Euclidean space $\mathbb{R}^n$ of n dimensions. A very viable way to classify the different cosmological models is by their symmetries. Symmetries or isometries in spacetime are transformations that leave the metric tensor, the physical and geometric properties invariant. The fields that generate these symmetries are called Killing's vector fields. These fields are defined in a Riemannian manifold, they are differentiable, and they have a differentiable and symmetric metric tensor. The Killing's vector fields are defined by means of the Lie derivative of the metric tensor equivalent to the nullity in some direction given by a Killing's field [29], in mathematical terms these fields comply with the Killing's equation:

$$\mathfrak{L}_X g_{\mu\nu} = 0. \quad (29)$$

The Bianchi's cosmological models are homogeneous, therefore, they have Killing's vectors associated with this symmetry. However, given the properties of the Lie's derivative, the Killing's vectors have the property:

$$[X_\mu, X_\nu] = C^\lambda_{\mu\nu} X_\lambda,$$

where $C^\lambda_{\mu\nu}$ are the structure constants (appendix B). Bianchi's models are classified according to the type of structure that characterizes them [30, 31].

3.1 General model

In Misner's notation, the metric of the Bianchi's models can be written as [5]

$$ds^2 = -N^2 dt^2 + e^{2\Omega(t)} e^{2\beta_{ij}(t)} \omega^i \omega^j, \qquad (30)$$

where $N(t)$ is the lapse function, ω^i are called the differential 1-forms, $e^{2\Omega(t)}$ is the scale factor of the universe and β_{ij} determines the anisotropic parameters $\beta_+(t)$ and $\beta_-(t)$ as follows

$$\beta_{ij} = \begin{pmatrix} \beta_+ + \sqrt{3}\beta_- & 0 & 0 \\ 0 & \beta_+ - \sqrt{3}\beta_- & 0 \\ 0 & 0 & -2\beta_+ \end{pmatrix}. \qquad (31)$$

In this general model of the Bianchi's models, the shift function is not stipulated in the metric of equation (30), consequently in the later developments for the Bianchi's cosmological models that will not appear as variable dynamics. Taking into account the multiplicand $h_{ij} = e^{2\Omega(t)} e^{\beta_{ij}(t)}$ of the second term of equation (30) and when comparing it with g_{ab} of the ADM formalism of general relativity (see appendix A), we can intuit that the dynamic variables for the Bianchi's models here will be Ω, β_+, β_-, since the lapse function it will set with the value $N = 1$; which is the physical norm.

Setting the lapse function equivalent to unity is necessary for the geometric Lagragian density $\mathcal{L}_G$ to coincide with the field equations of gravitation in vacuum and to be able to use the Hamiltonian density, From $\mathcal{H}$, we can extract the dynamics of the model.

The non-zero components of extrinsic curvature; using equations (30) and (31) and equation (147), they are given by:

$$\begin{aligned} K_{11} &= \tfrac{1}{N}\left(\tfrac{d\Omega}{dt} + \tfrac{d\beta_+}{dt} + \sqrt{3}\tfrac{d\beta_-}{dt}\right)\exp\left[2\left(\Omega + \beta_+ + \sqrt{3}\beta_-\right)\right], \\ K_{22} &= \tfrac{1}{N}\left(\tfrac{d\Omega}{dt} + \tfrac{d\beta_+}{dt} - \sqrt{3}\tfrac{d\beta_-}{dt}\right)\exp\left[2\left(\Omega + \beta_+ - \sqrt{3}\beta_-\right)\right], \\ K_{33} &= \tfrac{1}{N}\left(\tfrac{d\Omega}{dt} - 2\tfrac{d\beta_+}{dt}\right)\exp\left[2\left(\Omega - 2\beta_+\right)\right]. \end{aligned} \qquad (32)$$

The trace of extrinsic curvature; that is, the equation $K = h^{ij}K_{ij}$ is given by

$$K = -\frac{3}{N}\frac{d\Omega}{dt}. \qquad (33)$$

Taking into account the calculation

$$\sqrt{\det(h_{ij})} = \sqrt{\exp(6\Omega)\exp\left(\beta_+ + \sqrt{3}\beta_-\right)\exp\left(\beta_+ - \sqrt{3}\beta_-\right)\exp(-2\beta_+)} = \exp[3\Omega(t)],$$

and inserting equations (32) and (33) in equation (151), we can ensure that the Lagrangian density is expressed by

$$\mathcal{L}_G = \frac{6\exp(3\Omega)}{N}\left[-\left(\frac{d\Omega}{dt}\right)^2 + \left(\frac{d\beta_+}{dt}\right)^2 + \left(\frac{d\beta_-}{dt}\right)^2\right] + N\exp(3\Omega)^{(3)}R. \qquad (34)$$

The conjugate moments for the dynamic variables Ω, β_+, β_- are given by

$$\begin{aligned} p_\Omega &= \tfrac{\partial\mathcal{L}_G}{\partial\dot{\Omega}} = -\tfrac{12}{N}\tfrac{d\Omega}{dt}\exp(3\Omega), \\ p_{\beta_+} &= \tfrac{\partial\mathcal{L}_G}{\partial\dot{\beta}_+} = \tfrac{12}{N}\tfrac{d\beta_+}{dt}\exp(3\Omega), \\ p_{\beta_-} &= \tfrac{\partial\mathcal{L}_G}{\partial\dot{\beta}_-} = \tfrac{12}{N}\tfrac{d\beta_-}{dt}\exp(3\Omega). \end{aligned} \qquad (35)$$

Using the Legendre's transformation [32,33], equation (34) and equations (35); we can notice that the Hamiltonian density can be calculated from the equation

$$\mathcal{H} = p_\Omega\frac{d\Omega}{dt} + p_{\beta_+}\frac{d\beta_+}{dt} + p_{\beta_-}\frac{d\beta_-}{dt} - \mathcal{L}_G,$$

resulting

$$\mathcal{H} = \frac{N}{24}\exp(-3\Omega)\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) - N\exp(3\Omega)^{(3)}R, \qquad (36)$$

where the three-dimensional curvature scalar is given by [?]

$$^{(3)}R = C^i_{jk}C^l_{mn}h_{il}h^{km}h^{jn} + 2C^i_{jk}C^k_{li}h^{jl} + 4C^i_{ik}C^j_{jm}h^{km}, \qquad (37)$$

where C^i_{jk} the structure constants and $h_{ij} = e^{2\Omega(t)}e^{\beta_{ij}(t)}$. The third term of equation (37) is not taken into account in the Bianchi's models belonging to class A; that is, class A of the Bianchi's models have structure constants $C^i_{ik} = 0$, therefore, the third term will only be used in class B.

Equation (36) constitutes a Hamiltonian constraint in the ADM formalism of general relativity. Therefore, $\mathcal{H} \approx 0$ must be satisfied to reproduce Einstein's field equations. In equation (30), that is, the general metric for the Bianchi's cosmological models does not appear the shift function N^a, therefore, the equation $\mathfrak{L}_N h_{ab}\pi^{ab} = -2h_{ac}N^c D_b\pi^{ab} = N^c\mathcal{H}_c$ will not be considered, therefore will have not generating constraints of difeomorphism for the Bianchi's models.

The classical Poisson brackets for the dynamic variables considered are

$$\begin{aligned} \{x_i, x_j\} &= 0, \\ \{p_i, p_j\} &= 0, \\ \{x_i, p_j\} &= \delta_{ij}, \end{aligned} \tag{38}$$

where $x_i = \Omega, \beta_+, \beta_-$ and $p_i = p_\Omega, p_{\beta_+}, p_{\beta_-}$ with $i = 1, 2, 3$.

Next, the formalism of the Bianchi's models of class A and B is developed [38].

3.2 Class A

Bianchi I

This Bianchi model is characterized by the differential 1-forms

$$\begin{aligned} \omega^1 &= dx, \\ \omega^2 &= dy, \\ \omega^3 &= dz. \end{aligned}$$

The constants of the Bianchi I are null, that is, $C^i_{jk} = 0$ [31]; so it is the simplest model. Therefore, from equation (36) and using equation (37), the Hamiltonian density is expressed by the equation

$$\mathcal{H}_I = \frac{\exp(-3\Omega)}{24}\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right), \tag{39}$$

where $N = 1$. From equation (39) we can find the equations of motion

$$\frac{d\Omega}{dt} = \{\Omega, \mathcal{H}_I\} = \frac{\partial \Omega}{\partial \Omega}\frac{\partial \mathcal{H}_I}{\partial p_\Omega} - \frac{\partial \mathcal{H}_I}{\partial \Omega}\frac{\partial \Omega}{\partial p_\Omega} = -\frac{\exp(-3\Omega)}{12} p_\Omega, \tag{40}$$

$$\frac{d\beta_+}{dt} = \{\beta_+, \mathcal{H}_I\} = \frac{\partial \beta_+}{\partial \beta_+}\frac{\partial \mathcal{H}_I}{\partial p_{\beta_+}} - \frac{\partial \mathcal{H}_I}{\partial \beta_+}\frac{\partial \beta_+}{\partial p_{\beta_+}} = \frac{\exp(-3\Omega)}{12} p_{\beta_+}, \tag{41}$$

$$\frac{d\beta_-}{dt} = \{\beta_-, \mathcal{H}_I\} = \frac{\partial \beta_-}{\partial \beta_-}\frac{\partial \mathcal{H}_I}{\partial p_{\beta_-}} - \frac{\partial \mathcal{H}_I}{\partial \beta_-}\frac{\partial \beta_-}{\partial p_{\beta_-}} = \frac{\exp(-3\Omega)}{12} p_{\beta_-}, \tag{42}$$

$$\frac{dp_\Omega}{dt} = \{p_\Omega, \mathcal{H}_I\} = \frac{\partial p_\Omega}{\partial \Omega}\frac{\partial \mathcal{H}_I}{\partial p_\Omega} - \frac{\partial \mathcal{H}_I}{\partial \Omega}\frac{\partial p_\Omega}{\partial p_\Omega} = \frac{\exp(-3\Omega)}{8}\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right), \tag{43}$$

$$\frac{dp_{\beta_+}}{dt} = \{p_{\beta_+}, \mathcal{H}_I\} = \frac{\partial p_{\beta_+}}{\partial \beta_+}\frac{\partial \mathcal{H}_I}{\partial p_{\beta_+}} - \frac{\partial \mathcal{H}_I}{\partial \beta_+}\frac{\partial p_{\beta_+}}{\partial p_{\beta_+}} = 0, \tag{44}$$

$$\frac{dp_{\beta_-}}{dt} = \{p_{\beta_-}, \mathcal{H}_I\} = \frac{\partial p_{\beta_-}}{\partial \beta_-}\frac{\partial \mathcal{H}_I}{\partial p_{\beta_-}} - \frac{\partial \mathcal{H}_I}{\partial \beta_-}\frac{\partial p_{\beta_-}}{\partial p_{\beta_-}} = 0. \tag{45}$$

Using the fact that equation (39) is a constraint, then we solve for p_Ω^2 from the Hamiltonian density in question; that is, we have the equation $p_\Omega^2 = p_{\beta_+}^2 + p_{\beta_-}^2$, and introduce it into equation (43) and finally integrating the ordinary differential equations (44) and (45), we obtains

$$\begin{aligned} p_\Omega &= p_{0\Omega} = constante, \\ p_{\beta_+} &= p_{0\beta_+} = constante, \\ p_{\beta_-} &= p_{0\beta_-} = constante. \end{aligned} \tag{46}$$

If we insert equations (46) into equations (40), (41) and (42) and then integrate in the time the differential equations in time, we obtain the solutions to the dynamic variables for this cosmological model:

$$\begin{aligned} \Omega(t) &= \tfrac{1}{3}\ln\left(-\tfrac{1}{4}\sqrt{p_{0\beta_+}^2 + p_{0\beta_-}^2}\,t + 3\Omega_0\right), \\ \beta_+(t) &= -\tfrac{1}{3}\frac{p_{0\beta_+}}{\sqrt{p_{0\beta_+}^2 + p_{0\beta_-}^2}}\ln\left(-\tfrac{1}{4}\sqrt{p_{0\beta_+}^2 + p_{0\beta_-}^2}\,t + 3\Omega_0\right) + C_1, \\ \beta_-(t) &= -\tfrac{1}{3}\frac{p_{0\beta_-}}{\sqrt{p_{0\beta_+}^2 + p_{0\beta_-}^2}}\ln\left(-\tfrac{1}{4}\sqrt{p_{0\beta_+}^2 + p_{0\beta_-}^2}\,t + 3\Omega_0\right) + C_2, \end{aligned} \tag{47}$$

where C_1 y C_2 are constants of integration.

Bianchi II

This Bianchi's model is characterized by the differential 1-forms

$$\begin{aligned} \omega^1 &= dx - zdy, \\ \omega^2 &= dy, \\ \omega^3 &= dz. \end{aligned}$$

The constants of the Bianchi II are [31]

$$C^1_{23} = -C^1_{32} = 1.$$

Using the structure constants and equation (37), the curvature scalar is determined by

$$^{(3)}R_{II} = -2\exp\left(-2\Omega + 4\beta_+ + 4\sqrt{3}\beta_-\right). \tag{48}$$

Introducing equation (48) into equation (36), the Hamiltonian density for the Bianchi II is determined by the equation

$$\mathcal{H}_{II} = \frac{\exp(-3\Omega)}{24}\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) + 2\exp\left(\Omega + 4\beta_+ + 4\sqrt{3}\beta_-\right), \tag{49}$$

where $N = 1$, this will be done in the next models.

From equation (49), we can obtain the equations of motion

$$\frac{d\Omega}{dt} = \{\Omega, \mathcal{H}_{II}\} = \frac{\partial \mathcal{H}_{II}}{\partial p_\Omega} = -\frac{\exp(-3\Omega)}{12} p_\Omega, \tag{50}$$

$$\frac{d\beta_+}{dt} = \{\beta_+, \mathcal{H}_{II}\} = \frac{\partial \mathcal{H}_{II}}{\partial p_{\beta_+}} = \frac{\exp(-3\Omega)}{12} p_{\beta_+}, \quad (51)$$

$$\frac{d\beta_-}{dt} = \{\beta_-, \mathcal{H}_{II}\} = \frac{\partial \mathcal{H}_{II}}{\partial p_{\beta_-}} = \frac{\exp(-3\Omega)}{12} p_{\beta_-}, \quad (52)$$

$$\frac{dp_\Omega}{dt} = \{p_\Omega, \mathcal{H}_{II}\} = -\frac{\partial \mathcal{H}_{II}}{\partial \Omega} = \frac{\exp(-3\Omega)}{8} \left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) - 2\exp\left(\Omega + 4\beta_+ + 4\sqrt{3}\beta_-\right), \quad (53)$$

$$\frac{dp_{\beta_+}}{dt} = \{p_{\beta_+}, \mathcal{H}_{II}\} = -\frac{\partial \mathcal{H}_{II}}{\partial \beta_+} = -8\exp\left(\Omega + 4\beta_+ + 4\sqrt{3}\beta_-\right), \quad (54)$$

$$\frac{dp_{\beta_-}}{dt} = \{p_{\beta_-}, \mathcal{H}_{II}\} = -\frac{\partial \mathcal{H}_{II}}{\partial \beta_-} = -8\sqrt{3}\exp\left(\Omega + 4\beta_+ + 4\sqrt{3}\beta_-\right). \quad (55)$$

Using the fact that equation (49) is a constraint, then we clear p_Ω^2 from the Hamiltonian density in question and substitute it into equation (45), we obtain the differential equation

$$\frac{dp_\Omega}{dt} = -8\exp\left(\Omega + 4\beta_+ + 4\sqrt{3}\beta_-\right). \quad (56)$$

By virtue of the Hamiltonian constraint $\mathcal{H}_{II} \approx 0$, the dynamics of the Bianchi II is considered below; according to the second term of equation (49). Assuming fixed anisotropic parameters β_+ and β_-, consequently, the last term of equation (49) containing $2\exp\left(\Omega + 4\beta_+ + 4\sqrt{3}\beta_-\right) \to 0$ as $\Omega \to -\infty$. From the preceding consideration and by virtue of equations (54), (55), and (56) taking into account that as $\Omega \to -\infty$, we found $p_\Omega = p_{\beta_+} = p_{\beta_-} = constant$ and $p_\Omega^2 = p_{\beta_+}^2 + p_{\beta_-}^2$.

Bianchis VI_0 y VII_0

These models have their 1-differential forms expressed in the form

$$\begin{aligned} \omega^1 &= \cosh z dx \mp \sinh z dy, \\ \omega^2 &= -\sinh z dx + \cosh z dy, \\ \omega^3 &= dz, \end{aligned}$$

where in the first of the previous equations the sign above indicates the model VI_0 and the sign below the Bianchi VII_0 model, respectively. The type VI_0 of the Bianchi's models has the structure constants [34]

$$C^1_{23} = -C^1_{32} = 1, \quad C^2_{31} = -C^2_{13} = -1.$$

The Bianchi VII_0 have structure constants given by [31, 34]:

$$C^1_{23} = -C^1_{32} = -1, \quad C^2_{31} = -C^2_{13} = -1.$$

With the structure constants and using equation (37), the curvature scalar is

$$^{(3)}R^{VI_0}_{VII_0} = -4\exp(-2\Omega + 4\beta_+)\left[\cosh\left(4\sqrt{3}\beta_-\right) \pm 1\right], \quad (57)$$

where the sign above indicates the Bianchi VI_0 and the sign below the Bianchi VII_0; this will be the case in the development of these two models. If we use equation (57), equation (36) becomes

$$\mathcal{H}^{VI_0}_{VII_0} = \frac{\exp(-3\Omega)}{24}\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) + 4\exp(\Omega + 4\beta_+)\left[\cosh\left(4\sqrt{3}\beta_-\right) \pm 1\right]. \quad (58)$$

With these two Hamiltonian densities; that is, equations (58), we can write the equations of motion

$$\frac{d\Omega}{dt} = \left\{\Omega, \mathcal{H}^{VI_0}_{VII_0}\right\} = \frac{\partial \mathcal{H}^{VI_0}_{VII_0}}{\partial p_\Omega} = -\frac{\exp(-3\Omega)}{12} p_\Omega, \quad (59)$$

$$\frac{d\beta_+}{dt} = \left\{\beta_+, \mathcal{H}^{VI_0}_{VII_0}\right\} = \frac{\partial \mathcal{H}^{VI_0}_{VII_0}}{\partial p_{\beta_+}} = \frac{\exp(-3\Omega)}{12} p_{\beta_+}, \quad (60)$$

$$\frac{d\beta_-}{dt} = \left\{\beta_-, \mathcal{H}^{VI_0}_{VII_0}\right\} = \frac{\partial \mathcal{H}^{VI_0}_{VII_0}}{\partial p_{\beta_-}} = \frac{\exp(-3\Omega)}{12} p_{\beta_-}, \quad (61)$$

$$\frac{dp_\Omega}{dt} = \left\{p_\Omega, \mathcal{H}^{VI_0}_{VII_0}\right\} = \frac{\exp(-3\Omega)}{8}\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) - 4\exp(\Omega + 4\beta_+)\left[\cosh\left(4\sqrt{3}\beta_+\right) \pm 1\right], \quad (62)$$

$$\frac{dp_{\beta_+}}{dt} = \left\{p_{\beta_+}, \mathcal{H}^{VI_0}_{VII_0}\right\} = -\frac{\partial \mathcal{H}^{VI_0}_{VII_0}}{\partial \beta_+} = 16\exp(\Omega + 4\beta_+)\left[\cosh\left(4\sqrt{3}\beta_+\right) \pm 1\right], \quad (63)$$

$$\frac{dp_{\beta_-}}{dt} = \left\{p_{\beta_-}, \mathcal{H}_{VII_0}^{VI_0}\right\} = -\frac{\partial \mathcal{H}_{VII_0}^{VI_0}}{\partial \beta_-} = 16\sqrt{3}\exp\left(\Omega + 4\beta_+\right)\sinh\left(4\sqrt{3}\beta_-\right). \tag{64}$$

Taking equation (58) and inserting it into equation (62) we obtain the equation of motion in terms of the dynamic variables Ω, β_+, β_-

$$\frac{dp_\Omega}{dt} = -\frac{\partial \mathcal{H}_{VII_0}^{VI_0}}{\partial \Omega} = -16\exp\left[4\left(\Omega + \beta_+\right)\right]\left[\cosh\left(4\sqrt{3}\beta_+\right) \pm 1\right]. \tag{65}$$

By virtue of the Hamiltonian constraint $\mathcal{H}_{VII_0}^{VI_0} \approx 0$, the dynamics of the Bianchi cosmological models VI_0 and VII_0 are shown below according to the second term of equation (58). Assuming the fixed anisotropic parameters β_+ y β_-, consequently, the last term of equation (58) tends to 0, as $\Omega \to -\infty$, where it turns out that each conjugate moment is constant and $p_\Omega^2 = p_{\beta_+}^2 + p_{\beta_-}^2$. By virtue of equations (63), (64) and (65) tend to zero as $\Omega \to -\infty$ and therefore $p_\Omega = p_{\beta_+} = p_{\beta_-} = constant$.

Bianchi VIII

In the Bianchi VIII the 1-differential forms are given by [39]

$$\begin{aligned}\omega^1 &= \cosh y \cos z dx - \sin z dy,\\ \omega^2 &= \cosh y \sin z dx + \cos z dy,\\ \omega^3 &= \sinh y dx + dz.\end{aligned}$$

For this cosmological model, the structure constants are [31,34]

$$\begin{aligned}C_{23}^1 &= -C_{32}^1 = -1,\\ C_{31}^2 &= -C_{13}^2 = -1,\\ C_{12}^3 &= -C_{21}^3 = 1.\end{aligned}$$

Using these structure constants and inserting them into equation (37), the curvature scalar is given by the scalar equation

$$\begin{aligned}{}^{(3)}R_{VIII} = &-4\exp\left(-2\Omega + 4\beta_+\right)\cosh\left(4\sqrt{3}\beta_+\right) - 2\exp\left(-2\Omega - 8\beta_+\right) - 4\exp\left(-2\Omega + 4\beta_+\right)\\ &+8\exp\left(-2\Omega - 2\beta_+\right)\cosh\left(2\sqrt{3}\beta_-\right),\end{aligned} \tag{66}$$

and, therefore, if we use equation (66) to substitute it in equation (36), the Hamiltonian density turns out to be

$$\mathcal{H}_{VIII} = \frac{\exp\left(-3\Omega\right)}{24}\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) + \exp\left(\Omega\right)\left[W\left(\beta_+, \beta_-\right) - 1\right], \tag{67}$$

with

$$W\left(\beta_+, \beta_-\right) = 1 + 4e^{4\beta_+}\cosh\left(4\sqrt{3}\beta_+\right) + 2e^{-8\beta_+} - 8e^{-2\beta_+}\cosh\left(2\sqrt{3}\beta_-\right) + 4e^{4\beta_+}.$$

From equation (67), we find the equations of motion:

$$\frac{d\Omega}{dt} = \left\{\Omega, \mathcal{H}_{VIII}\right\} = \frac{\partial \mathcal{H}_{VIII}}{\partial p_\Omega} = -\frac{\exp\left(-3\Omega\right)}{12}p_\Omega, \tag{68}$$

$$\frac{d\beta_+}{dt} = \left\{\beta_+, \mathcal{H}_{VIII}\right\} = \frac{\partial \mathcal{H}_{VIII}}{\partial p_{\beta_+}} = \frac{\exp\left(-3\Omega\right)}{12}p_{\beta_+}, \tag{69}$$

$$\frac{d\beta_-}{dt} = \left\{\beta_-, \mathcal{H}_{VIII}\right\} = \frac{\partial \mathcal{H}_{VIII}}{\partial p_{\beta_-}} = \frac{\exp\left(-3\Omega\right)}{12}p_{\beta_-}, \tag{70}$$

$$\begin{aligned}\frac{dp_\Omega}{dt} = \left\{p_\Omega, \mathcal{H}_{VIII}\right\} = &\frac{\exp\left(-3\Omega\right)}{8}\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right)\\ &-\exp\left(\Omega\right)\left[W\left(\beta_+, \beta_-\right) - 1\right],\end{aligned} \tag{71}$$

$$\frac{dp_{\beta_+}}{dt} = \left\{p_{\beta_+}, \mathcal{H}_{VIII}\right\} = -\frac{\partial \mathcal{H}_{VIII}}{\partial \beta_+} = -\exp\left(\Omega\right)\frac{\partial W}{\partial \beta_+}, \tag{72}$$

$$\frac{dp_{\beta_-}}{dt} = \left\{p_{\beta_-}, \mathcal{H}_{VIII}\right\} = -\frac{\partial \mathcal{H}_{VIII}}{\partial \beta_-} = -\exp\left(\Omega\right)\frac{\partial W}{\partial \beta_-}. \tag{73}$$

Using equation (67) and inserting it into equation (71), we obtain the differential equation

$$\frac{dp_\Omega}{dt} = -4\exp\left(\Omega\right)\left[W\left(\beta_+, \beta_-\right) - 1\right]. \tag{74}$$

With the Hamiltonian constraint $\mathcal{H}_{VIII} \approx 0$, the dynamics of the Bianchi VIII can be seen as the dynamics of a particle at a time-dependent potential. The simplest motions are obtained by assuming the fixed anisotropic parameters β_+ and β_-, consequently, the last term of equation (67) containing $W(\beta_+, \beta_-)$ tends to zero at the limit $\Omega \to -\infty$. From the preceding consideration and equations (72), (73) and (74), we obtains $p_\Omega = p_{\beta_+} = p_{\beta_-} = constant$ and $p_\Omega^2 = p_{\beta_+}^2 + p_{\beta_-}^2$.

For large values of β of $W(\beta_+, \beta_-)$, it can be found that in the limit $\beta_+ \to -\infty$ the value of $W(\beta_+, \beta_-)$, from equation (67), behaves as

$$W(\beta_+ \to -\infty, \beta_-) \sim 2\exp(-8\beta_+) - 8\exp(-2\beta_+) \times \cosh\left(2\sqrt{3}\beta_-\right),$$

and for the limit $\beta \to +\infty$ taking into account $\beta_- \ll 1$, the anisotropic potential behaves in the way

$$W(\beta_+ \to +\infty, \beta_-) \sim 1 + 4\left(2 + 24\beta_-^2\right)\exp(4\beta_+).$$

Bianchi IX

This cosmological model have the 1-differential forms expressed by [39]:

$$\begin{aligned} \omega_1 &= \cos z \sin y dx - \sin z dy \\ \omega_2 &= \sin z \sin y dx + \cos z dy \\ \omega_3 &= \cos y dx + dz. \end{aligned}$$

This cosmological model has the following structure constants [31, 34]

$$\begin{aligned} C_{23}^1 &= -C_{32}^1 = 1, \\ C_{31}^2 &= -C_{13}^2 = 1, \\ C_{12}^3 &= -C_{21}^3 = 1. \end{aligned}$$

If we substitute these structure constants in equation (37), we obtain the three-dimensional curvature scalar

$$^{(3)}R_{IX} = -2\exp(-2\Omega - 8\beta_+) + 8\exp(-2\Omega - 2\beta_+) \times \cosh\left(2\sqrt{3}\beta_-\right) - 4\exp(-2\Omega + 4\beta_+)\left[\cosh\left(4\sqrt{3}\beta_+\right) + 1\right]. \quad (75)$$

and then equation (77), that is, the equation that represents the scalar of spatial curvature, we replace it in equation (36) we get to

$$\mathcal{H}_{IX} = \frac{\exp(-3\Omega)}{24}\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) + \exp(\Omega)\left[V(\beta_+, \beta_-) - 1\right], \quad (76)$$

where

$$V(\beta_+, \beta_-) = 1 + 2e^{-8\beta_+} - 8e^{-2\beta_+}\cosh\left(2\sqrt{3}\beta_-\right) + 4e^{4\beta_+} \times \left[\cosh\left(4\sqrt{3}\beta_-\right) + 1\right].$$

With equation (76) we can write the equations of motion as:

$$\frac{d\Omega}{dt} = \{\Omega, \mathcal{H}_{IX}\} = \frac{\partial \mathcal{H}_{IX}}{\partial p_\Omega} = -\frac{\exp(-3\Omega)}{12}p_\Omega, \quad (77)$$

$$\frac{d\beta_+}{dt} = \{\beta_+, \mathcal{H}_{IX}\} = \frac{\partial \mathcal{H}_{IX}}{\partial p_{\beta_+}} = \frac{\exp(-3\Omega)}{12}p_{\beta_+}, \quad (78)$$

$$\frac{d\beta_-}{dt} = \{\beta_-, \mathcal{H}_{IX}\} = \frac{\partial \mathcal{H}_{IX}}{\partial p_{\beta_-}} = \frac{\exp(-3\Omega)}{12}p_{\beta_-}, \quad (79)$$

$$\frac{dp_\Omega}{dt} = \{p_\Omega, \mathcal{H}_{IX}\} = -\frac{\partial \mathcal{H}_{IX}}{\partial \Omega} = \frac{\exp(-3\Omega)}{8} \times \left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) - \exp(\Omega)\left[V(\beta_+, \beta_-) - 1\right], \quad (80)$$

$$\frac{dp_{\beta_+}}{dt} = \left\{p_{\beta_+}, \mathcal{H}_{IX}\right\} = -\frac{\partial \mathcal{H}}{\partial \beta_+} = -\exp(\Omega)\frac{\partial V}{\partial \beta_+}, \quad (81)$$

$$\frac{dp_{\beta_-}}{dt} = \left\{p_{\beta_-}, \mathcal{H}_{IX}\right\} = -\frac{\partial \mathcal{H}}{\partial \beta_-} = -\exp(\Omega)\frac{\partial V}{\partial \beta_-}. \quad (82)$$

Using the fact that equation (76) is a constraint, then we clear p_Ω^2 from the Hamiltonian density in question and substitute it into equation (80), we get the differential equation

$$\frac{dp_\Omega}{dt} = -4\exp(\Omega)\left[V(\beta_+, \beta_-) - 1\right]. \quad (83)$$

The condition $\mathcal{H}_{IX} \approx 0$ must be fulfilled to reproduce Einstein's equations. Consequently, the dynamics of the Bianchi IX can be viewed as the dynamics of a particle at a time-dependent potential. Simple motions are obtained by assuming fixed anisotropic parameters β_+ and β_-, consequently, the last term of equation (76) containing the anisotropic potential $V(\beta_+, \beta_-)$ is negligible, accordingly $\Omega \to -\infty$, where each conjugate moment is constant and $p_\Omega^2 = p_{\beta_+}^2 + p_{\beta_-}^2$.

From the preceding limit in the Hamiltonian constraint (76) it was found that the conjugated moments are constant in that limit. Another viable way to verify such a statement could be done by taking the limit when $\Omega \to -\infty$ in equations (81), (82) and (83), and consequently we have the result $p_\Omega = p_{\beta_+} = p_{\beta_-} = constant$.

For the asymptotic description; that is, for large β, it can be found that in the limit $\beta_+ \to -\infty$, the value of the anisotropic potential of equation (76) behaves as

$$V(\beta_+ \to -\infty, \beta_-) \sim 2\exp(-8\beta_+) - 8\exp(-2\beta_+) \times \cosh\left(2\sqrt{3}\beta_-\right),$$

and finally for the opposite case, in addition to taking into account that $\beta_- \ll 1$, the anisotropic potential behaves in the way

$$V(\beta_+ \to +\infty, \beta_-) \sim 1 + 96\beta_-^2 \exp(4\beta_+).$$

3.3 Class B

Bianchi III

The structure constants of the Bianchi III are [31, 40]

$$C^1_{13} = -C^1_{31} = 1.$$

Using the structure constants and equation (37), the curvature scalar is determined by

$${}^{(3)}R_{III} = 2C^1_{13}C^1_{31}h_{11}h^{33}h^{11} + 2C^1_{31}C^1_{31}h^{33} + 4C^i_{ik}C^j_{jm}h^{km},$$

equation of which when using the values of the structure constants given for this Bianchi's model, we find

$${}^{(3)}R_{III} = 4\exp(-2\Omega + 4\beta_+). \tag{84}$$

Taking equation (84) and substituting it in equation (36), we find the Hamiltonian density expressed by:

$$\mathcal{H}_{III} = \frac{\exp(-3\Omega)}{24}\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) - 4\exp(\Omega + 4\beta_+), \tag{85}$$

With the previous Hamiltonian constraint, that is, equation (85) we can write the equations of motion

$$\frac{d\Omega}{dt} = \{\Omega, \mathcal{H}_{III}\} = \frac{\partial \mathcal{H}_{III}}{\partial p_\Omega} = -\frac{\exp(-3\Omega)}{12}p_\Omega, \tag{86}$$

$$\frac{d\beta_+}{dt} = \{\beta_+, \mathcal{H}_{III}\} = \frac{\partial \mathcal{H}_{III}}{\partial p_{\beta_+}} = \frac{\exp(-3\Omega)}{12}p_{\beta_+}, \tag{87}$$

$$\frac{d\beta_-}{dt} = \{\beta_-, \mathcal{H}_{III}\} = \frac{\partial \mathcal{H}_{III}}{\partial p_{\beta_-}} = \frac{\exp(-3\Omega)}{12}p_{\beta_-}, \tag{88}$$

$$\frac{dp_\Omega}{dt} = -\frac{\partial \mathcal{H}_{III}}{\partial \Omega} = \frac{\exp(-3\Omega)}{8}\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) + 4\exp(\Omega + 4\beta_+), \tag{89}$$

$$\frac{dp_{\beta_+}}{dt} = \left\{p_{\beta_+}, \mathcal{H}_{III}\right\} = -\frac{\partial \mathcal{H}_{III}}{\partial \beta_+} = 16\exp(\Omega + 4\beta_+), \tag{90}$$

$$\frac{dp_{\beta_-}}{dt} = \left\{p_{\beta_-}, \mathcal{H}_{III}\right\} = -\frac{\partial \mathcal{H}_{III}}{\partial \beta_-} = 0. \tag{91}$$

If we insert equation (85) into equation (89), we obtain an equation of motion in terms of the dynamic variables Ω, β_+, β_-

$$\frac{dp_\Omega}{dt} = -\frac{\partial \mathcal{H}_{III}}{\partial \Omega} = 16\exp(\Omega + 4\beta_+). \tag{92}$$

Using the Hamiltonian constraint $\mathcal{H}_{III} \approx 0$, the dynamics of the Bianchi III can be unraveled according to the second term of the Hamiltonian constriction. Assuming fixed anisotropic parameters β_+ and β_-, consequently, the last term of equation (85) tends to zero, as $\Omega \to -\infty$; in other words, the last term in equation (85) becomes very small if Ω becomes very large. From the above it follows that each conjugate moment is constant and $p_\Omega^2 = p_{\beta_+}^2 + p_{\beta_-}^2$. Since equations (90), (91) and (92) tend to zero as $\Omega \to -\infty$ and therefore $p_\Omega = p_{\beta_+} = p_{\beta_-} = constant$ (for the solution of this cosmological model in vacuum, see [41]).

Bianchi IV

This cosmological model has the structure constants expressed by equations [31, 40]

$$C^1_{13} = -C^1_{31} = 1,$$
$$C^1_{23} = -C^1_{32} = 1,$$
$$C^2_{23} = -C^2_{32} = 1.$$

Using equation (37), we obtain the relation

$${}^{(3)}R_{IV} = 2C^1_{23}C^1_{32}h_{11}h^{33}h^{22} + 4C^i_{ik}C^j_{jm}h^{km},$$

from which we finally obtain that the intrinsic curvature scalar for the Bianchi IV is given by

$${}^{(3)}R_{IV} = -2\exp\left(-2\Omega + 4\beta_+ + 4\sqrt{3}\beta_-\right) + 8\exp(-2\Omega + 4\beta_+). \tag{93}$$

Using equation (93) and we substitute it in equation (36) to then find the Hamiltonian constraint

$$\mathcal{H}_{IV} = \frac{\exp(-3\Omega)}{24}\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) + 2\exp\left(\Omega + 4\beta_+ + 4\sqrt{3}\beta_-\right) - 8\exp(\Omega + 4\beta_+). \tag{94}$$

From equation (94) we can write the equations of motion

$$\frac{d\Omega}{dt} = \{\Omega, \mathcal{H}_{IV}\} = \frac{\partial \mathcal{H}_{IV}}{\partial p_\Omega} = -\frac{\exp(-3\Omega)}{12}p_\Omega, \tag{95}$$

$$\frac{d\beta_+}{dt} = \{\beta_+, \mathcal{H}_{IV}\} = \frac{\partial \mathcal{H}_{IV}}{\partial p_{\beta_+}} = \frac{\exp(-3\Omega)}{12}p_{\beta_+}, \tag{96}$$

$$\frac{d\beta_-}{dt} = \{\beta_-, \mathcal{H}_{IV}\} = \frac{\partial \mathcal{H}_{IV}}{\partial p_{\beta_-}} = \frac{\exp(-3\Omega)}{12}p_{\beta_-}, \tag{97}$$

$$\frac{dp_\Omega}{dt} = -\frac{\partial \mathcal{H}_{IV}}{\partial \Omega} = \frac{\exp(-3\Omega)}{8}\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) - 2\left[\exp\left(4\sqrt{3}\beta_-\right) - 4\right]\exp\left(\Omega + 4\beta_+\right), \quad (98)$$

$$\frac{dp_{\beta_+}}{dt} = \left\{p_{\beta_+}, \mathcal{H}_{IV}\right\} = -\frac{\partial \mathcal{H}_{IV}}{\partial \beta_+} = 8\left[\exp\left(4\sqrt{3}\beta_-\right) - 4\right]\exp\left(\Omega + 4\beta_+\right), \quad (99)$$

$$\frac{dp_{\beta_-}}{dt} = \left\{p_{\beta_-}, \mathcal{H}_{IV}\right\} = -\frac{\partial \mathcal{H}_{IV}}{\partial \beta_-} = 8\sqrt{3}\exp\left(\Omega + 4\beta_+ + 4\sqrt{3}\beta_-\right). \quad (100)$$

Equation (94) replacing it in equation (98), we obtain an equation in terms of the dynamic variables Ω, β_+, β_- given by the expression

$$\frac{dp_\Omega}{dt} = -\frac{\partial \mathcal{H}_{IV}}{\partial \Omega} = 4\left[\exp\left(4\sqrt{3}\beta_-\right) - 4\right]\exp\left(\Omega + 4\beta_+\right). \quad (101)$$

Let's now analyze the Hamiltonian constraint $\mathcal{H}_{IV} \approx 0$. That is, the dynamics of the cosmological model can be unraveled according to the second and third terms of Hamiltonian constraint. Assuming fixed anisotropic parameters β_+ and β_-, consequently, the last two terms of equation (94) tend to zero as $\Omega \to -\infty$; in other words, the last two terms of equation (94) become very small if Ω becomes very large. Taking into consideration the previous analysis, from equations (99), (100) and (101), we find that $\frac{dp_\Omega}{dt} = \frac{dp_{\beta_+}}{dt} = \frac{dp_{\beta_-}}{dt} = 0$ as $\Omega \to -\infty$; therefore, we conclude that according to these conditions $p_\Omega = p_{\beta_+} = p_{\beta_-} = constant$.

Bianchi V

This cosmological model is characterized by the following structure constants [31, 40]

$$C^1_{13} = -C^1_{31} = 1,$$
$$C^2_{23} = -C^2_{32} = 1.$$

Using equation (37) once again, we find the following relationship of the three-dimensional scalar of curvature for the previous structure constants

$$^{(3)}R_V = 8\exp\left(-2\Omega + 4\beta_+\right). \quad (102)$$

If we substitute equation (102) in equation (36), we find the Hamiltonian density

$$\mathcal{H}_V = \frac{\exp(-3\Omega)}{24}\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) - 8\exp\left(-2\Omega + 4\beta_+\right). \quad (103)$$

Using equation (103), we find the Poisson brackets expressed by

$$\frac{d\Omega}{dt} = \{\Omega, \mathcal{H}_V\} = \frac{\partial \mathcal{H}_V}{\partial p_\Omega} = -\frac{\exp(-3\Omega)}{12}p_\Omega, \quad (104)$$

$$\frac{d\beta_+}{dt} = \{\beta_+, \mathcal{H}_V\} = \frac{\partial \mathcal{H}_V}{\partial p_{\beta_+}} = \frac{\exp(-3\Omega)}{12}p_{\beta_+}, \quad (105)$$

$$\frac{d\beta_-}{dt} = \{\beta_-, \mathcal{H}_V\} = \frac{\partial \mathcal{H}_V}{\partial p_{\beta_-}} = \frac{\exp(-3\Omega)}{12}p_{\beta_-}, \quad (106)$$

$$\frac{dp_\Omega}{dt} = \{p_\Omega, \mathcal{H}_V\} = -\frac{\partial \mathcal{H}_V}{\partial \Omega} = \frac{\exp(-3\Omega)}{8} \times \left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) + 4\exp\left(\Omega + 4\beta_+\right), \quad (107)$$

$$\frac{dp_{\beta_+}}{dt} = \left\{p_{\beta_+}, \mathcal{H}_V\right\} = -\frac{\partial \mathcal{H}_V}{\partial \beta_+} = 16\exp\left(\Omega + 4\beta_+\right), \quad (108)$$

$$\frac{dp_{\beta_-}}{dt} = \left\{p_{\beta_-}, \mathcal{H}_V\right\} = -\frac{\partial \mathcal{H}_V}{\partial \beta_-} = 0. \quad (109)$$

If we substitute equation (103) in equation (107) we obtain the equation of motion

$$\frac{dp_\Omega}{dt} = -\frac{\partial \mathcal{H}_V}{\partial \Omega} = 16\exp\left(\Omega + 4\beta_+\right). \quad (110)$$

Using the Hamiltonian constraint$\mathcal{H}_V \approx 0$, the dynamics of the Bianchi V can be unraveled according to the second term of said Hamiltonian constraint. Assuming the fixed anisotropic parameter β_+, consequently, the last term of equation (103) tends to zero, as $\Omega \to -\infty$. Since equations (108), (109) and (110) tend to zero as $\Omega \to -\infty$, then the result is $p_\Omega = p_{\beta_+} = p_{\beta_-} = constant$.

Bianchi VI$_h$

In the Bianchi VI$_h$ the non-zero structure constants are [31, 40]

$$C^1_{23} = -C^1_{32} = 1, \quad C^2_{31} = -C^2_{13} = -1$$
$$C^1_{13} = -C^1_{31} = 1, \quad C^2_{23} = -C^2_{32} = h.$$

With the previous structure constants, substituting them in equation (37), we find an equation for the intrinsic curvature scalar expressed by

$$^{(3)}R_{VI_h} = 2C^1_{23}C^1_{32}h_{11}h^{33}h^{22} + 2C^2_{13}C^2_{31}h_{22}h^{11}h^{33} + 4C^1_{32}C^2_{31}h^{33} + +4\left[\left(C^1_{13}\right)^2 + \left(C^2_{23}\right)^2 + 2C^1_{13}C^2_{23}\right]h^{33},$$

then, we obtain

$$^{(3)}R_{VI_h} = -4\exp\left(-2\Omega + 4\beta_+\right)\left[\cosh\left(4\sqrt{3}\beta_-\right) - 1\right] +4\left(1+h\right)^2\exp\left(-2\Omega + 4\beta_+\right) \quad (111)$$

From equation (111), we find the Hamiltonian density through equation (36):

$$\mathcal{H}_{VI_h} = \tfrac{1}{24}\exp\left(-3\Omega\right)\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) + 4\exp\left(\Omega + 4\beta_+\right)\left[\cosh\left(4\sqrt{3}\beta_-\right) - 1\right] - 4\left(1+h\right)^2\exp\left(\Omega + 4\beta_+\right). \quad (112)$$

With this Hamiltonian density; that is, the equation (112), we can write the equations of motion

$$\frac{d\Omega}{dt} = \{\Omega, \mathcal{H}_{VI_h}\} = \frac{\partial \mathcal{H}_{VI_h}}{\partial p_\Omega} = -\frac{\exp\left(-3\Omega\right)}{12}p_\Omega, \quad (113)$$

$$\frac{d\beta_+}{dt} = \{\beta_+, \mathcal{H}_{VI_h}\} = \frac{\partial \mathcal{H}_{VI_h}}{\partial p_{\beta_+}} = \frac{\exp\left(-3\Omega\right)}{12}p_{\beta_+}, \quad (114)$$

$$\frac{d\beta_-}{dt} = \{\beta_-, \mathcal{H}_{VI_h}\} = \frac{\partial \mathcal{H}_{VI_h}}{\partial p_{\beta_-}} = \frac{\exp\left(-3\Omega\right)}{12}p_{\beta_-}, \quad (115)$$

$$\begin{aligned}\tfrac{dp_\Omega}{dt} = -\tfrac{\partial \mathcal{H}_{VI_h}}{\partial \Omega} = \tfrac{1}{8}\exp\left(-3\Omega\right)\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) - 4\exp\left(\Omega + 4\beta_+\right)\cosh\left(4\sqrt{3}\beta_-\right) \\ +4\exp\left(\Omega + 4\beta_+\right) + 4\left(1+h\right)^2\exp\left(\Omega + 4\beta_+\right),\end{aligned} \quad (116)$$

$$\frac{dp_{\beta_+}}{dt} = -\frac{\partial \mathcal{H}_{VI_h}}{\partial \beta_+} = -16\exp\left(\Omega + 4\beta_+\right)\left[\cosh\left(4\sqrt{3}\beta_-\right) - 1\right] + 16\left(1+h\right)^2\exp\left(\Omega + 4\beta_+\right), \quad (117)$$

$$\frac{dp_{\beta_-}}{dt} = \left\{p_{\beta_-}, \mathcal{H}_{VI_h}\right\} = -\frac{\partial \mathcal{H}_{VI_h}}{\partial \beta_-} = -16\sqrt{3}\exp\left(\Omega + 4\beta_+\right)\sinh\left(4\sqrt{3}\beta_-\right). \quad (118)$$

If we use the Hamiltonian constraint (112), consequently we can transform equation (116) to the equation of motion

$$\frac{dp_\Omega}{dt} = -\frac{\partial \mathcal{H}_{VI_h}}{\partial \Omega} = -16\exp\left(\Omega + 4\beta_+\right)\left[\cosh\left(4\sqrt{3}\beta_-\right) - 1\right] + 16\left(1+h\right)^2\exp\left(\Omega + 4\beta_+\right). \quad (119)$$

We consider the Hamiltonian constraint, then, the dynamics of the cosmological model of Bianchi VI_h can be unraveled according to the second and third terms of said Hamiltonian constraint. Assuming fixed anisotropic parameters β_+ and β_-, consequently, the last two terms of equation (112) tend to zero as $\Omega \to -\infty$. Taking into consideration the previous analysis, from equations (117), (118) and (119) we find that $\frac{dp_\Omega}{dt} = \frac{dp_{\beta_+}}{dt} = \frac{dp_{\beta_-}}{dt} = 0$ as $\Omega \to -\infty$; therefore, we conclude that according to these conditions $p_\Omega = p_{\beta_+} = p_{\beta_-} = constant$.

Bianchi VII_h

In the Bianchi VII_h the non-zero structure constants are [42]

$$\begin{aligned}C^1_{23} = -C^1_{32} = -1, \quad C^2_{31} = -C^2_{13} = -1 \\ C^1_{13} = -C^1_{31} = h, \quad C^2_{23} = -C^2_{32} = h.\end{aligned}$$

From the previous structure constants, applying them to equation (37), we find the intrinsic curvature scalar expressed by the equation

$$\begin{aligned}^{(3)}R_{VII_h} = 2C^1_{23}C^1_{32}h_{11}h^{33}h^{22} + 2C^2_{13}C^2_{31}h_{22}h^{11}h^{33} + 4C^1_{32}C^2_{31}h^{33} + \\ +4\left[\left(C^1_{13}\right)^2 + \left(C^2_{23}\right)^2 + 2C^1_{13}C^2_{23}\right]h^{33},\end{aligned}$$

or

$$^{(3)}R_{VII_h} = -4\exp\left(-2\Omega + 4\beta_+\right)\left[\cosh\left(4\sqrt{3}\beta_-\right) + 1\right] + 4h^2\exp\left(-2\Omega + 4\beta_+\right). \quad (120)$$

From equations (36) and (120), it can be found that the Hamiltonian density is expressed by the equation

$$\begin{aligned}\mathcal{H}_{VII_h} = \tfrac{1}{24}\exp\left(-3\Omega\right)\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) + 4\exp\left(\Omega + 4\beta_+\right)\left[\cosh\left(4\sqrt{3}\beta_-\right) + 1\right] \\ -4h^2\exp\left(\Omega + 4\beta_+\right).\end{aligned} \quad (121)$$

With this Hamiltonian density; that is, the equations (121), we can write the equations of motion

$$\frac{d\Omega}{dt} = \{\Omega, \mathcal{H}_{VII_h}\} = \frac{\partial \mathcal{H}_{VII_h}}{\partial p_\Omega} = -\frac{\exp(-3\Omega)}{12} p_\Omega, \tag{122}$$

$$\frac{d\beta_+}{dt} = \{\beta_+, \mathcal{H}_{VII_h}\} = \frac{\partial \mathcal{H}_{VII_h}}{\partial p_{\beta_+}} = \frac{\exp(-3\Omega)}{12} p_{\beta_+}, \tag{123}$$

$$\frac{d\beta_-}{dt} = \{\beta_-, \mathcal{H}_{VII_h}\} = \frac{\partial \mathcal{H}_{VII_h}}{\partial p_{\beta_-}} = \frac{\exp(-3\Omega)}{12} p_{\beta_-}, \tag{124}$$

$$\begin{aligned}\frac{dp_\Omega}{dt} = -\frac{\partial \mathcal{H}_{VII_h}}{\partial \Omega} = \tfrac{1}{8}\exp(-3\Omega)\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) - 4\exp(\Omega + 4\beta_+)\cosh\left(4\sqrt{3}\beta_-\right) \\ -4\exp(\Omega + 4\beta_+) + 4h^2\exp(\Omega + 4\beta_+),\end{aligned} \tag{125}$$

$$\frac{dp_{\beta_+}}{dt} = -\frac{\partial \mathcal{H}_{VII_h}}{\partial \beta_+} = -16\exp(\Omega + 4\beta_+)\left[\cosh\left(4\sqrt{3}\beta_-\right) + 1\right] + 16h^2\exp(\Omega + 4\beta_+), \tag{126}$$

$$\frac{dp_{\beta_-}}{dt} = \left\{p_{\beta_-}, \mathcal{H}_{VII_h}\right\} = -\frac{\partial \mathcal{H}_{VII_h}}{\partial \beta_-} = -16\sqrt{3}\exp(\Omega + 4\beta_+)\sinh\left(4\sqrt{3}\beta_-\right). \tag{127}$$

If we use the Hamiltonian constriction (121), we can transform equation (125) to the equation of motion

$$\frac{dp_\Omega}{dt} = -\frac{\partial \mathcal{H}_{VII_h}}{\partial \Omega} = -16\exp(\Omega + 4\beta_+)\left[\cosh\left(4\sqrt{3}\beta_-\right) + 1\right] + 16h^2\exp(\Omega + 4\beta_+). \tag{128}$$

In the Hamiltonian constraint $\mathcal{H}_{VII_h} \approx 0$, we can treat the dynamics of the Bianchi VII$_h$ according to the second and third terms of said Hamiltonian constraint. Assuming fixed anisotropic parameters β_+ y β_-, consequently, the last two terms of equation (121) tend to zero as $\Omega \to -\infty$. Taking into consideration the previous analysis, from equations (126), (127) and (128) we find that $\frac{dp_\Omega}{dt} = \frac{dp_{\beta_+}}{dt} = \frac{dp_{\beta_-}}{dt} = 0$ as $\Omega \to -\infty$; therefore, we conclude that according to these conditions $p_\Omega = p_{\beta_+} = p_{\beta_-} = constant$.

4 Classification of Bianchi's cosmological models

4.1 Jacobi's identity

The Lie's bracket of infinitesimal differential operators related to the quantity $C^\lambda_{\rho\sigma}$ is given by

$$[X_\rho, X_\sigma] = X_\rho X_\sigma - X_\sigma X_\rho = C^\lambda_{\rho\sigma} X_\lambda,$$

with

$$X_\lambda = U^\mu_\lambda \frac{\partial}{\partial x^\mu}.$$

It can be shown that for certain types of arbitrary structure constants a group exists, if the structure constants have the antisymmetric property

$$C^\lambda_{\rho\sigma} = -C^\lambda_{\sigma\rho}, \tag{129}$$

this property can be verified in the Lie's bracket and they satisfy the Jacobi-Lie identity [43]

$$C^\lambda_{\rho\mu} C^\mu_{\sigma\tau} + C^\lambda_{\sigma\mu} C^\mu_{\tau\rho} + C^\lambda_{\tau\mu} C^\mu_{\rho\sigma} = 0, \tag{130}$$

which is deduced from the Jacobi's identity [43]

$$[X_\rho, [X_\sigma, X_\tau]] + [X_\sigma, [X_\tau, X_\rho]] + [X_\tau [X_\rho, X_\sigma]] = 0. \tag{131}$$

Example

As an example, we have the group of rotations of a flat three-dimensional space with Killing's vectors given by

$$U^\mu_1 = (y, -x, 0), \quad U^\mu_2 = (z, 0, -x), \quad U^\mu_3 = (0, z, -y). \tag{132}$$

Therefore, the differential operators X_λ when inserting the Killing vectors (equations 132) are determined by

$$\begin{aligned} X_1 = y\partial/\partial x - x\partial/\partial y, \quad X_2 = z\partial/\partial x - x\partial/\partial z, \\ X_3 = z\partial/\partial y - y\partial/\partial z.\end{aligned} \tag{133}$$

Also by making use of equations (133); that is, of the differential operators associated with the Killing's vectors, the Lie's brackets are

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2,$$

therefore, the structure constants are given by $C^3_{12} = C^1_{23} = C^2_{31} = 1$, from which the rotations of flat space do not commute. These brackets correspond to the operators of quantum mechanics and their commutation rules.

4.2 Structure constants of the groups $\mathbf{G}_3$

The movement groups of a group are characterized by the number of its Killing's vectors, the structure of the group, and the regions of transitivity. Establishing all non-isomorphic groups G_r of r Killing's vectors, of groups whose structure constants cannot be converted into some other by linear transformations of the base, is a purely mathematical problem of group theory.

Each group with two elements is an Abelian group if

$$[X_1, X_2] = 0, \tag{134}$$

or else you have

$$[X_1, X_2] = \alpha X_1 + \beta X_2, \tag{135}$$

where $\alpha \neq 0$. If we consider the second case, that is, in a non-Abelian group, we can arrive at a new commutation rule with structure constant $C^1_{23} = 1$, that characterizes the two non-isomorphic G_2 groups.

We turn our attention now to homogeneous three-dimensional cosmological models of the universe; that is, where all the points of the three-dimensional universe are equivalent. A set of non-isomorphic groups G_3 can be obtained from the relation

$$\frac{1}{2}\epsilon^{\rho\sigma\lambda} C^{\mu}_{\rho\sigma} = A^{\lambda\mu}, \tag{136}$$

where $\rho, \sigma = 1, 2, 3$, and $\epsilon^{\rho\sigma\lambda}$ is the Levi-Civita symbol, which is defined by [44]

$$\epsilon^{\rho\sigma\lambda} = \begin{cases} 0, & \text{there is repetition of two indices,} \\ 1, & (\rho, \sigma, \lambda) \text{ an even permutation of } (1, 2, 3), \\ -1, & (\rho, \sigma, \lambda) \text{ an odd permutation of } (1, 2, 3). \end{cases} \tag{137}$$

Since $A^{\mu\lambda}$ is a 3×3 matrix, then we can separate it into two parts, in other words, we decompose it into the symmetric and antisymmetric parts, respectively. Its symmetric part is represented by the matrix $n^{(\mu\lambda)}$ and the antisymmetric part by $\epsilon^{\lambda\mu\rho} A_\rho$, where A_ρ is a vector. Therefore, we can write this matrix using the equation

$$A^{\lambda\mu} = n^{(\lambda\mu)} + \epsilon^{\lambda\mu\rho} A_\rho. \tag{138}$$

Substituting equation (138) in equation (136); after some manipulations, we get the mathematical relation

$$C^{\lambda}_{\rho\sigma} = \epsilon_{\mu\rho\sigma} n^{(\lambda\mu)} + \delta^{\lambda}_{\sigma} A_\rho - \delta^{\lambda}_{\rho} A_\sigma, \tag{139}$$

where δ^{λ}_{ρ} is the Kronecker delta and is defined as [44]

$$\delta^{\lambda}_{\rho} = \begin{cases} 1, & \text{if } \lambda = \rho, \\ 0, & \text{if } \lambda \neq \rho. \end{cases} \tag{140}$$

Using equation (139) and substituting it in the Jacobi-Lie identity we obtain

$$n^{(\rho\sigma)} A_\rho = 0,$$

where the index of the internal multiplication can be applied to any of the two indices of $n^{(\rho\sigma)}$, because it is a symmetric quantity.

The basis of the Killing's vector space can be chosen in such a way that $A^{\mu\lambda}$ is a diagonal matrix, that is, $n^{(\rho\sigma)} = \text{diag}\,(n_1, n_2, n_3)$ and also have the vector $A_\rho = (a, 0, 0)$, from which we have $an_1 = 0$. From the above, we have a G_3

$$\begin{aligned} [X_1, X_2] &= n_3 X_3 + a X_2, \\ [X_2, X_3] &= n_1 X_1, \qquad a n_1 = 0, \\ [X_3, X_1] &= n_2 X_2 - a X_3, \quad n_i = 0, \pm 1. \end{aligned} \tag{141}$$

In class B, it is introduced a scalar h with the equation

$$A_\rho A_\sigma = \frac{1}{2} h \epsilon_{\rho\mu\nu} \epsilon_{\sigma\lambda\tau} n^{(\mu\lambda)} n^{(\nu\tau)}. \tag{142}$$

Using $A_\rho = (a, 0, 0)$ and $n^{\mu\lambda} = \text{diag}\,(n_1, n_2, n_3)$, we obtain from equation (142) the quantity

$$a^2 = h n_2 n_3,$$

from where the condition $n_2 n_3 \neq 0$ is deduced.

FLRW cosmological models can only be generalized to some Bianchi's models. The Bianchi's type I and VII_0 are a generalization of the Euclidian FLRW model ($k = 0$), the Bianchi IX for the spherical FLRW cosmological model ($k = 1$) and the Bianchis V and VII_h are for the hyperbolic FLRW model ($k = -1$). The rest of Bianchi's cosmological models do not contain the FLRW cosmological models as a particular case.

4.3 Classification tables of Bianchi's models

The previous analysis allows the following classification of the 11 Bianchi's cosmological models in the Table 1 [45,46].

As shown in Table 1, there are eleven types of G_3 groups, which are distributed through the so-called Bianchi's cosmological models from I to IX.

And according to the structure constants [31,40], not to the parameters a, n_1, n_2, n_3, we obtains the Table 2.

The Bianchi's cosmological models have as a limit case the Bianchi I by keeping the parameters β_+ y β_- fixed and taking the limit $\Omega \to -\infty$.

Class	A	A	A	A	A	A	B	B	B	B	B
Type Bianchi	I	II	VI_0	VII_0	VIII	IX	V	IV	III	VII_h	VI_h
a	0	0	0	0	0	0	1	1	1	a	a
n_1	0	1	1	1	1	1	0	0	0	0	0
n_2	0	0	-1	1	1	1	0	0	1	1	1
n_3	0	0	0	0	-1	1	0	1	-1	1	-1

Table 1: Classification of Bianchi's models according to the parameters a, n_1, n_2, n_3.

Class	Type	Structure constants
A	I	$C^{\lambda}_{\rho\sigma} = 0,$
A	II	$C^1_{23} = -C^1_{32} = 1,$
B	III	$C^1_{13} = -C^1_{13} = 1,$
B	IV	$C^1_{13} = -C^1_{31} = 1, \quad C^1_{23} = -C^1_{32} = 1, \quad C^2_{23} = -C^2_{32} = 1$
B	V	$C^1_{13} = -C^1_{31} = 1, \quad C^2_{23} = -C^2_{32} = 1$
A	VI_0	$C^1_{23} = -C^1_{32} = 1, \quad C^2_{31} = -C^2_{13} = -1$
A	VII_0	$C^1_{23} = -C^1_{32} = -1, \quad C^2_{31} = -C^2_{13} = -1$
A	VIII	$C^1_{23} = -C^1_{32} = -1, \quad C^3_{12} = -C^3_{21} = 1, \quad C^2_{31} = -C^2_{13} = -1$
A	IX	$C^1_{23} = -C^1_{32} = 1, \quad C^2_{31} = -C^2_{13} = 1, \quad C^3_{12} = -C^3_{21} = 1$
B	VI_h	$C^1_{23} = -C^1_{32} = 1, \quad C^2_{31} = -C^2_{13} = -1,$ $C^1_{13} = -C^1_{31} = 1, \quad C^2_{23} = -C^2_{32} = h$
B	VII_h	$C^1_{23} = -C^1_{32} = -1, \quad C^2_{31} = -C^2_{13} = -1,$ $C^1_{13} = -C^1_{31} = h, \quad C^2_{23} = -C^2_{32} = h$

Table 2: Classification of Bianchi models according to the structure constants.

5 Concluding remarks

We show the way to construct the Lagrangian density and the Hamiltonian density for each cosmological model of Bianchi, in a vacuum, without cosmological constant and also, without scalar field. As previously mentioned, from the Hamiltonian density it was possible for us to analyze each of the Bianchi's space-times. However, it has not been mentioned that the curvature scalar ${}^{(3)}R$ is the one that was always the main argument to calculate all the Hamiltonian densities $\mathcal{H}$, this scalar according to equation (37) depends on the structure constants $C^{\lambda}_{\mu\nu}$. The structure constants are of the utmost importance in this work since they are the ones that provide an algebraic classification of the Bianchi's models in accordance with group theory, as shown in tables 1 and 2. In particular, table 2 has been the basis for our analysis of each Bianchi spacetime.

We show the way to construct the Lagrangian density and the Hamiltonian density for each cosmological model of Bianchi, in a vacuum, without cosmological constant and also, without scalar field. As previously mentioned, from the Hamiltonian density it was possible for us to analyze each of the Bianchi's space-times. However, it has not been mentioned that the curvature scalar ${}^{(3)}R$ is the one that was always the main argument to calculate all the Hamiltonian densities $\mathcal{H}$, this scalar according to equation (37) depends on the structure constants $C^{\lambda}_{\mu\nu}$. The structure constants are of the utmost importance in this work since they are the ones that provide an algebraic classification of the Bianchi's models in accordance with group theory, as shown in tables 1 and 2. In particular, table 2 has been the basis for our analysis of each Bianchi spacetime.

We conclude, as seen in the section on the classification of Bianchi cosmological models, that FLRW cosmological models can only be generalized to some Bianchi's models. The Bianchi's type I and VII_0 are a generalization of the Euclidian FLRW model ($k = 0$), the Bianchi IX for the spherical FLRW cosmological model ($k = 1$) and the Bianchis V and VII_h are for the hyperbolic FLRW model ($k = -1$). The rest of Bianchi's cosmological models do not contain the FLRW cosmological models as a particular case.

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Appendix A: ADM Formalism of General Relativity

One way to unravel the dynamics of General Relativity is to see it as a Cauchy problem, that is, to analyze the dynamics of the evolution of a three-dimensional hypersurface where the fields are defined. This way of treating General Relativity was formulated by R. Arnowitt, S. Deser and C.W. Misner [47–57]; it is known as the ADM formalism of General Relativity [58, 59].

Decomposition of space-time

Let's get started an analysis by describing some quantities on the hypersurface. Let us consider a vector flow t^μ, which we decompose into its normal part and tangential to the hypersurface as

$$t^\mu = Nn^\mu + N^\mu, \tag{143}$$

where $n^\mu$is a unit vector to the hypersurface and N^μ is a tangent vector. The scalar N is called the "lapse" function, and the N^μ function is called the "shift" function. These, together with the metric g_{ab} constitute the ADM variables. The lapse function represents how far one hypersurface is separated from another, in other words, it measures the ratio of the proper time flux τ with respect to the function t, as the normal movement to the hypersurface is performed, and therefore we have $d\tau = Ndt$. On the other hand, the spatial part of the shift function measures the amount of tangential displacement for the hypersurface contained in the vector field t^μ.

Geometrically, the vector flux t^μ can be interpreted as follows: Let us consider two infinitesimally close hypersurfaces, as explained in the preceding paragraph, the term Nn^μ tells us how much we move perpendicular to the hypersurface, on the other hand, the vector N^μ can be said to indicate how much we move tangentially to the hypersurface (see figure 1).

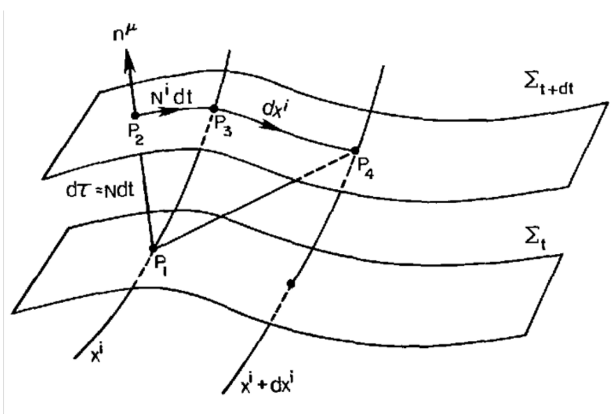

Figure 1: $3 + 1$ decomposition of the manifold, with lapse function N, and shift vector N^i.

The metric tensor g_{ab} of the hypersurface

$$^{(3)}ds^2 = g_{ab}dx^a dx^b,$$

and the metric tensor of spacetime is related by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(Ndx^0\right)^2 + g_{ab} \times \left(dx^a + N^a dx^0\right)\left(dx^b + N^b dx^0\right), \tag{144}$$

where $(dx^a + N^a dx^0)$ is the displacement on the base hypersurface and Ndt is the proper time between them, or, rearranging terms

$$ds^2 = \left(N^a N_a - N^2\right)\left(dx^0\right)^2 + 2N_a dx^a dx^0 + g_{ab}dx^a dx^b,$$

where the space-time have signature $(-, +, +, +)$. From the last equation it can be seen that the components of the metric tensor are given by

$$g_{\mu\nu} = \begin{pmatrix} N_a N^a - N^2 & N_b \\ N_a & g_{ab} \end{pmatrix}, \tag{145}$$

where g_{ab} denotes the spatial metric tensor. The contravariant components of the metric tensor are found by inverting the matrix $g_{\mu\nu}$, so that we have

$$g^{\mu\nu} = \begin{pmatrix} -1/N^2 & N^b/N^2 \\ N^a/N^2 & g_{ab} - N^a N^b/N^2 \end{pmatrix}. \tag{146}$$

Extrinsic curvature

For an arbitrary vector u_μ at a point p belonging to the hypersurface, we construct a covariant derivative D_μ associated with the metric tensor $h^{\mu\nu}$ by

$$D_\mu u_\nu = h^\rho_\mu h^\sigma_\nu \nabla_\rho u_\sigma = h^\rho_\mu h^\sigma_\nu \left(\frac{\partial u_\sigma}{\partial x^\rho} - \Gamma^\lambda_{\rho\sigma} u_\lambda\right).$$

An extrinsic curvature can be defined, which describes how hypersurfaces $\sum_t$ curve with respect to the 4-dimensional manifold. The above is represented mathematically by

$$K_{\mu\nu} = \frac{1}{2N} h^\rho_\mu h^\sigma_\nu \left(\frac{\partial h_{\rho\sigma}}{\partial t} - \nabla_\rho N_\sigma - \nabla_\sigma N_\rho\right),$$

or

$$K_{\mu\nu} = \frac{1}{2N}\left(\frac{\partial h_{\mu\nu}}{\partial t} - D_\mu N_\nu - D_\nu N_\mu\right). \tag{147}$$

Note that $K_{\mu\nu}$ does not depend on the derivatives with respect to t of N^μ.

Curvature scalar

The Riemann tensor is defined by

$$[\nabla_\mu, \nabla_\nu] u_\rho = R^\sigma_{\mu\nu\rho} u_\sigma, \tag{148}$$

and the curvature scalar

$$R = R_{\mu\nu\rho\sigma} g^{\mu\rho} g^{\nu\sigma},$$

or, if we do some mathematical tricks

$$^{(3)}R = R + K^2 - K_{\mu\nu}K^{\mu\nu} + 2\nabla_\mu \left(n^\nu \nabla_\nu n^\mu - n^\mu \nabla_\nu n^\nu\right). \tag{149}$$

Next, we define the intrinsic curvature scalar in the hypersurfaces $\sum_t$, related to the 4-dimensional curvature scalar, in the form:

$$^{(3)}R = R + K^2 - K_{\mu\nu}K^{\mu\nu} + 2\nabla_\mu \left(\Delta^\mu\right),$$

where $\Delta^\mu = n^\nu \nabla_\nu n^\mu - n^\mu \nabla_\nu n^\nu$. This equation is called the Codazzi's equation and shows the relationship between the curvature scalar of the hypersurface and the curvature scalar of space-time. The last term of this equation is a covariant derivative of the term and when introduced into action, by means of the divergence theorem, it does not have dynamic information and we can ignore it.

Hamiltonian formulation

Taking the Codazzi equation, we can rewrite the action for the gravitational field in the form

$$S[g_{ab}, N, N^a] = \int dt \int d^3x N \sqrt{\det(h)} \left({}^{(3)}R - K^2 + K_{\mu\nu}K^{\mu\nu}\right), \tag{150}$$

with

$$\mathcal{L}_G = N \sqrt{\det(h)} \left({}^{(3)}R - K^2 + K_{\mu\nu}K^{\mu\nu}\right). \tag{151}$$

The action we propose includes the action of Einstein's gravity, a cosmological constant, matter and scalar potential

$$S = \frac{1}{16\pi G} \int dt \int d^3x N\sqrt{h} \left({}^{(3)}R - K^2 + K_{\mu\nu}K^{\mu\nu} - 2\Lambda\right) + \int d^4x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \frac{\partial \Phi}{\partial x^\mu} \frac{\partial \Phi}{\partial x^\nu} - V(\Phi)\right]. \tag{152}$$

So far we have rewritten the action of the gravitational field so that we can find the field equations in a vacuum, taking the variation of the action and setting it equal to zero ($\delta S = 0$)

$$0 = \int dt \int d^3x \left(\frac{\delta \mathcal{L}_{total}}{\delta \dot{h}_{ab}} \delta \dot{h}_{ab} + \frac{\delta \mathcal{L}_{total}}{\delta \dot{N}^a} \delta \dot{N}^a + \frac{\delta \mathcal{L}_{total}}{\delta \dot{N}} \delta \dot{N} + \frac{\delta \mathcal{L}_{total}}{\delta \dot{\Phi}} \delta \dot{\Phi}\right),$$

where the conjugated moments are

$$\pi^{ab} = \frac{\delta \mathcal{L}_{total}}{\delta \dot{h}_{ab}} = \sqrt{\det(h)} \left(K^{ab} - K q^{ab}\right), \tag{153}$$

$$\pi_\Phi = \frac{\delta \mathcal{L}_{total}}{\delta \dot{\Phi}} = \frac{\sqrt{h}}{N} \left(\frac{\partial \Phi}{\partial t} - N^i \frac{\partial \Phi}{\partial x^i}\right), \tag{154}$$

$$\pi^a = \frac{\delta \mathcal{L}_G}{\delta \dot{N}_a} = 0, \tag{155}$$

$$\pi = \frac{\delta \mathcal{L}_G}{\delta \dot{N}} = 0. \tag{156}$$

The cancellation of the conjugated moments indicates that the system has first class constrictions, this is Dirac's terminology [60].

So the action is expressed by

$$S[g_{ab}, N, N^a] = \int dt \int d^3x \left\{\dot{h}_{ab}\pi^{ab} + \dot{N}^a \pi_a + \dot{N}\pi - N^a \mathcal{H}_a - N\mathcal{H}\right\}, \tag{157}$$

with

$$\mathcal{H} = \frac{\sqrt{h}}{8\pi G} \left(K^{ab}K_{ab} - \frac{1}{2}K^2\right) - \frac{\sqrt{h}}{16\pi G} \left[{}^{(3)}R - 2\Lambda + \frac{1}{2}\sqrt{h} \left(\frac{\pi_\Phi^2}{h} + h^{ab} \frac{\partial \Phi}{\partial x^a} \frac{\partial \Phi}{\partial x^b} + 2V\right)\right], \tag{158}$$

$$\mathcal{H}_a = -2h_{ac} D_b \pi^{bc} + h_{ab} \pi_\Phi \frac{\partial \Phi}{\partial x_b}. \tag{159}$$

The “lapse” and “shift” functions act as Lagrange multipliers, varying the action (158) with respect to the lapse function, N, we obtain the Hamiltonian constraint $\mathcal{H}_a \approx 0$. On the other hand, varying the action with respect to the "shift" function, N_a, leads to the moment constraint, $\mathcal{H}_a \approx 0$. These constraints are simply the components (00) and (0i) of the Eisntein's equations; in Dirac's terms, they are secondary constrains [60]. The analysis of each Bianchi model presented in this article, in accordance with the formalism presented in this appendix, can be extended to the case where matter, cosmological constant and a scalar field are considered (to analysis of some Bianchi's models, see [61–68]).

Appendix B: Structure constants

Let us consider the case of a group of r-parameters and n variables. The starting point is given by

$$x_0^\mu = f^\mu\left(\mathbf{x_0};\mathbf{0}\right),$$

where $\mu = 1, 2, ..., n$.

We can obtain x^μ by the transformation

$$x^\mu = f^\mu\left(\mathbf{x}_0;\mathbf{a}\right).$$

We could go to $x^\mu + dx^\mu$through transformation

$$x^\mu + dx^\mu = f^\mu\left(\mathbf{x_0};\mathbf{a}+d\mathbf{a}\right).$$

However, we can also go from x^μ a $x^\mu + dx^\mu$ by a parametric infinitesimal change $\delta\mathbf{a}$, that is,

$$x^\mu + dx^\mu = f^\mu\left(\mathbf{x};\delta\mathbf{a}\right).$$

Expanding, the preceding result gives

$$dx^\mu = \sum_{\sigma=1}^{r} \left.\frac{\partial f^\mu\left(\mathbf{x};\mathbf{a}\right)}{\partial a^\sigma}\right|_{\mathbf{a}=\mathbf{0}} \delta a^\sigma,$$

where $\sigma = 1, 2, ..., r$, or equivalently

$$dx^\mu = U_\sigma^\mu\left(\mathbf{x}\right)\delta a^\sigma, \qquad (160)$$

where

$$U_\sigma^\mu\left(\mathbf{x}\right) = \left.\frac{\partial f^\mu\left(\mathbf{x};\mathbf{a}\right)}{\partial a^\sigma}\right|_{\mathbf{a}=\mathbf{0}}.$$

The connection between da^σ and δa^ρ can be established by the equation

$$a^\sigma + da^\sigma = \varphi^\sigma\left(\mathbf{a};\delta\mathbf{a}\right),$$

and therefore

$$da^\sigma = \left.\frac{\partial\varphi^\sigma\left(\mathbf{a};\mathbf{b}\right)}{\partial b^\rho}\right|_{\mathbf{b}=\mathbf{0}} \delta a^\rho = V_\rho^\sigma\left(\mathbf{a}\right)\delta a^\rho, \qquad (161)$$

where $\rho = 1, 2, ..., r$.

The inverse matrix V_ρ^σ will be λ_τ^ρ, where $\lambda_\tau^\rho V_\rho^\sigma = \delta_\tau^\sigma$. The inverse of the transformation established in equation (161) is given by

$$\delta a^\rho = \lambda_\tau^\rho\left(\mathbf{a}\right)da^\tau. \qquad (162)$$

Substituting equation (162) into equation (160), we find

$$dx^\mu = U_\sigma^\mu\left(\mathbf{x}\right)\lambda_\rho^\sigma\left(\mathbf{a}\right)da^\rho,$$

o well

$$\frac{\partial x^\mu}{\partial a^\rho} = U_\sigma^\mu\left(\mathbf{x}\right)\lambda_\rho^\sigma\left(\mathbf{a}\right). \qquad (163)$$

The infinitesimal transformation $\mathbf{x}\to\mathbf{x}+d\mathbf{x}$ induces in $F\left(\mathbf{x}\right)$ the transformation $F\left(\mathbf{x}\right) \to F\left(\mathbf{x}\right) + dF\left(\mathbf{x}\right)$. Therefore

$$dF\left(\mathbf{x}\right) = \frac{\partial F}{\partial x^\mu}dx^\mu = \frac{\partial F}{\partial x^\mu}U_\sigma^\mu\delta a^\sigma = \delta a^\sigma U_\sigma^\mu\frac{\partial F}{\partial x^\sigma} = \delta a^\sigma X_\sigma F,$$

where

$$X_\sigma = U_\sigma^\mu\frac{\partial}{\partial x^\mu}, \qquad (164)$$

they are called the infinitesimal operators of the group.

Equation (163) describes the change in the point $\mathbf{x}$ generated by an infinitesimal displacement from its initial position $\mathbf{x}\left(\mathbf{0}\right)$, where $\mathbf{a}=\mathbf{0}$.

In order to obtain a finite displacement, equation (163) is required to be integrable, that is, the condition

$$\frac{\partial^2 x^\mu}{\partial a^\tau\partial a^\rho} = \frac{\partial^2 x^\mu}{\partial a^\rho\partial a^\tau}. \qquad (165)$$

Substituting equation (163) into equation (165), we find the result

$$U_\sigma^\mu\left(\mathbf{x}\right)\frac{\partial\lambda_\rho^\sigma\left(\mathbf{a}\right)}{\partial a^\tau} + \frac{\partial U_\sigma^\mu\left(\mathbf{x}\right)}{\partial x^\beta}\frac{\partial x^\beta}{\partial a^\tau}\lambda_\rho^\sigma\left(\mathbf{a}\right) = U_\sigma^\mu\left(\mathbf{x}\right)\frac{\partial\lambda_\tau^\sigma\left(\mathbf{a}\right)}{\partial a^\rho} + \frac{\partial U_\sigma^\mu\left(\mathbf{x}\right)}{\partial x^\beta}\frac{\partial x^\beta}{\partial a^\rho}\lambda_\tau^\sigma\left(\mathbf{a}\right).$$

Using Eq. (163) once more and rearranging terms, we obtain

$$\left[U_\nu^\beta\left(\mathbf{x}\right)\frac{\partial U_\sigma^\mu\left(\mathbf{x}\right)}{\partial x^\beta} - U_\sigma^\beta\left(\mathbf{x}\right)\frac{\partial U_\nu^\mu\left(\mathbf{x}\right)}{\partial x^\beta}\right]\lambda_\tau^\nu\left(\mathbf{a}\right)\lambda_\rho^\sigma\left(\mathbf{a}\right) + U_\sigma^\mu\left(\mathbf{x}\right)\left[\frac{\partial\lambda_\rho^\sigma\left(\mathbf{a}\right)}{\partial a^\tau} - \frac{\partial\lambda_\tau^\sigma\left(\mathbf{a}\right)}{\partial a^\rho}\right] = 0.$$

Multiplying the above equation by $U_\xi^\tau U_\eta^\rho$; we obtain, and for brevity suppressing $\mathbf{x}$ and $\mathbf{a}$,

$$\left[U_\nu^\beta \frac{\partial U_\sigma^\mu}{\partial x^\beta} - U_\sigma^\beta \frac{\partial U_\nu^\mu}{\partial x^\beta}\right] \lambda_\tau^\nu \lambda_\rho^\sigma U_\xi^\tau U_\eta^\rho = \left[\frac{\partial \lambda_\tau^\sigma}{\partial a^\rho} - \frac{\partial \lambda_\rho^\sigma}{\partial a^\tau}\right] U_\xi^\tau U_\eta^\rho U_\sigma^\mu = C_{\xi\eta}^\sigma \left(\mathbf{x};\mathbf{a}\right) U_\sigma^\mu,$$

and using the equation $U_\xi^\tau \lambda_\tau^\nu = \delta_\xi^\nu$, we can write

$$U_\xi^\beta \frac{\partial U_\eta^\mu}{\partial x^\beta} - U_\eta^\beta \frac{\partial U_\xi^\mu}{\partial x^\beta} = \left[\frac{\partial \lambda_\tau^\sigma}{\partial a^\rho} - \frac{\partial \lambda_\rho^\sigma}{\partial a^\tau}\right] U_\xi^\tau U_\eta^\rho U_\sigma^\mu = C_{\xi\eta}^\sigma \left(\mathbf{x};\mathbf{a}\right) U_\sigma^\mu. \tag{166}$$

The term $U_\sigma^\mu(\mathbf{x})$ is independent of $\mathbf{a}$, and therefore if we differentiate equation (166) with respect to a^κ, we find

$$\frac{\partial}{\partial a^\kappa}\left[U_\xi^\beta \frac{\partial U_\eta^\mu}{\partial x^\beta} - U_\eta^\beta \frac{\partial U_\xi^\mu}{\partial x^\beta}\right] = \left[\frac{\partial^2 \lambda_\tau^\sigma}{\partial a^\kappa \partial a^\rho} - \frac{\partial^2 \lambda_\rho^\sigma}{\partial a^\kappa \partial a^\tau}\right] U_\xi^\tau U_\eta^\rho U_\sigma^\mu = \frac{\partial}{\partial a^\kappa}\left[C_{\xi\eta}^\sigma\left(\mathbf{x};\mathbf{a}\right)\right] U_\sigma^\mu,$$

and therefore the constants $C_{\xi\eta}^\sigma(\mathbf{x};\mathbf{a})$ are independent of the parameters $\mathbf{a}$.

Lie brackets are given by

$$[X_\rho, X_\sigma] = X_\rho X_\sigma - X_\sigma X_\rho,$$

taking into account equation (164), we can write the equation

$$[X_\rho, X_\sigma] = U_\rho^\mu \frac{\partial}{\partial x^\mu} U_\sigma^\nu \frac{\partial}{\partial x^\nu} - U_\sigma^\nu \frac{\partial}{\partial x^\nu} U_\rho^\mu \frac{\partial}{\partial x^\mu} = \left(U_\rho^\mu \frac{\partial U_\sigma^\nu}{\partial x^\mu} - U_\sigma^\mu \frac{\partial U_\rho^\mu}{\partial x^\mu}\right)\frac{\partial}{\partial x^\nu},$$

that when compared with equation (166)

$$[X_\rho, X_\sigma] = C_{\rho\sigma}^\lambda U_\lambda^\nu \frac{\partial}{\partial x^\nu},$$

or equivalently according to equation (164), we find [69,70]

$$[X_\rho, X_\sigma] = C_{\rho\sigma}^\lambda X_\lambda.$$

Given the antisymmetry of the Lie bracket, then the structure constants must be antisymmetric at the lower indices.

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