ARTÍCULO ORIGINAL

Revista de Investigación de Física **25(2)**, (May-Ago 2022)
Doi: 10.15381/rif.v25i2.21449

eISSN:1728-2977

On the relativistic theory of the assymetric field

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Recibido 23 Oct 2021 – Aceptado 15 Ago 2022 – Publicado 20 Ago 2022

Abstract
In this paper, we have given a presentation of the relativistic theory of asymmetric field. It is our aim to present a theory of gravitation and electomagnetism by a generalization of the concepts and mathematical methods of the general relativity. We look for the formally most simple expression for the law of gravitation in the absence of an electromagnetic field, and then the most natural generalization of this law. This theory contain Maxwell's theory in the Lambda transformation.

Keywords: General Relativity, Curvature Tensor, field equations, Bianchi's identities..

Sobre la teoría relativista del campo asimétrico

Resumen
En este trabajo, damos una presentación de una teoría relativista del campo asimétrico. Es nuestro objetivo presentar una teoría de la gravitación y el electromagnetismo mediante una generalización de los conceptos y métodos matemáticos de la relatividad general. Buscamos la expresión formalmente más simple para la ley de la gravitación en ausencia de un campo electromagnético, y luego la generalización más natural de esta ley. Esta teoría contiene la teoría de Maxwell en la tranformación Lambda.

Palabras clave: Relatividad General, tensor de curvatura, ecuaciones de campo, identidades de Bianchi..

1 Introduction

The first attempts of unification of Einstein [1] and Kaluza [2], other types of interactions different from gravity and electromagnetism, such as weak interaction and strong interaction, have been the subject of various attempts at unification, and by the end of the 1960s the electroweak theory was formulated [3–6]. In fact, it is a unified field theory of electromagnetism and weak interaction. Attempts to unify the theory of strong interaction [7, 8] with the electroweak model and with gravity have since remained one of the still pending challenges of physicists.

In the beginning of 20th century, the mathematical theories essential for the creation of the general relativity [9–17,17,19] were based on the Riemann metric, which was considered as the fundamental concept of general relativity. Although, it was later pointed out, correctly, that the element of the theory that allows to avoid the inertial system, it is rather the field of infinitesimal displacement. It replaces the inertial system to the extent that the comparison of vectors at infinitesimally close points becomes possible.

In section 2, we present the parallel transport equation in order to define the covariant derivative of a tensor of rank one and the infinitesimal displacement field (affine connection). Using the parallel transport equation, the covariant derivative and the transformation law of a second rank mixed tensor, we calculate the transformation equation of the affine connection. Additionally, we define the symmetric and antisymmetric part of the displacement field, noting that if the affine connection is symmetric, we arrive at the relativistic theory of the gravitational field. In Section 3, by virtue of the affine connection transformation law, we deduce the curvature tensor and define the contracted curvature tensor. In

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section 4, a new field is defined by a Lambda function, l, where the curvature equations of the theory are invariant under this transformation. In section 5, we construct a new curvature tensor in terms of the pseudo tensor $U^{\rho}_{\eta\sigma}$. In section 6, by defining the variational principle, we generalize the Einstein field equations, of General Relativity, in the present theory, by means of the new curvature tensor presented in section 5, additionally resulting in the pseudo tensor $\sqrt{-g}n^{\mu\nu}_{\lambda}$. In section 7, we compute the Bianchi differential identities, the differential identity resulting from the Lambda transformation (which shows us the role of the electromagnetic vector potential and the current density equation).

2 Infinitesimal displacement field

If A^{ν} is a vector in P_1, and $A^{\nu}+dA^{\nu}$ is a vector shifted at P_2 along the interval dx^{ν}, then δA^{ν} is an infinitesimal quantity that indicates how much has been displaced the vector field under consideration. Consequently, we can write the equation used to express the infinitesimal displacement field:

$$\delta A^{\lambda} = -\Gamma^{\lambda}_{\mu\nu} A^{\mu} dx^{\nu}, \tag{1}$$

where the Γ are functions of x.

On the other hand, if A^{μ} is a vector field, the components of A^{μ} at point P_2 are equal to $A^{\mu}+dA^{\mu}$, where

$$dA^{\mu} = \frac{\partial A^{\mu}}{\partial x^{\nu}} dx^{\nu}.$$

The difference $dA^{\mu} - \delta A^{\mu}$ in the vicinity of the point $A^{\lambda}+dA^{\lambda}$ is a vector

$$A^{\lambda}_{\nu} dx^{\nu} = \left(\frac{\partial A^{\lambda}}{\partial x^{\nu}} + \Gamma^{\lambda}_{\beta\nu} A^{\beta} \right) dx^{\nu},$$

which relates the components of the vector field at two infinitely close points. Furthermore, the quantity in parentheses is the covariant derivative of the vector field A^{λ}. The tensor character of an arbitrary second rank tensor, A^{μ}_{ν}, determines the affine transformation law for the infinitesimal displacement field. First, let us consider the transformation law of that tensor

$$A^{\rho *}_{\sigma} = \frac{\partial x^{\rho *}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x^{\sigma *}} A^{\mu}_{\nu}.$$

We substitute the covariant derivatives of both sides in the transformation equation, for the original coordinate system, and in the prime coordinate system, to find the equation

$$\frac{\partial}{\partial x^{\sigma *}} \left(\frac{\partial x^{\rho *}}{\partial x^{\eta}} A^{\eta} \right) + \Gamma^{\rho *}_{\tau\sigma} \frac{\partial x^{\tau *}}{\partial x^{\xi}} A^{\xi} = \frac{\partial x^{\rho *}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x^{\sigma *}} \frac{\partial A^{\mu}}{\partial x^{\nu}} + \frac{\partial x^{\rho *}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x^{\sigma *}} \Gamma^{\mu}_{\lambda\nu} A^{\lambda},$$

where we have used the transformation law $A^{\rho *} = \frac{\partial x^{\rho *}}{\partial x^{\eta}} A^{\eta}$. Due to the chain rule, the partial derivative of the first term of the first member can be transformed, and as a result we find the equation

$$\frac{\partial x^{\alpha}}{\partial x^{\sigma *}} \frac{\partial^2 x^{\rho *}}{\partial x^{\alpha} \partial x^{\eta}} A^{\eta} + \frac{\partial x^{\alpha}}{\partial x^{\sigma *}} \frac{\partial x^{\rho *}}{\partial x^{\eta}} \frac{\partial A^{\eta}}{\partial x^{\alpha}} + \Gamma^{\rho *}_{\tau\sigma} \frac{\partial x^{\tau *}}{\partial x^{\xi}} A^{\xi} = \frac{\partial x^{\rho *}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x^{\sigma *}} \frac{\partial A^{\mu}}{\partial x^{\nu}} + \frac{\partial x^{\rho *}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x^{\sigma *}} \Gamma^{\mu}_{\lambda\nu} A^{\lambda},$$

that after algebraic manipulations:

$$\Gamma^{\rho *}_{\eta\sigma} = \frac{\partial x^{\rho *}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x^{\sigma *}} \frac{\partial x^{\lambda}}{\partial x^{\eta *}} \Gamma^{\mu}_{\lambda\nu} - \frac{\partial^2 x^{\rho *}}{\partial x^{\alpha} \partial x^{\beta}} \frac{\partial x^{\alpha}}{\partial x^{\sigma *}} \frac{\partial x^{\beta}}{\partial x^{\eta *}} \tag{2}$$

or

$$\Gamma^{\rho *}_{\eta\sigma} = \frac{\partial x^{\rho *}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x^{\sigma *}} \frac{\partial x^{\lambda}}{\partial x^{\eta *}} \Gamma^{\mu}_{\lambda\nu} + \frac{\partial x^{\rho *}}{\partial x^{\alpha}} \frac{\partial^2 x^{\alpha}}{\partial x^{\eta *} \partial x^{\sigma *}}. \tag{3}$$

We call such equations, a pseudo tensor, like the law of affine transformation. For linear transformations it changes like a tensor, while for non-linear transformations a term appears that does not contain the expression to be transformed, but that only depends on the transformation coefficients.

Making equations (2) and (3) symmetric or antisymmetric with respect to the lower indices, we obtain the two equations

$$\Gamma^{\rho *}_{\underline{\eta\sigma}} \left(= \frac{1}{2} \left(\Gamma^{\rho *}_{\eta\sigma} + \Gamma^{\rho *}_{\sigma\eta} \right) \right) = \frac{\partial x^{\rho *}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x^{\sigma *}} \frac{\partial x^{\lambda}}{\partial x^{\eta *}} \Gamma^{\mu}_{\underline{\lambda\nu}} - \frac{\partial^2 x^{\rho *}}{\partial x^{\alpha} \partial x^{\beta}} \frac{\partial x^{\alpha}}{\partial x^{\sigma *}} \frac{\partial x^{\beta}}{\partial x^{\eta *}} \tag{4}$$

$$\Gamma^{\rho *}_{\underset{\vee}{\eta\sigma}} \left(= \frac{1}{2} \left(\Gamma^{\rho *}_{\eta\sigma} - \Gamma^{\rho *}_{\sigma\eta} \right) \right) = \frac{\partial x^{\rho *}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x^{\sigma *}} \frac{\partial x^{\lambda}}{\partial x^{\eta *}} \Gamma^{\mu}_{\underset{\vee}{\lambda\nu}}. \tag{5}$$

Therefore, the two parts (symmetric and antisymmetric) transform independently. On the other hand, the lower indices of the displacement field play completely different roles in the definition equation (1), so there are no reasons that force us to limit the displacement field by symmetry with respect to the lower indices. However, if done, this leads to the relativistic theory of gravitation.

3 Curvature

Although the G-field does not itself have tensor character, it implies the existence of a tensor. The latter is most easily obtained by displacing a vector A^λ according to $\delta A^\lambda = -\Gamma^\lambda_{\rho\sigma} A^\rho dx^\sigma$ along the circumference of an infinitesimal two-dimensional surface element and computing its change in one circuit. This change has vector character.

Let $\underset{0}{x}^\rho$ be the co-ordinates of a fixed point and x^τ those of another point on the circumference. Then $\xi^\tau = x^\tau - \underset{0}{x}^\tau$ is small for all points of the circumference and can be used as a basis for the definition of orders of magnitude. The integral $\oint \delta A^\lambda$ to be computed is then in more explicit notation

$$-\oint \underline{\Gamma}^\lambda_{\sigma\tau} \underline{A}^\sigma dx^\tau$$

or

$$-\oint \underline{\Gamma}^\lambda_{\sigma\tau} \underline{A}^\sigma d\xi^\tau.$$

Underlining of the quantities in the integrand indicates that they are to be taken for successive points of the circumference (and not for the initial point, $\xi^\tau = 0$).

We first compute in the lowest approximation the value of A^λ for an arbitrary point ξ^τ of the circumference. This lowest approximation is obtained by replacing in the integral, extended now over an open path, $\underline{\Gamma}^\lambda_{\sigma\tau}$ and $\underline{A}^\sigma$ by the values $\Gamma^\lambda_{\sigma\tau}$ and A^σ for the initial point of integration ($\xi^\tau = 0$). The integration gives then

$$\underline{A}^\lambda = A^\lambda - \Gamma^\lambda_{\sigma\tau} A^\sigma \int d\xi^\tau = A^\lambda - \Gamma^\lambda_{\sigma\tau} A^\sigma \xi^\tau.$$

What are neglected here, are terms of second or higher order in ξ. With the same approximation we obtain immediately

$$\underline{\Gamma}^\lambda_{\sigma\tau} = \Gamma^\lambda_{\sigma\tau} + \frac{\partial \Gamma^\lambda_{\sigma\tau}}{\partial x^\eta} \xi^\eta.$$

Inserting these expressions in the integral above we obtain first, with an appropriate choice of the summation indices,

$$-\oint \left(\Gamma^\lambda_{\sigma\tau} + \frac{\partial \Gamma^\lambda_{\sigma\tau}}{\partial x^\eta} \xi^\eta \right) \left(A^\lambda - \Gamma^\sigma_{\mu\nu} A^\mu \xi^\nu \right) d\xi^\tau$$

where all quantities, with the exception of ξ, have to be taken for the initial point of integration. We then find

$$-\Gamma^\lambda_{\sigma\tau} A^\sigma \oint d\xi^\tau - \frac{\partial \Gamma^\lambda_{\sigma\tau}}{\partial x^\eta} A^\sigma \oint \xi^\eta d\xi^\tau + \Gamma^\lambda_{\sigma\tau} \Gamma^\sigma_{\mu\nu} A^\mu \oint \xi^\nu d\xi^\tau$$

where the integrals are extended over the closed circumference. (The first term vanishes because its integral vanishes.) The term proportional to $(\xi)^2$ is omitted since it is of higher order. The two other terms may be combined into

$$\left[-\frac{\partial \Gamma^\lambda_{\eta\nu}}{\partial x^\mu} + \Gamma^\lambda_{\alpha\nu} \Gamma^\alpha_{\mu\eta} \right] A^\eta \oint \xi^\mu d\xi^\nu$$

This is the change ΔA^λ of the vector A^λ after displacement along the circumference. We have

$$\oint \xi^\mu d\xi^\nu = \oint d\left(\xi^\mu \xi^\nu\right) - \oint \xi^\nu d\xi^\mu = -\oint \xi^\nu d\xi^\mu.$$

This integral is thus antisymmetric in μ and ν, and in addition it has tensor character. We denote it by $f^{\underset{\vee}{\mu\nu}}$. If $f^{\underset{\vee}{\mu\nu}}$ were an arbitrary tensor, then the vector character of ΔA^λ would imply the tensor character of the bracketed expression in the last but one formula. As it is, we can only infer the tensor character of the bracketed expression if antisymmetrized with respect to μ and ν. This is the curvature tensor

$$R^\beta_{\xi\lambda\nu} \equiv \frac{\partial \Gamma^\beta_{\xi\lambda}}{\partial x^\nu} - \frac{\partial \Gamma^\beta_{\xi\nu}}{\partial x^\lambda} + \Gamma^\alpha_{\xi\lambda} \Gamma^\beta_{\alpha\nu} - \Gamma^\alpha_{\xi\nu} \Gamma^\beta_{\alpha\lambda}. \qquad (6)$$

and

$$R^{\rho *}_{\eta\sigma\kappa} \equiv \frac{\partial \Gamma^{\rho *}_{\eta\sigma}}{\partial x^{\kappa *}} - \frac{\partial \Gamma^{\rho *}_{\eta\kappa}}{\partial x^{\sigma *}} + \Gamma^{\alpha *}_{\eta\sigma} \Gamma^{\rho *}_{\alpha\kappa} - \Gamma^{\alpha *}_{\eta\kappa} \Gamma^{\rho *}_{\alpha\sigma}, \qquad (7)$$

where the transformation law is:

$$R^{*\tau}_{\rho\sigma\eta} = \frac{\partial x^{*\tau}}{\partial x^\lambda} \frac{\partial x^\mu}{\partial x^{*\rho}} \frac{\partial x^\nu}{\partial x^{*\sigma}} \frac{\partial x^\kappa}{\partial x^{*\eta}} R^\lambda_{\mu\nu\kappa}.$$

These equations are called the curvature tensor.

Also, we can contract the curvature tensor with respect to ρ and κ to obtain the second rank covariant tensor:

$$R_{\eta\sigma} = R^\rho_{\eta\sigma\rho} \equiv \frac{\partial \Gamma^\rho_{\eta\sigma}}{\partial x^\rho} + \Gamma^\alpha_{\eta\sigma} \Gamma^\rho_{\alpha\rho} - \left(\frac{\partial \Gamma^\rho_{\eta\rho}}{\partial x^\sigma} + \Gamma^\alpha_{\eta\rho} \Gamma^\rho_{\alpha\sigma} \right), \qquad (8)$$

which is often called a contracted curvature tensor.

4 The λ transformation

Curvature has an important property. For a displacement field a new field is defined using the formula

$$\Gamma^{\rho*}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + \delta^{\rho}_{\mu}\frac{\partial\lambda}{\partial x^{\nu}}, \tag{9}$$

where the Lambda's function, is an arbitrary function of the coordinates, and δ^{λ}_{μ} is the Kronecker's tensor defined by

$$\delta^{\rho}_{\mu} = \begin{cases} 1, & \text{if } \rho = \mu, \\ 0, & \text{if } \rho \neq \mu. \end{cases} \tag{10}$$

If the curvature tensor is formed in terms of $\Gamma^{\rho*}_{\mu\nu}$ by the second member of equation (1), then the function λ vanishes, that is, the equations are satisfied

$$R^{\lambda}_{\mu\nu\sigma}\left(\Gamma^{*}\right) = R^{\lambda}_{\mu\nu\sigma}\left(\Gamma\right),$$

$$R_{\mu\nu}\left(\Gamma^{*}\right) = R_{\mu\nu}\left(\Gamma\right).$$

The curvature equations are invariant under the transformation λ. From this we can say, a theory that contains only the infinitesimal displacement field in the curvature tensor cannot completely determine this field, but only a function λ, which remains arbitrary. In such a theory $\Gamma^{\rho*}_{\mu\nu}$ and $\Gamma^{\rho}_{\mu\nu}$ represent the same field.

5 The pseudo tensor $U^{\rho}_{\eta\sigma}$

It turns out that a new tensor can be formed from the contracted curvature tensor, by introducing a pseudo-tensor $U^{\rho}_{\eta\sigma}$ instead of $\Gamma^{\rho}_{\eta\sigma}$. In equation (8) the two terms that are linear in the displacement field can be formally combined into one, to define a new pseudo tensor, by the equation

$$U^{\rho}_{\eta\sigma} \equiv \Gamma^{\rho}_{\eta\sigma} - \delta^{\rho}_{\sigma}\Gamma^{\beta}_{\eta\beta}. \tag{11}$$

By contraction with respect to ρ y σ

$$U^{\rho}_{\eta\rho} = -3\Gamma^{\rho}_{\eta\rho},$$

we obtain the relation of $\Gamma^{\rho}_{\eta\sigma}$ in terms of $U^{\rho}_{\eta\sigma}$

$$\Gamma^{\rho}_{\eta\sigma} \equiv U^{\rho}_{\eta\sigma} - \frac{1}{3}\delta^{\rho}_{\sigma}U^{\beta}_{\eta\beta}. \tag{12}$$

Substituting equation (12) in equation (8), we find the expression:

$$S_{\eta\sigma} = \frac{\partial U^{\rho}_{\eta\sigma}}{\partial x^{\rho}} - \frac{1}{3}U^{\alpha}_{\eta\sigma}U^{\beta}_{\alpha\beta} + \frac{1}{9}U^{\beta}_{\eta\beta}U^{\beta}_{\sigma\beta} - U^{\alpha}_{\eta\rho}U^{\rho}_{\alpha\sigma} + \frac{1}{3}\left(U^{\beta}_{\eta\beta}U^{\alpha}_{\alpha\sigma} + U^{\alpha}_{\eta\rho}U^{\beta}_{\alpha\beta}\delta^{\rho}_{\sigma}\right) - \frac{1}{9}U^{\beta}_{\eta\beta}U^{\lambda}_{\alpha\lambda}\delta^{\alpha}_{\rho}\delta^{\rho}_{\sigma}$$

and we find the tensor, after doing index manipulations:

$$S_{\eta\sigma} \equiv \frac{\partial U^{\rho}_{\eta\sigma}}{\partial x^{\rho}} - U^{\alpha}_{\eta\rho}U^{\rho}_{\alpha\sigma} + \frac{1}{3}U^{\beta}_{\eta\beta}U^{\alpha}_{\alpha\sigma}, \tag{13}$$

where the new contracted curvature tensor is in terms of $U^{\rho}_{\eta\sigma}$.

If in equation (9) the $\Gamma^{\rho}_{\eta\sigma}$ are replaced by the $U^{\rho}_{\eta\sigma}$, then we obtain

$$U^{\rho*}_{\eta\sigma} = U^{\rho}_{\eta\sigma} + \delta^{\rho}_{\eta}\frac{\partial\lambda}{\partial x^{\sigma}} - \delta^{\rho}_{\sigma}\frac{\partial\lambda}{\partial x^{\eta}}. \tag{14}$$

This equation defines the transformation λ for $U^{\rho}_{\eta\sigma}$. If in equation (3), we replace $\Gamma^{\rho*}_{\mu\nu}$ by $U^{\rho*}_{\mu\nu}$ with the help of (5.2) we can calculate the equation

$$U^{\rho*}_{\eta\sigma} = \frac{\partial x^{\rho*}}{\partial x^{\mu}}\frac{\partial x^{\nu}}{\partial x^{\sigma*}}\frac{\partial x^{\xi}}{\partial x^{\eta*}}U^{\mu}_{\xi\nu} + \frac{\partial x^{\rho*}}{\partial x^{\alpha}}\frac{\partial^{2}x^{\alpha}}{\partial x^{\eta*}\partial x^{\sigma*}} + \frac{1}{3}U^{\beta*}_{\eta\beta}\delta^{\rho*}_{\sigma} - \frac{\partial x^{\rho*}}{\partial x^{\mu}}\frac{\partial x^{\nu}}{\partial x^{\sigma*}}\frac{\partial x^{\xi}}{\partial x^{\eta*}}\left(\frac{1}{3}U^{\beta}_{\xi\beta}\delta^{\mu}_{\nu}\right),$$

and when we use the definition of equation (10), the third term of the preceding equation is transformed, to arrive at the expression

$$U^{\rho*}_{\eta\sigma} = \frac{\partial x^{\rho*}}{\partial x^{\mu}}\frac{\partial x^{\nu}}{\partial x^{\sigma*}}\frac{\partial x^{\xi}}{\partial x^{\eta*}}U^{\mu}_{\xi\nu} + \frac{\partial x^{\rho*}}{\partial x^{\alpha}}\frac{\partial^{2}x^{\alpha}}{\partial x^{\eta*}\partial x^{\sigma*}} + \frac{1}{3}U^{\beta*}_{\eta\beta}\delta^{\rho*}_{\sigma} - \frac{1}{3}\delta^{\rho*}_{\sigma}\frac{\partial x^{\xi}}{\partial x^{\eta*}}U^{\beta}_{\xi\beta}.$$

By virtue of the equation $\Gamma^{\rho*}_{\eta\rho} - \frac{\partial x^{\xi}}{\partial x^{\eta*}}\Gamma^{\nu}_{\xi\nu} = \frac{\partial x^{\rho*}}{\partial x^{\alpha}}\frac{\partial^{2}x^{\alpha}}{\partial x^{\eta*}\partial x^{\rho*}}$, finally we find the equations

$$U^{\rho*}_{\eta\sigma} = \frac{\partial x^{\rho*}}{\partial x^{\mu}}\frac{\partial x^{\nu}}{\partial x^{\sigma*}}\frac{\partial x^{\lambda}}{\partial x^{\eta*}}U^{\mu}_{\lambda\nu} + \frac{\partial x^{\rho*}}{\partial x^{\alpha}}\frac{\partial^{2}x^{\alpha}}{\partial x^{\eta*}\partial x^{\sigma*}} - \delta^{\rho*}_{\sigma*}\frac{\partial x^{\xi*}}{\partial x^{\alpha}}\frac{\partial^{2}x^{\alpha}}{\partial x^{\eta*}\partial x^{\xi*}}. \tag{15}$$

Transformation equation for the pseudo tensor $U^{\mu}_{\lambda\nu}$ in two arbitrary coordinate systems. Note that the indexes for both systems take the values from 1 to 4 independently of each other, even if the same letter is used. Observing this formula, it is worth noting that according to the last term it is not a symmetric transposition with respect to the indices η and ξ. This circumstance can be clarified by showing that this transformation can be considered as a composition of a coordinate transformation, which is a symmetric transposition, and a λ transformation. To see it we write the last term in the form:

$$-\frac{1}{2}\left[\delta^{\rho*}_{\sigma*}\frac{\partial x^{\tau*}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x^{\tau*}\partial x^{\eta*}}+\delta^{\rho*}_{\eta*}\frac{\partial x^{\tau*}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x^{\sigma*}\partial x^{\tau*}}\right]+\frac{1}{2}\left[\delta^{\rho*}_{\eta*}\frac{\partial x^{\tau*}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x^{\sigma*}\partial x^{\tau*}}-\delta^{\rho*}_{\sigma*}\frac{\partial x^{\tau*}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x^{\tau*}\partial x^{\eta*}}\right].$$

The first of these two terms is a symmetric transposition. We combine this term with the first two terms of the second member of (15) in an expression $K^{\rho*}_{\eta\sigma}$. Now let us consider, what we get, if the transformation

$$U^{\rho*}_{\eta\sigma}=K^{\rho*}_{\eta\sigma}$$

is followed by the transformation λ

$$U^{\rho**}_{\eta\sigma}=U^{\rho*}_{\eta\sigma}+\delta^{\rho*}_{\eta*}\frac{\partial\lambda}{\partial x^{\sigma*}}-\delta^{\rho*}_{\sigma*}\frac{\partial\lambda}{\partial x^{\eta*}}.$$

The composition gives

$$U^{\rho**}_{\eta\sigma}=K^{\rho*}_{\eta\sigma}+\delta^{\rho*}_{\eta*}\frac{\partial\lambda}{\partial x^{\sigma*}}-\delta^{\rho*}_{\sigma*}\frac{\partial\lambda}{\partial x^{\eta*}}.$$

This implies that (15) can be considered as such a composition provided that the term

$$\frac{1}{2}\left[\delta^{\rho*}_{\eta*}\frac{\partial x^{\tau*}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x^{\sigma*}\partial x^{\tau*}}-\delta^{\rho*}_{\sigma*}\frac{\partial x^{\tau*}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x^{\tau*}\partial x^{\eta*}}\right]$$

can be expressed in the form $\delta^{\rho*}_{\eta*}\frac{\partial\lambda}{\partial x^{\sigma*}}-\delta^{\rho*}_{\sigma*}\frac{\partial\lambda}{\partial x^{\eta*}}$. For which, it is sufficient to show that there exists a λ such that

$$\frac{1}{2}\frac{\partial x^{\tau*}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x^{\eta*}\partial x^{\tau*}}=\frac{\partial\lambda}{\partial x^{\eta*}}\tag{16}$$

and

$$\frac{1}{2}\frac{\partial x^{\tau*}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x^{\sigma*}\partial x^{\tau*}}=\frac{\partial\lambda}{\partial x^{\sigma*}}.\tag{17}$$

To transform these equations. First, we have to express $\frac{\partial x^{\tau*}}{\partial x^{\sigma}}$ by the inverse transformation coefficients, $\frac{\partial x^{\alpha}}{\partial x^{\beta*}}$. On one side,

$$\frac{\partial x^{\lambda}}{\partial x^{\tau*}}\frac{\partial x^{\tau*}}{\partial x^{\sigma}}=\delta^{\lambda}_{\sigma}.\tag{18}$$

For other

$$\frac{\partial x^{\lambda}}{\partial x^{\tau*}}V^{\sigma}_{\tau*}=\frac{\partial x^{\lambda}}{\partial x^{\tau*}}\frac{\partial D}{\partial\left(\frac{\partial x^{\sigma}}{\partial x^{\tau*}}\right)}=D\delta^{\lambda}_{\sigma}.\tag{19}$$

Here $V^{\sigma}_{\tau*}$ represents the factor that accompanies $\frac{\partial x^{\sigma}}{\partial x^{\tau*}}$ and can be expressed as the derivative of the determinant $D=\left|\frac{\partial x^{\alpha}}{\partial x^{\beta*}}\right|$ with respect to $\frac{\partial x^{\sigma}}{\partial x^{\tau*}}$. Therefore, it has

$$\frac{\partial x^{\lambda}}{\partial x^{\tau*}}\frac{\partial\log D}{\partial\left(\frac{\partial x^{\sigma}}{\partial x^{\tau*}}\right)}=\delta^{\lambda}_{\sigma}.\tag{20}$$

From (18) and (19) it follows that

$$\frac{\partial x^{\tau*}}{\partial x^{\sigma}}=\frac{\partial\log D}{\partial\left(\frac{\partial x^{\sigma}}{\partial x^{\tau*}}\right)}.$$

With this relationship, the first member of equations (16) and (17) can be written as

$$\frac{1}{2}\frac{\partial\log D}{\partial\left(\frac{\partial x^{\alpha}}{\partial x^{\tau*}}\right)}\frac{\partial}{\partial x^{\sigma*}}\left(\frac{\partial x^{\alpha}}{\partial x^{\tau*}}\right)=\frac{1}{2}\frac{\partial\log D}{\partial x^{\sigma*}}.$$

This implies that equations (16) and (17) are satisfied by

$$\lambda = \frac{1}{2}\log D.$$

Which proves that the transformation (15) can be considered as a composition of the symmetric transpose transformation

$$U^{\rho *}_{\eta\sigma} = \frac{\partial x^{\rho *}}{\partial x^{\mu}}\frac{\partial x^{\nu}}{\partial x^{\sigma *}}\frac{\partial x^{\lambda}}{\partial x^{\eta *}}U^{\mu}_{\lambda\nu} + \frac{\partial x^{\rho *}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x^{\eta *}\partial x^{\sigma *}} - \frac{1}{2}\left[\delta^{\rho *}_{\sigma *}\frac{\partial x^{\tau *}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x^{\tau *}\partial x^{\eta *}} + \delta^{\rho *}_{\eta *}\frac{\partial x^{\tau *}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x^{\sigma *}\partial x^{\tau *}}\right], \quad (21)$$

and a transformation λ. Equation (21) can be taken instead of (15) as the transformation formula for U. Any transformation of the field U, which only changes its representation form, can be expressed as a composition of a coordinate transformation according to (21) and a transformation λ.

6 Variational principle and field equations

The task of finding the field equations, of a variational principle, has the advantage that the compatibility of the resulting systems of equations is ensured, and that the differential identities related to covariance, the "Bianchi identities", in addition to the laws of divergence, result in a systematic way. When considering the action integral, it is required as integrating at a scalar density. We build this density in such a way that an action is postulated of the form

$$S\left(= \int d^4x\sqrt{-g}g^{\mu\nu}S_{\mu\nu}\right) = \int dt\int d^3x\mathscr{L}\left(\sqrt{-g}g^{\mu\nu}, U^{\rho}_{\eta\sigma}, \frac{\partial U^{\rho}_{\eta\sigma}}{\partial x^{\lambda}}\right),$$

where $\sqrt{-|g_{\mu\nu}|} = \sqrt{-g}$.

The variational principle is

$$\delta S = \int dt\int d^3x\delta\left(\sqrt{-g}g^{\mu\nu}S_{\mu\nu}\right) = 0,$$

where $\sqrt{-g}g^{\mu\nu}$ and $U^{\rho}_{\eta\sigma}$ vary independently. The variations of these fields cancel out at the border of the intergration domain and the variation of the Lagrangian density, $\mathcal{L}_{campo} = \sqrt{-g}g^{\mu\nu}S_{\mu\nu}$, gives:

$$\int dt\int d^3x\left\{\delta\left(\sqrt{-g}g^{\eta\sigma}\right)S_{\eta\sigma} - \sqrt{-g}n^{\mu\nu}_{\lambda}\delta U^{\lambda}_{\mu\nu} + \frac{\partial}{\partial x^{\xi}}\left[\sqrt{-g}g^{\mu\nu}\delta U^{\xi}_{\mu\nu}\right]\right\} = 0,$$

where $S_{\mu\nu}$ is the contracted curvature tensor and $n^{\mu\nu}_{\lambda}$ is a pseudo tensor. The last term does not provide any information, since $\delta U^{\rho}_{\eta\sigma}$ vanishes at the boundary. From this we obtain the field equations

$$S_{\eta\sigma} \equiv \frac{\partial U^{\rho}_{\eta\sigma}}{\partial x^{\rho}} - U^{\alpha}_{\eta\rho}U^{\rho}_{\alpha\sigma} + \frac{1}{3}U^{\beta}_{\eta\beta}U^{\alpha}_{\alpha\sigma} = 0, \quad (22)$$

$$\sqrt{-g}n^{\mu\nu}_{\lambda} = \frac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\mu\nu}\right) + \sqrt{-g}g^{\eta\nu}\left(U^{\mu}_{\eta\lambda} - \frac{1}{3}\delta^{\mu}_{\lambda}U^{\xi}_{\eta\xi}\right) + \sqrt{-g}g^{\mu\eta}\left(U^{\nu}_{\eta\lambda} - \frac{1}{3}\delta^{\mu}_{\lambda}U^{\xi}_{\eta\xi}\right), \quad (23)$$

which are invariant with respect to coordinate transformations and the transformation λ.

In the case of the symmetric field we obtain the fields equations most simply in the following manner. We use as Lagrangian function the scalar density

$$\mathscr{L}_G = \sqrt{-g}g^{\mu\nu}R_{\mu\nu} \quad (24)$$

where $R_{\mu\nu}$is the curvature tensor in the relativistic theory of gravitation.

If we vary the volume integral of $\mathscr{L}_G$, i. e.

$$\begin{aligned}\delta\int\mathscr{L}_G d^4x = -\int d^4x\left[\tfrac{\partial}{\partial x^{\mu}}\left(\sqrt{-g}g^{\mu\nu}\right) + \sqrt{-g}g^{\lambda\nu}\Gamma^{\mu}_{\lambda\rho} + \sqrt{-g}g^{\mu\lambda}\Gamma^{\nu}_{\rho\lambda} - \sqrt{-g}g^{\mu\nu}\Gamma^{\lambda}_{\rho\lambda}\right]\delta\Gamma^{\rho}_{\mu\nu} \\ + \int d^4x\delta^{\nu}_{\rho}\left[\tfrac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\mu\nu}\right) + \sqrt{-g}g^{\lambda\sigma}\Gamma^{\lambda}_{\lambda\sigma}\right]\delta\Gamma^{\rho}_{\mu\nu} - \int d^4x\sqrt{-g}\left(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R\right)\delta g_{\mu\nu}\end{aligned}$$

independiently with respect to Γ and g, then variation with respect to Γ yields

$$\begin{aligned}-\left[\tfrac{\partial}{\partial x^{\mu}}\left(\sqrt{-g}g^{\mu\nu}\right) + \sqrt{-g}g^{\lambda\nu}\Gamma^{\mu}_{\lambda\rho} + \sqrt{-g}g^{\mu\lambda}\Gamma^{\nu}_{\rho\lambda} - \sqrt{-g}g^{\mu\nu}\Gamma^{\lambda}_{\rho\lambda}\right] \\ +\delta^{\nu}_{\rho}\left[\tfrac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\mu\nu}\right) + \sqrt{-g}g^{\lambda\sigma}\Gamma^{\lambda}_{\lambda\sigma}\right] = 0\end{aligned}$$

or

$$\frac{\partial g_{\mu\nu}}{\partial x^{\rho}} - g_{\alpha\nu}\Gamma^{\alpha}_{\mu\rho} - g_{\mu\alpha}\Gamma^{\alpha}_{\rho\nu} = 0 \tag{25}$$

and variation with respect to g yields the equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$, or $R_{\mu\nu} = 0$.

The variation of the gravitational action, $\int d^4x\mathscr{L}_G$, with respect to $g_{\mu\nu}$ leads to the Einstein's field equations of general relativity, and the variation with respect to the affine connection, $\Gamma^{\lambda}_{\mu\nu}$, reveals that the connection is necessarily the metric connection.

Another way of treating the Lagrangian density of the action proposed in the relativistic theory of the asymmetric field can be characterized as follows. From $\sqrt{-g}g^{\mu\nu}$ and $R_{\mu\nu}$ construct a Lagrangian density-function $\mathscr{L}$ whose integral we vary independiently with respect to $\sqrt{-g}g^{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$. The variation of the integral of $\mathscr{L}$

$$\begin{aligned}\delta\int\mathscr{L}d^4x = &-\int d^4x\left[\tfrac{\partial}{\partial x^{\mu}}\left(\sqrt{-g}g^{\mu\nu}\right) + \sqrt{-g}g^{\lambda\nu}\Gamma^{\mu}_{\lambda\rho} + \sqrt{-g}g^{\mu\lambda}\Gamma^{\nu}_{\rho\lambda} - \sqrt{-g}g^{\mu\nu}\Gamma^{\lambda}_{\rho\lambda}\right]\delta\Gamma^{\rho}_{\mu\nu}\\ &+\int d^4x\delta^{\nu}_{\rho}\left[\tfrac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\mu\nu}\right) + \sqrt{-g}g^{\lambda\sigma}\Gamma^{\lambda}_{\lambda\sigma}\right]\delta\Gamma^{\rho}_{\mu\nu} + \int d^4x\delta\left(\sqrt{-g}g^{\mu\nu}\right)R_{\mu\nu}\end{aligned}$$

with respect to $\sqrt{-g}g^{\mu\nu}$yields the 16 equations

$$R_{\mu\nu} = 0 \tag{26}$$

the variation with respect to the $\Gamma^{\alpha}_{\mu\nu}$ at first the 64 equations (see Appendix A)

$$\frac{\partial\left(\sqrt{-g}g^{\mu\nu}\right)}{\partial x^{\rho}} + \sqrt{-g}g^{\lambda\nu}\Gamma^{\mu}_{\lambda\rho} + \sqrt{-g}g^{\mu\lambda}\Gamma^{\nu}_{\rho\lambda} - \sqrt{-g}g^{\mu\nu}\Gamma^{\lambda}_{\rho\lambda} - \delta^{\nu}_{\rho}\left[\frac{\partial\left(\sqrt{-g}g^{\mu\sigma}\right)}{\partial x^{\sigma}} + \sqrt{-g}g^{\lambda\sigma}\Gamma^{\mu}_{\lambda\sigma}\right] = 0. \tag{27}$$

If there were no bracket in (27) would imply the vanishing of

$$\frac{\partial\left(\sqrt{-g}g^{\mu\nu}\right)}{\partial x^{\rho}} + \sqrt{-g}g^{\lambda\nu}\Gamma^{\mu}_{\lambda\rho} + \sqrt{-g}g^{\mu\lambda}\Gamma^{\nu}_{\rho\lambda} - \sqrt{-g}g^{\mu\nu}\Gamma^{\lambda}_{\rho\lambda}, \tag{28}$$

however, this would require the vanishing $\Gamma^{\lambda}_{\underset{\vee}{\nu\lambda}}$. We can resolve this difficulty in the following manner. We can compute the equations of (27)

$$\begin{aligned}&\tfrac{\partial}{\partial x^{\rho}}\left(\sqrt{-g}g^{\underline{\mu\nu}}\right) + \sqrt{-g}g^{\underline{\lambda\nu}}\Gamma^{\mu}_{\underline{\lambda\rho}} + \sqrt{-g}g^{\underset{\vee}{\lambda\nu}}\Gamma^{\mu}_{\underset{\vee}{\lambda\rho}} + \sqrt{-g}g^{\underline{\mu\lambda}}\Gamma^{\nu}_{\underline{\rho\lambda}} + \sqrt{-g}g^{\underset{\vee}{\mu\lambda}}\Gamma^{\nu}_{\underset{\vee}{\rho\lambda}} - \sqrt{-g}g^{\underline{\mu\nu}}\Gamma^{\lambda}_{\underline{\rho\lambda}}\\ &-\sqrt{-g}g^{\underset{\vee}{\mu\nu}}\Gamma^{\lambda}_{\underset{\vee}{\rho\lambda}} - \delta^{\nu}_{\rho}\left[\tfrac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\underline{\mu\lambda}}\right) + \sqrt{-g}g^{\underline{\lambda\sigma}}\Gamma^{\mu}_{\underline{\lambda\sigma}} + \sqrt{-g}g^{\underset{\vee}{\lambda\sigma}}\Gamma^{\mu}_{\underset{\vee}{\lambda\sigma}}\right] = 0,\end{aligned} \tag{29}$$

and

$$\begin{aligned}&\tfrac{\partial}{\partial x^{\rho}}\left(\sqrt{-g}g^{\underset{\vee}{\mu\nu}}\right) + \sqrt{-g}g^{\underline{\lambda\nu}}\Gamma^{\mu}_{\underset{\vee}{\lambda\rho}} + \sqrt{-g}g^{\underset{\vee}{\lambda\nu}}\Gamma^{\mu}_{\underline{\lambda\rho}} + \sqrt{-g}g^{\underline{\mu\lambda}}\Gamma^{\nu}_{\underset{\vee}{\rho\lambda}} + \sqrt{-g}g^{\underset{\vee}{\mu\lambda}}\Gamma^{\nu}_{\underline{\rho\lambda}} - \sqrt{-g}g^{\underline{\mu\nu}}\Gamma^{\lambda}_{\underset{\vee}{\rho\lambda}}\\ &-\sqrt{-g}g^{\underset{\vee}{\mu\nu}}\Gamma^{\lambda}_{\underline{\rho\lambda}} - \delta^{\nu}_{\rho}\left[\tfrac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\underset{\vee}{\mu\lambda}}\right) + \sqrt{-g}g^{\underline{\lambda\sigma}}\Gamma^{\mu}_{\underset{\vee}{\lambda\sigma}} + \sqrt{-g}g^{\underset{\vee}{\lambda\sigma}}\Gamma^{\mu}_{\underline{\lambda\sigma}}\right] = 0.\end{aligned} \tag{30}$$

If we contract the equation (30) with respect to ν and ρ

$$\frac{3}{2}\frac{\partial}{\partial x^{\lambda}}\left(\sqrt{-g}g^{\underset{\vee}{\mu\lambda}}\right) + \sqrt{-g}g^{\underline{\mu\lambda}}\Gamma^{\rho}_{\underset{\vee}{\lambda\rho}} = 0. \tag{31}$$

From this, we can deduce that the necessary and sufficient condition for $\Gamma^{\rho}_{\underset{\vee}{\rho\lambda}} = 0$ is that $\partial_{\rho}\left(\sqrt{-g}g^{\underset{\vee}{\mu\rho}}\right) = 0$. In order to satisfy this identically it suffices to asume

$$\sqrt{-g}g^{\underset{\vee}{\mu\nu}} = \frac{\partial}{\partial x^{\rho}}\left(\sqrt{-g}g^{\mu\nu\rho}\right) \tag{32}$$

where $\sqrt{-g}g^{\underset{\vee}{\mu\nu}}$ is a tensor density which is antisymmetric in all three indices. That is, we require that $\sqrt{-g}g^{\underset{\vee}{\mu\nu}}$ be derived from a "vector potential". Therefore, we substitute in the Lagrangian density

$$\sqrt{-g}g^{\mu\nu} = \sqrt{-g}g^{\underline{\mu\nu}} + \frac{\partial}{\partial x^{\rho}}\left(\sqrt{-g}g^{\mu\nu\rho}\right) \tag{33}$$

and vary independiently with respect to the Γ then yields

$$\frac{\partial g_{\mu\nu}}{\partial x^{\rho}} - g_{\alpha\nu}\Gamma^{\alpha}_{\mu\rho} - g_{\mu\alpha}\Gamma^{\alpha}_{\rho\nu} = 0. \tag{34}$$

The variation respect to the $\sqrt{-g}g^{\underline{\mu\nu}}$ and $\sqrt{-g}g^{\mu\lambda\tau}$ yields the equations

$$R_{\underline{\mu\nu}} = 0, \tag{35}$$

$$\frac{\partial}{\partial x^{\lambda}} R_{\underset{\vee}{\mu\nu}} + \frac{\partial}{\partial x^{\mu}} R_{\underset{\vee}{\nu\lambda}} + \frac{\partial}{\partial x^{\nu}} R_{\underset{\vee}{\lambda\mu}} = 0. \tag{36}$$

If we omit $\frac{\partial}{\partial x^{\nu}}\left(\sqrt{-g}g^{\underset{\vee}{\mu\nu}}\right) = 0$, then the system of field equations not weakened is therefore:

$$R_{\underline{\mu\nu}} = 0,$$

$$\frac{\partial}{\partial x^{\lambda}} R_{\underset{\vee}{\mu\nu}} + \frac{\partial}{\partial x^{\mu}} R_{\underset{\vee}{\nu\lambda}} + \frac{\partial}{\partial x^{\nu}} R_{\underset{\vee}{\lambda\mu}} = 0,$$

$$\Gamma^{\lambda}_{\underset{\vee}{\mu\lambda}} = 0,$$

$$\frac{\partial g_{\mu\nu}}{\partial x^{\rho}} - g_{\alpha\nu}\Gamma^{\alpha}_{\mu\rho} - g_{\mu\alpha}\Gamma^{\alpha}_{\rho\nu} = 0.$$

The field equations (22) and (23) are the field equations of the relativistic asymmetric field theory. Furthermore, the system of field equations not weakened is equivalent to the system (22) and (23), since it has been deduced from the same integral by the variational method [20–24] (see other versions of non-symmetric theories of gravitation [20–33] and quantum gravity [34–37]).

7 Differential Identities

The field equations are not independent of each other. To do this, we take an infinitesimal transformation defined of the form

$$x^{\mu} \longrightarrow x^{\mu *} = x^{\mu} + \xi^{\mu}, \tag{37}$$

where ξ^{μ} is an infinitesimal vector.

Under the transformation law

$$\sqrt{-g}g^{\rho\sigma *} = \frac{\partial x^{\rho *}}{\partial x^{\mu}}\frac{\partial x^{\sigma *}}{\partial x^{\nu}}\sqrt{-g}g^{\mu\nu}\left|\frac{\partial x^{\tau}}{\partial x^{\tau *}}\right|,$$

and with the help of equation (37), we obtain the variation (see appendix B)

$$\delta\left(\sqrt{-g}g^{\mu\nu}\right) = \sqrt{-g}g^{\lambda\nu}\frac{\partial \xi^{\mu}}{\partial x^{\lambda}} + \sqrt{-g}g^{\mu\lambda}\frac{\partial \xi^{\nu}}{\partial x^{\lambda}} - \sqrt{-g}g^{\mu\nu}\frac{\partial \xi^{\lambda}}{\partial x^{\lambda}} + \left[-\frac{\partial\left(\sqrt{-g}g^{\mu\nu}\right)}{\partial x^{\lambda}}\xi^{\lambda}\right], \tag{38}$$

where $\delta\left(\sqrt{-g}g^{\mu\nu}\right) = \sqrt{-g}g^{\mu\nu *} - \sqrt{-g}g^{\mu\nu}$.

The variation for the pseudo tensor $U^{\rho}_{\mu\nu}$; with the help of equation (37) and equation (15), it is given by (see appendix C):

$$\delta U^{\rho}_{\eta\sigma} = U^{\rho *}_{\eta\sigma} - U^{\rho}_{\eta\sigma} = U^{\lambda}_{\eta\sigma}\frac{\partial \xi^{\rho}}{\partial x^{\lambda}} - U^{\rho}_{\lambda\sigma}\frac{\partial \xi^{\lambda}}{\partial x^{\eta}} - U^{\rho}_{\eta\lambda}\frac{\partial \xi^{\lambda}}{\partial x^{\sigma}} - \frac{\partial^{2}\xi^{\mu}}{\partial x^{\eta}\partial x^{\sigma}} + \left[-\frac{\partial U^{\rho}_{\eta\sigma}}{\partial x^{\lambda}}\xi^{\lambda}\right]. \tag{39}$$

In the variational calculus, the variations (38) and (39) represent the variations for fixed points of the coordinates. To obtain these, the terms in the parentheses have to be added. If these transform variations are substituted; that is, equations (38) and (39), in the integral

$$\int dt \int d^{3}x \left\{\delta\left(\sqrt{-g}g^{\eta\sigma}\right)S_{\eta\sigma} - \sqrt{-g}n^{\mu\nu}_{\lambda}\delta U^{\lambda}_{\mu\nu}\right\},$$

it becomes, therefore, identically null. Considering that this integral depends linearly and homogeneously on ξ^{μ} and its derivatives, it can be represented as follows

$$\int dt \int d^3x \sqrt{-g} m_\mu \xi^\mu = 0,$$

and through repeated integration by parts, the differential identities of the integrand are deduced, ($\sqrt{-g} m_\mu \equiv 0$), i. e.,

$$\sqrt{-g} n_\rho^{\eta\sigma} \frac{\partial U_{\eta\sigma}^\rho}{\partial x^\nu} + \frac{\partial}{\partial x^\lambda} \left(\sqrt{-g} n_\nu^{\eta\sigma} U_{\eta\sigma}^\lambda \right) - \frac{\partial}{\partial x^\eta} \left(\sqrt{-g} n_\rho^{\eta\sigma} U_{\nu\sigma}^\rho \right) - \frac{\partial}{\partial x^\sigma} \left(\sqrt{-g} n_\rho^{\eta\sigma} U_{\eta\nu}^\rho \right)$$
$$- \frac{\partial^2 \left(\sqrt{-g} n_\nu^{\eta\sigma} \right)}{\partial x^\sigma \partial x^\eta} - \frac{\partial \mathfrak{g}^{\eta\sigma}}{\partial x^\nu} S_{\eta\sigma} - \frac{\partial}{\partial x^\lambda} \left(\mathfrak{g}^{\lambda\sigma} S_{\nu\sigma} \right) - \frac{\partial}{\partial x^\lambda} \left(\mathfrak{g}^{\eta\lambda} S_{\eta\nu} \right) + \frac{\partial}{\partial x^\nu} \left(\mathfrak{g}^{\eta\sigma} S_{\eta\sigma} \right) = 0, \tag{40}$$

or

$$-\sqrt{-g} n_\rho^{\eta\sigma} \frac{\partial U_{\eta\sigma}^\rho}{\partial x^\nu} + \frac{\partial}{\partial x^\eta} \left[\sqrt{-g} n_\nu^{\lambda\sigma} U_{\lambda\sigma}^\eta - \sqrt{-g} n_\rho^{\eta\sigma} U_{\eta\nu}^\rho - \sqrt{-g} n_\rho^{\eta\sigma} U_{\nu\sigma}^\rho - \frac{\partial \left(\sqrt{-g} n^{\eta\sigma} \right)}{\partial x^\sigma} \right]$$
$$- \frac{\partial \mathfrak{g}^{\eta\sigma}}{\partial x^\nu} S_{\eta\sigma} - \frac{\partial}{\partial x^\lambda} \left(\mathfrak{g}^{\lambda\sigma} S_{\nu\sigma} + \mathfrak{g}^{\eta\lambda} S_{\eta\nu} - \delta_\nu^\lambda \mathfrak{g}^{\eta\sigma} S_{\eta\sigma} \right) = 0, \tag{41}$$

with $\mathfrak{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$. These are four differential identities for the first terms of the field equations (22) and (23), which are equivalent to the Bianchi identities.

There is a fifth identity corresponding to the invariance of the action integral with respect to infinitesimal λ transformations. By substituting $\delta \left(\sqrt{-g} g^{\mu\nu} \right) = 0$ and $\delta U_{\mu\nu}^\rho = \delta_\mu^\rho \partial_\nu \lambda - \delta_\nu^\rho \partial_\mu \lambda$ in the integral

$$\int dt \int d^3x \left\{ \delta \left(\sqrt{-g} g^{\eta\sigma} \right) S_{\eta\sigma} - \sqrt{-g} n_\lambda^{\mu\nu} \delta U_{\mu\nu}^\lambda \right\},$$

we find,

$$\int \sqrt{-g} n_\rho^{\mu\nu} \left(\delta_\mu^\rho \frac{\partial \lambda}{\partial x^\nu} - \delta_\nu^\rho \frac{\partial \lambda}{\partial x^\mu} \right),$$

after an integration by parts:

$$2 \int \frac{\partial}{\partial x^\lambda} \left(\sqrt{-g} n_\sigma^{\underset{\vee}{\lambda\sigma}} \right) \lambda d^4x = 0$$

the desired identity

$$\frac{\partial \left(\sqrt{-g} n_\sigma^{\underset{\vee}{\lambda\sigma}} \right)}{\partial x^\lambda} \equiv 0. \tag{42}$$

For $\sqrt{-g} n_\eta^{\underset{\vee}{\mu\eta}}$, from field equations (23), we find

$$\sqrt{-g} n_\eta^{\underset{\vee}{\mu\eta}} = \frac{\partial \left(\sqrt{-g} g^{\underset{\vee}{\mu\eta}} \right)}{\partial x^\eta} = 0,$$

equation that expresses the nullity of the magnetic density and $g^{\underset{\vee}{\mu\eta}}$ plays the role of the electromagnetic potential vector. If we name

$$(G_\lambda \equiv) \frac{\partial}{\partial x^\alpha} \left(\sqrt{-g} g^{\underset{\vee}{\mu\rho}} \right) = 0 \tag{43}$$

then

$$\left(G_{\underset{\vee}{\mu\nu}} \equiv \right) \frac{\partial^2 \left(\sqrt{-g} g^{\underset{\vee}{\mu\nu}} \right)}{\partial x^\alpha \partial x^\alpha} = 0. \tag{44}$$

We now have the identity

$$\frac{\partial G_{\underset{\vee}{\mu\nu}}}{\partial x^\nu} - \frac{\partial^3 \left(\sqrt{-g} g^{\underset{\vee}{\mu\nu}} \right)}{\partial x^\nu \partial x^\alpha \partial x^\alpha} \equiv 0$$

or

$$\frac{\partial G_{\underset{\vee}{\mu\nu}}}{\partial x^{\nu}} - \frac{\partial^2 G_{\mu}}{\partial x^{\alpha}\partial x^{\alpha}} \equiv 0. \tag{45}$$

After differenciate equation (45) with to respect to ρ, we found the next expression

$$\frac{\partial G_{\underset{\vee}{\mu\nu}}}{\partial x^{\rho}} - \frac{\partial^2}{\partial x^{\alpha}\partial x^{\alpha}}\left[\frac{\partial\left(\sqrt{-g}g^{\underset{\vee}{\mu\nu}}\right)}{\partial x^{\rho}}\right] = 0. \tag{46}$$

After applying two cyclic permutations of the indices μ, ν and ρ two further equations appear. Then, we obtain

$$\frac{\partial G_{\underset{\vee}{\mu\nu}}}{\partial x^{\rho}} + \frac{\partial G_{\underset{\vee}{\rho\mu}}}{\partial x^{\nu}} + \frac{\partial G_{\underset{\vee}{\nu\rho}}}{\partial x^{\mu}} - \frac{\partial^2}{\partial x^{\alpha}\partial x^{\alpha}}\left[\frac{\partial\left(\sqrt{-g}g_{\underset{\vee}{\mu\nu}}\right)}{\partial x^{\rho}} + \frac{\partial\left(\sqrt{-g}g_{\underset{\vee}{\rho\mu}}\right)}{\partial x^{\nu}} + \frac{\partial\left(\sqrt{-g}g_{\underset{\vee}{\nu\rho}}\right)}{\partial x^{\mu}}\right] \equiv 0. \tag{47}$$

Therefore, the equations which according to field equations hold for an antisymmetric field are

$$\frac{\partial}{\partial x^{\alpha}}\left(\sqrt{-g}g^{\underset{\vee}{\mu\rho}}\right) = 0 \tag{48}$$

$$\frac{\partial^2}{\partial x^{\alpha}\partial x^{\alpha}}\left[\frac{\partial\left(\sqrt{-g}g_{\underset{\vee}{\mu\nu}}\right)}{\partial x^{\rho}} + \frac{\partial\left(\sqrt{-g}g_{\underset{\vee}{\rho\mu}}\right)}{\partial x^{\nu}} + \frac{\partial\left(\sqrt{-g}g_{\underset{\vee}{\nu\rho}}\right)}{\partial x^{\mu}}\right] \equiv 0. \tag{49}$$

If, in the equation (49), the expression inside the parentheses would itself vanish, then we would have Maxwell's equations for empty space.

If this is taken, the expression

$$\frac{\partial g_{\underset{\vee}{\mu\nu}}}{\partial x^{\eta}} + \frac{\partial g_{\underset{\vee}{\nu\eta}}}{\partial x^{\mu}} + \frac{\partial g_{\underset{\vee}{\eta\mu}}}{\partial x^{\nu}} = 0, \tag{50}$$

expresses the current density. Furthermore, the divergence of this magnitude becomes identically zero [12–14].

The system (50) thus contains essentially four equations which are written out as follows:

$$\frac{\partial g_{\underset{\vee}{23}}}{\partial x^{0}} + \frac{\partial g_{\underset{\vee}{30}}}{\partial x^{2}} + \frac{\partial g_{\underset{\vee}{02}}}{\partial x^{3}} = 0, \tag{51}$$

$$\frac{\partial g_{\underset{\vee}{30}}}{\partial x^{1}} + \frac{\partial g_{\underset{\vee}{01}}}{\partial x^{3}} + \frac{\partial g_{\underset{\vee}{13}}}{\partial x^{0}} = 0, \tag{52}$$

$$\frac{\partial g_{\underset{\vee}{01}}}{\partial x^{2}} + \frac{\partial g_{\underset{\vee}{12}}}{\partial x^{0}} + \frac{\partial g_{\underset{\vee}{20}}}{\partial x^{1}} = 0, \tag{53}$$

$$\frac{\partial g_{\underset{\vee}{12}}}{\partial x^{3}} + \frac{\partial g_{\underset{\vee}{23}}}{\partial x^{1}} + \frac{\partial g_{\underset{\vee}{31}}}{\partial x^{2}} = 0. \tag{54}$$

This system correspond to the second of Maxwell's system of equations. We recognize this at once by setting

$$\begin{aligned} g_{\underset{\vee}{23}} &= H_x, \quad g_{\underset{\vee}{31}} = H_y, \quad g_{\underset{\vee}{12}} = H_z, \\ g_{\underset{\vee}{10}} &= E_x, \quad g_{\underset{\vee}{20}} = E_y, \quad g_{\underset{\vee}{30}} = E_z. \end{aligned} \tag{55}$$

Then in place of (51), (52), (53) and (54) we may set, in the usual notation of the three-dimensional vector analysis

$$-\frac{\partial \overrightarrow{H}}{\partial t} = \nabla \times \overrightarrow{E}, \tag{56}$$

$$\nabla \cdot \overrightarrow{H} = 0. \tag{57}$$

Now, we take $0 = \frac{\partial}{\partial x^{\nu}} g_{\underset{\vee}{\mu\nu}}$ we obtain

$$\nabla \cdot \overrightarrow{E} = 0, \tag{58}$$

$$\nabla \times \overrightarrow{H} = \frac{\partial \overrightarrow{E}}{\partial t}. \tag{59}$$

Therefore, we deduce the Maxwel's first system. Thus (48) and (49) are substantially the Maxwel's equations [38–41] of empty space.

8 Concluding remarks

A generalization of the relativistic field theory has been presented, starting from the infinitesimal displacement field, avoiding the concept of inertial system. Also, with the transformation law of $\Gamma^{\lambda}_{\mu\nu}$, we calculate the curvature tensor and define the contracted curvature tensor. However, later, we define a pseudo tensor $U^{\rho}_{\eta\sigma}$ in terms of the infinitesimal displacement field, with which, we find another curvature tensor. With this curvature tensor and the variational principle, we deduce the field equations, the Bianchi differential identities and the differential identity of the electromagnetic vector potential.

Therefore, we obtain equation (48), the first system of Maxwell's equations. If, in the equation (49), the expression inside the parentheses would itself vanish, then we would have Maxwell's equations for empty space, whose solutions therefore satisfy our equations. Maxwell's equations of empty space seem to be too weak, however, it is not a (justified) objection to the theory since we do not know to which solutions of the field equations there correspond rigorous solutions which are regular in the entire space. It is clear from the start that in a consistent field theory which claims to be complete (in contrast e.g. to the pure theory of gravitation) only those solutions are to be considered which are regular in the entire space.

Acknowledgements

I thanks the IF-UNAM, for being an associate student while I have elaborated this work.

Appendix A: Variational principle

Hamilton's principle

The variational principle consists in finding the extremes of the Lagrangian density $\mathscr{L}$, for the relativistic theory of the asymmetric field. The action must be expressed as an integral over space-time (with the volume-invariant element) of a scalar function. The variational principle for the relativistic theory of the asymmetric field is expressed in the form

$$S = \int \mathscr{L} d^4x, \tag{60}$$

where $\mathscr{L} = R\sqrt{-g}$ is the scalar Lagrangian density and g is the determinant of the metric tensor $g_{\mu\nu}$. Making small variations $\delta g_{\mu\nu}$ in the metric tensor $g_{\mu\nu}$ and keeping the tensor $g_{\mu\nu}$ and its first derivatives constant on the boundary, in effect, we can find that $\delta S = 0$ for $\delta g_{\mu\nu}$ gives the field equations in the absence of matter.

Palatini's proposal

Palatini's proposal is very elegant and consists of the idea of treating the metric and the connection as independent fields in the Einstein Lagrangian. To be more specific, let's change $\mathscr{L}$ as a functional of $\mathfrak{g}^{\mu\nu}$ and $\Gamma^\sigma_{\mu\nu}$ and its derivatives only

$$\mathscr{L} = \mathscr{L}\left(\mathfrak{g}^{\mu\nu}, \Gamma^\sigma_{\mu\nu}, \frac{\partial \Gamma^\sigma_{\mu\nu}}{\partial x^\rho}\right),$$

thus the Ricci tensor depends only on $\Gamma^\sigma_{\mu\nu}$ and its derivatives. So, if we carry out a variation with respect to $\mathfrak{g}^{\mu\nu}$ and the principle of stationary action immediately gives the field equations in vacuum $R_{\mu\nu}$. Starting from the action in the form

$$S = \int \mathfrak{g}^{\mu\nu} R_{\mu\nu} d^4x,$$

we carry out a variation and use the Leibniz rule for products, to obtain

$$\delta S = \int \left(\delta \mathfrak{g}^{\mu\nu} R_{\mu\nu} + \mathfrak{g}^{\mu\nu} \delta R_{\mu\nu}\right) d^4x,$$

where we have written $\mathfrak{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$.

Now we use the Palatini equation [42]

$$\delta R_{\mu\nu} = \nabla_\nu \left(\delta \Gamma^\sigma_{\mu\sigma}\right) - \nabla_\sigma \left(\delta \Gamma^\sigma_{\mu\nu}\right),$$

in consequense

$$\delta S = \int \delta \mathfrak{g}^{\mu\nu} R_{\mu\nu} d^4x + \int \mathfrak{g}^{\mu\nu} \left[-\nabla_\sigma \left(\delta \Gamma^\sigma_{\mu\nu}\right) + \nabla_\nu \left(\delta \Gamma^\sigma_{\mu\sigma}\right)\right] d^4x,$$

the second term vanishes, since the covariant derivative of $\mathfrak{g}^{\mu\nu}$ vanishes identically, in other words, by making use of the divergence theorem, this new quantity vanishes because the variations are assumed null at the frontier, to obtain

$$\delta S = -\int \sqrt{-g} \left(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R\right) \delta g_{\alpha\beta} d^4x,$$

where we have used the equation

$$\delta \sqrt{-g} = \frac{1}{2}\sqrt{-g} g^{\alpha\beta} \delta g_{\alpha\beta}.$$

By virtue of the steady-state action principle, we obtain the field equations

$$R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = 0 \tag{61}$$

or

$$R_{\mu\nu} = 0.$$

Next we do a variation with respect to $\Gamma^\sigma_{\mu\nu}$, so

$$\delta S = \int \mathfrak{g}^{\mu\nu} \left[\nabla_\nu \left(\delta \Gamma^\rho_{\mu\rho}\right) - \nabla_\rho \left(\delta \Gamma^\rho_{\mu\nu}\right)\right] d^4x.$$

Integrating by parts and discarding the divergence term by the usual argument, we get

$$\delta S = \int \left[\delta^\nu_\rho \nabla_\sigma \mathfrak{g}^{\mu\sigma} - \nabla_\rho \mathfrak{g}^{\mu\nu}\right] \delta \Gamma^\rho_{\mu\nu} d^4x.$$

Since δS vanishes for an arbitrary volume, the integrand must vanish, i.e.

$$\delta^\nu_\rho \nabla_\sigma \mathfrak{g}^{\mu\sigma} - \nabla_\rho \mathfrak{g}^{\mu\nu} = 0.$$

The $\delta \Gamma^\rho_{\mu\nu}$ variations are arbitrary. In consecuense

$$\frac{\partial\left(\sqrt{-g}g^{\mu\nu}\right)}{\partial x^{\rho}}+\sqrt{-g}g^{\lambda\nu}\Gamma^{\mu}_{\lambda\rho}+\sqrt{-g}g^{\mu\lambda}\Gamma^{\nu}_{\rho\lambda}-\sqrt{-g}g^{\mu\nu}\Gamma^{\lambda}_{\rho\lambda}-\delta^{\nu}_{\rho}\left[\frac{\partial\left(\sqrt{-g}g^{\mu\sigma}\right)}{\partial x^{\sigma}}+\sqrt{-g}g^{\lambda\sigma}\Gamma^{\mu}_{\lambda\sigma}\right]=0. \tag{62}$$

Appendix B: Variation of $\sqrt{-g}g^{\mu\nu}$

If we consider the special case of an infinitesimal transformation of coordinates, i. e., equation (37), where ξ^{a} is a vector field. Then, when differentiating the proposed transformation, we find

$$\frac{\partial x^{*\mu}}{\partial x^{\nu}}=\delta^{\mu}_{\nu}+\frac{\partial\xi^{\mu}}{\partial x^{\nu}}.$$

Substituting in the transformation equation

$$\sqrt{-g}g^{\rho\sigma *}=\frac{\partial x^{\rho *}}{\partial x^{\mu}}\frac{\partial x^{\sigma *}}{\partial x^{\nu}}\sqrt{-g}g^{\mu\nu}\left|\frac{\partial x^{\tau}}{\partial x^{\tau *}}\right|$$

and using the Taylor's theorem to first order, we obtain

$$\sqrt{-g}g^{\mu\nu *}\left(x*\right)\approx\left(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma}+\delta^{\mu}_{\rho}\frac{\partial\xi^{\nu}}{\partial x^{\sigma}}+\delta^{\nu}_{\sigma}\frac{\partial\xi^{\mu}}{\partial x^{\rho}}\right)\left[\sqrt{-g}g^{\rho\sigma}\left(x*\right)-\frac{\partial\left(\sqrt{-g}g^{\rho\sigma}\right)}{\partial x^{\lambda *}}\xi^{\lambda}\right]\left|1-\frac{\partial\xi^{\tau}}{\partial x^{\tau *}}\right|.$$

Rearranging terms

$$\mathfrak{g}^{\mu\nu *}\left(x*\right)=\left[\mathfrak{g}^{\mu\nu}\left(x*\right)+\mathfrak{g}^{\mu\sigma}\frac{\partial\xi^{\nu}}{\partial x^{\sigma}}+\mathfrak{g}^{\rho\nu}\frac{\partial\xi^{\mu}}{\partial x^{\rho}}-\frac{\partial\mathfrak{g}^{\mu\nu}}{\partial x^{\lambda}}\xi^{\lambda}\right]\left|1-\frac{\partial\xi^{\tau}}{\partial x^{\tau *}}\right|,$$

where $\mathfrak{g}^{\mu\nu}=\sqrt{-g}g^{\mu\nu}$. We consider terms to first order and subtracting $\mathfrak{g}^{\mu\nu}\left(x*\right)$ from each side, it follows that

$$\mathfrak{g}^{\mu\nu *}\left(x*\right)-\mathfrak{g}^{\mu\nu}\left(x*\right)\approx\mathfrak{g}^{\mu\sigma}\frac{\partial\xi^{\nu}}{\partial x^{\sigma}}+\mathfrak{g}^{\rho\nu}\frac{\partial\xi^{\mu}}{\partial x^{\rho}}-\frac{\partial\mathfrak{g}^{\mu\nu}}{\partial x^{\lambda}}\xi^{\lambda}-\mathfrak{g}^{\mu\nu}\frac{\partial\xi^{\tau}}{\partial x^{\tau *}}$$

or

$$\delta\mathfrak{g}^{\mu\nu *}=\mathfrak{g}^{\mu\sigma}\frac{\partial\xi^{\nu}}{\partial x^{\sigma}}+\mathfrak{g}^{\rho\nu}\frac{\partial\xi^{\mu}}{\partial x^{\rho}}-\frac{\partial\mathfrak{g}^{\mu\nu}}{\partial x^{\lambda}}\xi^{\lambda}+\left[-\mathfrak{g}^{\mu\nu}\frac{\partial\xi^{\tau}}{\partial x^{\tau *}}\right].$$

The preceding equation is equivalent to equation (38).

Appendix C: Variation of pseudo tensor $U^{\lambda}_{\mu\nu}$

Differentiating the transformation (37), it follows that

$$\frac{\partial x^{*\mu}}{\partial x^{\nu}}=\delta^{\mu}_{\nu}+\frac{\partial\xi^{\mu}}{\partial x^{\nu}}.$$

Substituting this equation in (21) for the pseudo tensor U and using the first-order Taylor's theorem, we obtain the expression

$$U^{\rho *}_{\eta\sigma}\left(x*\right)\approx\left(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma}\delta^{\lambda}_{\eta}-\delta^{\mu}_{\rho}\delta^{\lambda}_{\eta}\frac{\partial\xi^{\nu}}{\partial x^{\sigma *}}+\delta^{\nu}_{\sigma}\delta^{\lambda}_{\eta}\frac{\partial\xi^{\rho}}{\partial x^{\mu}}-\delta^{\rho}_{\mu}\delta^{\nu}_{\sigma}\frac{\partial\xi^{\lambda}}{\partial x^{\eta *}}\right)\left[U^{\mu}_{\lambda\nu}\left(x*\right)-\frac{\partial U^{\mu}_{\lambda\nu}}{\partial x^{\lambda *}}\xi^{\lambda}\right]$$
$$-\left(\delta^{\rho}_{\alpha}+\frac{\partial\xi^{\beta}}{\partial x^{\alpha}}\right)\frac{\partial^{2}\xi^{\alpha}}{\partial x^{\eta}\partial x^{\sigma}}+\delta^{\rho *}_{\sigma *}\left(\delta^{\beta}_{\alpha}+\frac{\partial\xi^{\beta}}{\partial x^{\alpha}}\right)\frac{\partial^{2}\xi^{\alpha}}{\partial x^{\eta *}\partial x^{\beta *}}$$

or, by neglecting second-order terms in ξ^{μ}:

$$U^{\rho *}_{\eta\sigma}\left(x*\right)=U^{\rho}_{\eta\sigma}\left(x*\right)+U^{\alpha}_{\eta\sigma}\frac{\partial\xi^{\rho}}{\partial x^{\alpha}}-U^{\rho}_{\lambda\sigma}\frac{\partial\xi^{\lambda}}{\partial x^{\eta *}}-U^{\rho}_{\eta\lambda}\frac{\partial\xi^{\lambda}}{\partial x^{\sigma *}}-\frac{\partial^{2}\xi^{\rho}}{\partial x^{\eta *}\partial x^{\sigma *}}-\frac{\partial U^{\rho}_{\eta\sigma}}{\partial x^{\lambda *}}\xi^{\lambda}.$$

Also, subtracting $U^{\rho}_{\eta\sigma}\left(x*\right)$ from each side, it follows that

$$\delta U^{\rho *}_{\eta\sigma}\left(x*\right)=U^{\alpha}_{\eta\sigma}\frac{\partial\xi^{\rho}}{\partial x^{\alpha}}-U^{\rho}_{\lambda\sigma}\frac{\partial\xi^{\lambda}}{\partial x^{\eta *}}-U^{\rho}_{\eta\lambda}\frac{\partial\xi^{\lambda}}{\partial x^{\sigma *}}-\frac{\partial^{2}\xi^{\rho}}{\partial x^{\eta *}\partial x^{\sigma *}}-\frac{\partial U^{\rho}_{\eta\sigma}}{\partial x^{\lambda *}}\xi^{\lambda}.$$

This equation is the variation of the pseudo tensor $U^{\rho}_{\eta\sigma}\left(x*\right)$.

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