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Production of relic particles

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Abstract
The last few decades have seen an explosive increase in the amount and precision of data obtained from cosmological observations. Theoretical cosmologists have not yet finished interpreting this amount of data. Likewise, the wonderful ideas we have for the beginning of the Universe have not yet been connected with concrete models of elementary particles. Currently, the existence of dark matter (DM) is accepted in the scientific community. Stars, planets, comets, cosmic dust and other forms of matter make up 5% of the matter in the Universe. The remaining 95% would be "dark matter" and "dark energy", if it is that, it is not known what it is, in fact they exist. In this work, the standard cosmological model is studied, as well as the production of relic particles with the Boltzmann equation applied to the decoupling of relic particles from a reservoir.

Keywords: Field equations, Friedmann equations, conservation of entropy, Boltzmann equation, density of relic particles..

Producción de partículas reliquia

Resumen
Las últimas décadas han visto un explosivo aumento en la cantidad y la precisión de los datos obtenidos a partir de las observaciones cosmológicas. Los cosmólogos teóricos todavía no han terminado de interpretar esta cantidad de datos. Así mismo, todavía no se han conectado las maravillosas ideas que tenemos para el inicio del Universo con modelos concretos de partículas elementales. En la actualidad, la existencia de materia oscura (DM, por sus siglas en inglés) es aceptada en la comunidad científica. Estrellas, planetas, cometas, polvo cósmico y otras formas de materia constituyen el 5% de materia del Universo. El 95% restante sería de "materia oscura" y "energía oscura", si es que, que no se sabe lo que es, de hecho existen. En este trabajo se estudia el modelo cosmológico estándar, así como la producción de partículas reliquia con la ecuación de Boltzmann aplicada al desacoplamiento de partículas reliquia de un reservorio.

Palabras clave: Ecuaciones de campo, ecuaciones de Friedmann, conservación de la entropía, Ecuación de Boltzmann, Densidad de partículas reliquia..

1 Standard Cosmology

Cosmology is defined as the branch of physics that studies the origin of the Universe on its largest scale. In 1687, Isaac Newton published his book entitled "Mathematical Principles of Natural Philosophy" [1], better known as "Principia", where he formulated the bases of classical mechanics through the laws that bear his name and his theory of gravitation, with which was born analytic cosmology.

In the year 1916 Einstein published general relativity in its complete and definitive version [2]. Soon after,

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various solutions of the field equations of general relativity [3] [4], which are the structure of modern cosmology, were published. The hypothesis of isotropy and homogeneity [1] applied to general relativity, opened the field of cosmology in the 20-th century with the construction of models that accept exact solutions, which are known as Friedmann-Lemaître-Robertson-Walker models, or FLRW. These types of models were developed by Alexander Friedmann [5] and later by Howard Percy Robertson [6] and Arthur Geoffrey Walker [7], among others [8]. In addition, these principles were based on the cosmological principle.

The cosmological principle establishes that on large scales the Universe is homogeneous and isotropic [9], that is, there are no privileged positions or directions in the Universe. The clearest evidence of this cosmological principle is found in observations of the cosmic microwave background [10] [11], which reveal the anisotropy of the Universe.

In this article, we will use units in which $c = 1$.

1.1 General relativity

In general relativity, the geometry of space-time is characterized by a second-order symmetric tensor, whose components in the coordinate system $\{x^{\mu}\}$, will be denoted by $g_{\mu\nu}$, where $\mu = 0, 1, 2, 3$. The square of the line element between two neighboring points in space-time will be given by the expression

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}, \tag{1}$$

where we will assume a metric tensor with signature $(+, -, -, -)$.

With a metric associated to the space-time manifold, a covariant derivative associated to this metric can be defined, denoted by the symbol ∇_{β}, whose action on a tensor is:

$$\nabla_{\beta} A^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_m} = \frac{\partial}{\partial x^{\beta}} A^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_m} - \sum_{i=1}^{m} \Gamma^{\alpha}_{\nu_i \beta} A^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_{i-1} \alpha \nu_{i+1} \cdots \nu_m} + \sum_{j=1}^{n} \Gamma^{\mu_j}_{\alpha\beta} A^{\mu_1 \cdots \mu_{j-1} \alpha \mu_{j+1} \cdots \mu_n}_{\nu_1 \cdots \nu_n}, \tag{2}$$

then we have the covariant derivative of the tensor $A^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_m}$, where the Γ are the Christoffel symbols, which in the case of the metric connection and imposing a metricity condition ($\nabla_{\lambda} g_{\mu\nu} \equiv 0$), are given by:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right). \tag{3}$$

The space-time curvature is characterized by the Riemann-Christoffel tensor [12] [13] whose components can be expressed in terms of the Christoffel symbols according to the expression:

$$R^{\lambda}_{\mu\nu\kappa} \equiv \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} + \Gamma^{\sigma}_{\mu\nu} \Gamma^{\lambda}_{\sigma\kappa} - \Gamma^{\sigma}_{\kappa\mu} \Gamma^{\lambda}_{\sigma\nu}. \tag{4}$$

The Riemann-Christoffel curvature tensor is defined for any manifold in terms of the connection regardless of the definition of a metric tensor. General relativity requires a metric since the concept of space-time distance from special relativity must be generalized.

The field equations relate the geometry of space-time to its matter content. These equations under non-relativistic conditions reproduce Newtonian mechanics and Poisson's equation. The geometry appears in the field equations through the Ricci tensor, defined by:

$$R_{\mu\nu} = R^{\lambda}_{\mu\nu\lambda}, \tag{5}$$

and the curvature scalar

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{6}$$

The relativistic field equations of gravitation are given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} \tag{7}$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor, $T_{\mu\nu}$ is the energy-momentum tensor [14] and the coupling constant is expressed by $\kappa = 8\pi G$.

Matter enters the field equations through the energy-momentum tensor, denoted by $T_{\mu\nu}$, whose time-time component corresponds to the energy density, the time-space components to the momentum density, and the space-space components to the stress tensor [14]. The field equations are the fundamental equations of the relativistic description of gravitation and are part of general relativity. In general relativity, gravity is the effect of the existence of a curvature in space-time. The field equations relate the presence of matter to the curvature of space-time. More precisely, the greater the concentration of matter, represented by the energy-momentum tensor, the greater the components of the Ricci tensor [15] [16].

When Einstein deduced his equations, he discovered that they did not allow a static solution, so he proposed to modify them by adding a constant Λ, known as the cosmological constant, obtaining a modification of the field equations. This cosmological constant can have two physical interpretations [17]. The first interpretation: Λ could be interpreted as a shift in energy, i. e., the right-hand side of the field equations is modified by considering a new effective energy-momentum tensor $Q_{\mu\nu}$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G \left(T_{\mu\nu} + Q_{\mu\nu} \right), \tag{8}$$

[1] Isotropy refers to the observation that the Universe is the same in every direction and sense. Instead, homogeneity refers to the observation that the Universe looks the same at all points.

where $Q_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu}$. In the second interpretation, gravity is represented by two constants, Newton's constant G and the cosmological constant Λ. The left-hand side of the field equations is modified, as shown:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}. \tag{9}$$

In this interpretation, space-time is curved even in the absence of matter since the equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 0$ does not support flat space-time as solutions.

1.2 FLRW cosmological model

The cosmological principle is one of the basic principles on which the cosmological standard model is based. The clearest evidence for the cosmological principle is found in observations of the cosmic microwave background [9] [10]. Under the conditions of homogeneity and isotropy, it follows that the Friedmann-Lamaître-Robertson-Walker metric (FLRW metric) describing the space-time of the Universe is given by

$$ds^2 = dt^2 - a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right], \tag{10}$$

where $k = 0, +1, -1$. Therefore, the three types of geometries for space described in the FLRW model metric are classified as:

- Open universe if $k = -1$, i.e., hyperbolic space.
- Plane if $k = 0$, i.e., Euclidean space.
- Closed if $k = 1$, i.e., spherical space.

The dimensionless parameter $a(t)$ is the scale factor of the Universe and its time dependence describes the cosmological expansion. Consequently, the scale factor $a(t)$ gives physical measure to the coordinate r. With this, the instantaneous physical (radial) distances are expressed by

$$R = a(t) \int_0^r \frac{dr}{\sqrt{1 - kr^2}}. \tag{11}$$

The function that is often used to denote the expansion of the Universe is the Hubble parameter [18]

$$H(t) = \frac{1}{a} \frac{da}{dt}, \tag{12}$$

the name receives it in honor of Edwin P. Hubble. Hubble in 1929 observed that the speed with which any object is moving away from us is proportional to its distance. Likewise, the Hubble parameter refers to the speed with which most distant galaxies are receding from us through Hubble's law

$$v = Hd, \tag{13}$$

where v is the speed and tells us how an object moves away or approaches and d is the distance between the observer and the distant galaxy that moves away.

1.3 Friedmann equations

Suppose now that the Universe is filled with an ideal fluid; adiabatic frictionless fluid, i.e. fluid characterized by the fact that in a local coordinate system of a fluid element there is only one isotropic pressure. Therefore, the energy-momentum tensor will be represented by:

$$T^{\mu\nu} = (\rho + p) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - p g^{\mu\nu}, \tag{14}$$

or covariant

$$T_{\mu\nu} = (\rho + p) g_{\mu\rho} g_{\nu\sigma} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} - p g_{\mu\nu}.$$

With this simple description of matter, if we make use of equation (10) and plug equation (14) into equation (9), then we get the Friedmann equations

$$\frac{2}{a} \frac{d^2 a}{dt^2} + \frac{1}{a^2} \left(\frac{da}{dt} \right)^2 + \frac{k}{a^2} - \Lambda = -8\pi G p, \tag{15}$$

$$3 \left[\frac{1}{a^2} \left(\frac{da}{dt} \right)^2 + \frac{k}{a^2} \right] - \Lambda = 8\pi G \rho. \tag{16}$$

A direct consequence of Eqs. (15) and (16) is the continuity equation. If we differentiate equation (16) with respect to t, then the resulting differential equation is divided by $a \frac{da}{dt}$, and finally this resulting equation is subtracted with equation (15) (without taking into account the cosmological constant), then we get

$$\frac{d\rho}{dt} + 3H(\rho + p) = 0. \tag{17}$$

There is another physically feasible way to find Eq. (17). This way would be through the application of the equation of conservation of momentum and energy for matter in the relativistic theory of the gravitational field. The conservation of energy is expressed in general relativity by nullifying the divergence of the energy-momentum tensor, that is, by the equation [19]

$$\frac{\partial \left(\sqrt{-g} T^\alpha_\sigma \right)}{\partial x^\alpha} - \Gamma^\alpha_{\sigma\beta} \sqrt{-g} g^{\rho\beta} T_{\rho\alpha} = 0. \tag{18}$$

Applying this conservation law, i.e. equation (18), to our hypothesis of the FLRW metric of equation (10) and the energy-momentum tensor for a perfect fluid (with $\mu = 0$), we obtain a single energy conservation equation, that is, equation (17).

This equation is not actually independent of the Friedmann equations, but it is necessary for consistency. This equation implies that the expansion of the Universe (as specified by H) can give rise to local changes in energy density. Note that there is no notion of total energy conservation, since energy can be exchanged between matter and geometric space.

1.4 Solution of the dynamical equations in a Euclidean Universe ($k = 0$) and without cosmological constant

In order to determine the solutions to the Friedmann differential equations, let us consider an equation of state for cosmological matter of the form $p = \omega\rho$, with $\omega = \omega(a)$, not necessarily constant . Integrating equation (17) we get

$$\rho(a) = \rho(a_0)\exp\left\{-3\int_{a_0}^{a}[1+\omega(u)]\frac{du}{u}\right\}. \tag{19}$$

If ω is constant, then we have

$$\rho(a) = \rho(a_0)\left(\frac{a}{a_0}\right)^{-3(1+\omega)}. \tag{20}$$

Equation (20) is studied for three types of matter that play a very important role in cosmology. The first type of matter is dust (non-relativistic matter), for which $\omega = 0$, the second type of matter is radiation (relativistic particle gas), where $\omega = \frac{1}{3}$, and finally, the so-called vacuum energy, with the value $\omega = -1$.

Let us consider a flat Universe, for this, we substitute equation (20) in equation (16) obtaining the differential equation

$$\frac{1}{a^2}\left(\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho_0\left(\frac{a}{a_0}\right)^{-3(1+\omega)}, \tag{21}$$

equation from which it follows that $\left(\frac{da}{dt}\right)^2 \propto a^{2-3(1+\omega)}$, which implies the evolution of the scale factor. The solution to the differential equation (21), considering the initial condition $a(t_0) = a_0$, is given by

$$a(t) = a_0\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+\omega)}}, \tag{22}$$

where a_0 is the current scale factor, unless $\omega = -1$, in which case we get $a(t) \propto \exp(Ht)$. It is important to note that the matter and global radiation in flat Universes start with $a = 0$, this is a singularity, known as the Big Bang. We can therefore calculate the age of the Universe in question. If we take into account equation (22), then

$$H(t) = \frac{1}{a(t)}\frac{da(t)}{dt} = \frac{2}{3(1+\omega)t}. \tag{23}$$

From equation (23), we calculate the evolution time of the Universe

$$t_0 = \frac{2}{3(1+\omega)H_0}. \tag{24}$$

Unless ω is close to -1, it is often convenient to approximate equation (23) to the quantity:

$$t_0 \sim H_0^{-1}. \tag{25}$$

This quantity is known as the Hubble time.

If we consider the previous analysis and use equations (20) and (22), we construct the following table:

Type of energy	$\rho(a)$	$a(t)$
Dust	a^{-3}	$t^{\frac{2}{3}}$
Radiation	a^{-4}	$t^{\frac{1}{2}}$
Cosmological Constant	constant	$\exp(Ht)$

Table 1: The behavior of the scale factor applies to the case of a flat Universe; with curvature $k = 0$, the behavior of the energy densities is perfectly general.

1.5 Cosmological parameters

The best known cosmological parameter is the Hubble parameter, defined in equation (12), whose value today is called the Hubble constant $H_0 = \frac{1}{a(t_0)}\frac{da}{dt}\big|_{t=t_0}$. The Hubble constant is given by

$$H_0 = 100h\,(\mathrm{km/seg/Mpc}). \tag{26}$$

The Hubble Telescope is one of the main instruments on extragalactic distances [20] [21]. From the observations and data collected by this telescope it is inferred that

$$h = 0.71 \pm 0.06. \tag{27}$$

From the Hubble parameter, the critical energy[2] density is defined for

$$\rho_0 = \frac{3H^2(t_0)}{8\pi G}, \tag{28}$$

and with the help of equations (26) and (27)

$$\rho_0 \simeq 1.88h^2 10^{-29}\frac{g}{cm^3} = 9.5\times 10^{-30}\frac{g}{cm^3}, \tag{29}$$

where this amount is approximately equivalent to 6 protons per cubic meter.

In terms of the energy density of the Universe and the Hubble parameter it is possible to define the density parameter by the mathematical equation

[2] This energy is taken into account for a flat Universe.

$$\Omega = \frac{\rho}{\rho_0} = \frac{8\pi G}{3H^2(t_0)}\rho, \tag{30}$$

where the sign can be used to determine the spatial curvature. The classification of the Universe, according to the definition of equation (30), can be done in the following way:

- For a closed Universe ($k = +1$) we have that $\Omega > 1$.
- Flat Universe ($k = 0$), the density parameter is equal to zero, that is, $\Omega = 0$.
- Open Universe ($k = -1$), It is true thats $\Omega < 1$.

Cosmological observations allow us to estimate the different density parameters as follows:

- Baryonic matter (matter made up of electrons, neutrons and protons): $\Omega_B \approx 0.04$ [22] [23].
- Dark matter: $\Omega_{DM} = \Omega_{CDM} + \Omega_{HDM} \approx 0.26$ [24].
- Dark energy (compatible with a cosmological constant): $\Omega_{DE} = \Omega_\Lambda = 0.7$ [24].
- Radiation: $\Omega_R \approx 5 \times 10^{-5}$.

An important consequence of the density parameters is the consistency relationship between the cosmological parameters [24]:

$$\Omega_B + \Omega_{DM} + \Omega_{DE} = 1. \tag{31}$$

If we substitute equation (16) into equation (15), then we have the equation $\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{8\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}$. From this equation another basic parameter is defined, the deceleration parameter [25]:

$$q = -a\left(\frac{da}{dt}\right)^{-2}\frac{d^2a}{dt^2} = \frac{4\pi G}{3H^2}(\rho + 3p) - \frac{\Lambda}{3H^2}. \tag{32}$$

The uniform expansion corresponds to $q = 0$ and requires a cancellation between matter and vacuum energy. For matter we have $q > 0$, otherwise, for the vacuum energy domain, $q < 0$. According to the current density parameter, the presence of radiation is negligible, but in the past radiation was dominant. At present, the total energy content is dominated by dark energy, similar to a cosmological constant, and therefore the expansion of the Universe at present is accelerating (see Figure 1).

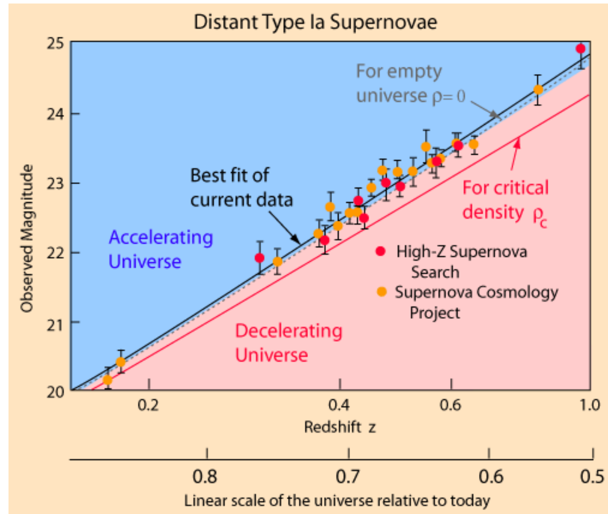

Figure 1: Magnitude-Redshift diagram to study the expansion of the Universe. At redshifts greater than $z = 0.1$, cosmological predictions begin to diverge, depending on the assumed cosmic densities of mass and vacuum energy. The red curves represent models with zero vacuum energy and mass densities ranging from the critical density ρ_0 to zero. The best fit is given by the blue line which assumes a mass density $\rho_0/3$ plus a vacuum energy density twice as high, which implies an accelerated expansion of the Universe [26].

1.6 Behavior of FLRW models

In this section we will study the behavior of the Friedman-Lemaître-Robertson-Walker models to understand the effect produced by their different components. As we will see below.

The energy conservation equation

$$\frac{d\rho}{dt} + \frac{3}{a}\frac{da}{dt}(\rho + p) = 0,$$

can be interpreted as the first law of thermodynamics $dQ = dU - pdV = d(\rho a^3) - pd(a^3) = 0$. To obtain the explicit solutions of $a(t)$ and k, equations (16) and (17) are often used. Furthermore, it is necessary to complement these equations with the equation of state $p = \omega\rho$ that relates the pressure and density of the fluid. With this it is possible to integrate equation (17) to obtain the relation (20). The most important fluids in cosmology are barotropic, that is, their pressure is proportional to their density, i. e, $p = \omega\rho$, and therefore $\frac{dp}{d\rho} = Constant$. In this case, equation (20) leads to $\rho \propto a^{-3(1+\omega)}$. If the Universe contains N fluids of different species with equations of state ω_i, this result holds for each species separately as long as they do not interact. If we denote by $\rho_{i,0}$ the current density of each species, then the total energy density of the Universe corresponding to the epoch $a(t)$, will be

$$\rho = \sum_{i=1}^{N} \frac{\rho_{i,0}}{a^{3(1+\omega_i)}},$$

where we have set $a_0 = 1$ for simplicity. Then, the Friedman equation (16), for the scale factor, can be written

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\sum_{i=1}^{N} \frac{\rho_{i,0}}{a^{3(1+\omega_i)}} - \frac{k}{a^2} + \frac{\Lambda}{3}.$$

Evaluating this equation in the present, and under the Hubble parameter, we get:

$$H_0^2 = \frac{8\pi G}{3}\sum_{i=1}^{N} \rho_{i,0} - \frac{k}{a^2} + \frac{\Lambda}{3}. \tag{33}$$

Equation (33) can be rewritten in the form

$$k = H_0^2 \left(\frac{8\pi G}{3H_0^2}\sum_{i=1}^{N} \rho_{i,0} + \frac{\Lambda}{3H_0^2} - 1\right). \tag{34}$$

In this way the curvature of space will be given by the energy content of the Universe and the value of the cosmological constant.

In terms of the critical density, equation (28), it is possible to define the density parameter corresponding to each species of the Universe

$$\Omega_{i,0} = \frac{\rho_{i,0}}{\rho_0} = \frac{8\pi G}{3H_0^2}\rho_{i,0},$$

and if we define

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2},$$

and

$$\Omega_k = \frac{k}{H_0^2},$$

it is possible to rewrite (34) in the form

$$\sum_{i=1}^{N} \Omega_{i,0} + \Omega_\Lambda + \Omega_k = 1. \tag{35}$$

Thus, the equation for the evolution of the scale factor can be rewritten as follows

$$\left(\frac{da}{dt}\right)^2 = H_0^2 \left\{\sum_{i=0}^{N} \Omega_{i,0}\left[a^{-(1+3\omega_i)} - 1\right] + \Omega_\Lambda\left(a^2 - 1\right) + 1\right\}. \tag{36}$$

In this way, knowing the energy content of the Universe, it is possible to obtain its temporal evolution. Assuming a flat Universe, without cosmological constant, containing a single species of equation of state $\omega_i \neq -1$, the time evolution of the scale factor is given by the proportionality relation, $a(t) \propto t^{\left(\frac{2}{3+3\omega_i}\right)}$.

It is also possible to use (38) to obtain another useful relation. Separating variables, integrating and assuming that $a = 0$ for $t = 0$, an expression is obtained for the age of the Universe given by

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{a da}{\sqrt{\sum_{i=1}^{N} \Omega_{i,0} a^{1-3\omega_i} + \Omega_\Lambda a^4 + \left(1 - \sum_{i=1}^{N} \Omega_{i,0} - \Omega_\Lambda\right) a^2}}. \tag{37}$$

As we have seen, the radiation density dominates the evolution of the Universe in its early stages, however, its contribution is negligible compared to that of matter or dark energy, which dominate its later stages. For this reason we will consider only Universes with these two components (assuming that $\omega_{DE} = -1$). In this way, equations (38) and (39) can be written as

$$\left(\frac{da}{dt}\right)^2 = H_0^2\left[\Omega_m\left(a^{-1} - 1\right) + \Omega_\Lambda\left(a^2 - 1\right) + 1\right], \tag{38}$$

and

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{da}{\sqrt{\Omega_m\left(a^{-1} - 1\right) + \Omega_{DE}\left(a^2 - 1\right) + 1}}. \tag{39}$$

1.7 Some solutions to the homogeneous and isotropic models

In this way it is possible to study some important particular cases for various combinations of the parameters (Ω_m, Ω_{DE}).

- • The Einstein-de Sitter model. This is a flat Universe with Ω_m and $\Omega_{DE}=0$. In this model

$$a(t)=\left(\frac{t}{t_0}\right)^{2/3},$$

$$t_0=\frac{2}{3H_0}.$$

- Open model without Dark Energy. In this universe $\Omega_m<1$ and $\Omega_{DE}=0$. In this model, the evolution of the scale factor is given parametrically by

$$a=\frac{\Omega_m}{2(1-\Omega_m)}(\cosh\theta-1),$$

$$H_0t=\frac{\Omega_m}{2(1-\Omega_m)^{3/2}}(\sinh\theta-\theta),$$

$$H_0t_0=\frac{\Omega_m}{2(1-\Omega_m)}-\frac{\Omega_m}{2(1-\Omega_m)^{\frac{3}{2}}}\ln\left(\frac{2-\Omega_m+2\sqrt{1-\Omega_m}}{\Omega_m}\right).$$

- Closed model without Dark Energy. In this universe $\Omega_m>1$ and $\Omega_{DE}=0$. In this model, the evolution of the scale factor can also be obtained in the form

$$a=\frac{\Omega_m}{2(1-\Omega_m)}(\cos\theta-1),$$

$$H_0t=\frac{\Omega_m}{2(1-\Omega_m)^{3/2}}(\sin\theta-\theta),$$

$$H_0t_0=\frac{\Omega_m}{2(1-\Omega_m)}-\frac{\Omega_m}{2(1-\Omega_m)^{\frac{3}{2}}}\ln\left(\frac{2-\Omega_m+2\sqrt{1-\Omega_m}}{\Omega_m}\right).$$

- Flat model with Dark Energy. In this model the contributions of Dark Matter and Dark Energy ($\omega_{DE}=-1$) combine to produce a flat Universe. In this case we get

$$a(t)=\left(\frac{\Omega_m}{1-\Omega_m}\right)^{\frac{1}{3}}\left[\sinh\left(\frac{3\sqrt{1-\Omega_m}}{2}H_0t\right)\right]^{2/3},$$

$$H_0t_0=\frac{1}{3\sqrt{1-\Omega_m}}\ln\left(\frac{2-\Omega_m+2\sqrt{1-\Omega_m}}{\Omega_m}\right).$$

1.8 Thermal equilibrium in an expanding Universe

The Universe has evolved from a very compact and extremely dense state. We now observe that the photon radiation around us has a strong thermal spectrum. Only part of the gas in the Universe is in thermal equilibrium, the rest has decoupled. In the description of a particle plasma in thermal equilibrium, the density in phase space, $f(\overrightarrow{p},t)$, is the fundamental object. By considering only the FLRW metrics, the density will not depend on either momentum or position. From this density function, the densities of n particles and of energy ρ and the pressure p are found, which correspond to a gas of a species of particles with g internal degrees of freedom

$$n=\frac{g}{(2\pi)^3}\int f(|\overrightarrow{p}|)\,d^3p, \tag{40}$$

$$\rho=\frac{g}{(2\pi)^3}\int E(|\overrightarrow{p}|)\,f(|\overrightarrow{p}|)\,d^3p, \tag{41}$$

$$p=\frac{g}{(2\pi)^3}\int\frac{|\overrightarrow{p}|^2}{3E}f(|\overrightarrow{p}|)\,d^3p, \tag{42}$$

where $E=\sqrt{|\overrightarrow{p}|^2+m^2}$. The density in phase space will be given by the Fermi-Dirac (FD) [27] [28] or Bose-Einstein (BE) [29–32] distributions, depending on the particles, whether they are fermions or bosons, respectively. These are:

$$f\left(|\overrightarrow{p}|^2\right)=\frac{1}{\exp\left[(E-\mu)/T\right]\pm1}, \tag{43}$$

where T is the equilibrium plasma temperature and μ is the chemical potential of the species, which introduces an asymmetry between particle and antiparticle. Additionally, if the species is in chemical equilibrium, the chemical potential, μ, is related to those of the species with which it interacts.

By plugging the distribution functions, i.e. equation (43), into equations (40), (41) and (42) and using the relation $d^3\overrightarrow{p} = 4\pi\left|\overrightarrow{p}\right|^2 dp = 4\pi\left|\overrightarrow{p}\right| EdE$, we find the expressions for the densities of n particles and for energy ρ and the pressure p

$$n = \frac{g}{2\pi^2}\int\limits_m^\infty \frac{\sqrt{E^2-m^2}}{\exp\left[\left(E-\mu\right)/T\right]\pm 1}EdE, \tag{44}$$

$$\rho = \frac{g}{2\pi^2}\int\limits_m^\infty \frac{\sqrt{E^2-m^2}}{\exp\left[\left(E-\mu\right)/T\right]\pm 1}E^2dE, \tag{45}$$

$$p = \frac{g}{6\pi^2}\int\limits_m^\infty \frac{\sqrt{\left(E^2-m^2\right)^3}}{\exp\left[\left(E-\mu\right)/T\right]\pm 1}dE. \tag{46}$$

The integrals of equations (44), (45) and (46) have not analytical solution, however, analytical results can be found for the following limits. First of all, taking $T \gg m$ y $T \gg \mu$, we have the non-degenerate relativistic limit, where differentiating bosons from fermions, we find:

$$\rho = \begin{cases} \frac{\pi^2}{30}gT^4 & (BE), \\ \frac{7}{8}\frac{\pi^2}{30}gT^4 & (FD), \end{cases} \tag{47}$$

$$n = \begin{cases} \frac{1}{\pi^2}\varsigma\left(3\right)gT^3 & (BE), \\ \frac{3}{4\pi^2}\varsigma\left(3\right)gT^3 & (FD), \end{cases} \tag{48}$$

$$p = \frac{\rho}{3}, \tag{49}$$

where $\varsigma\left(3\right) = 1.202...$ is the Riemann zeta function of 3. Second, the other limit corresponds to non-relativistic particles, $m \gg T$:

$$n = g\sqrt{\left(\frac{mT}{2\pi}\right)^3}\exp\left[-\frac{\left(m-\mu\right)}{T}\right], \tag{50}$$

$$\rho = g\sqrt{\left(\frac{mT}{2\pi}\right)^3}m\exp\left[-\frac{\left(m-\mu\right)}{T}\right], \tag{51}$$

$$p = g\sqrt{\left(\frac{mT}{2\pi}\right)^3}T\exp\left[-\frac{\left(m-\mu\right)}{T}\right] \ll \rho, \tag{52}$$

corresponding to the Maxwell-Boltzmann statistic [33–36].

The energy density and pressure of non-relativistic species are exponentially smaller than those of a relativistic species when the Universe is composed of pure radiation. When calculating the total energy density, it is convenient to express the density of each species as a function of the photon temperature, T,

$$\rho_R = T^4\left[\sum_{i=bosons}\frac{\pi^2}{30}g_i\left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_{i=fermions}\frac{\pi^2}{30}g_i\left(\frac{T_i}{T}\right)^4\right],$$

where T_i is the temperature of each boson or fermion, also to simplify the previous equation, we define the parameter g_* (which counts the effective relativistic degrees of freedom)

$$g_* = \sum_{i=bosons}g_i\left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_{i=fermions}g_i\left(\frac{T_i}{T}\right)^4, \tag{53}$$

getting the equation

$$\rho_R = \frac{\pi^2}{30}g_*T^4. \tag{54}$$

With these results we can calculate H, t and a, depending on the temperature of the photon reservoir, for a Euclidean Universe, dominated by radiation. With the help of the quantity $\omega = 1/3$ (for radiation, substituting in the equation of state), of equations (47), (16), (23), $a\left(t\right) \propto \sqrt{t}$ and (22) we can find the expressions

$$H = \sqrt{\frac{8\pi G}{3}\rho_R} = \sqrt{\frac{4\pi^3}{45}}\sqrt{g_*}\frac{T^2}{M_P}, \tag{55}$$

$$t = \frac{1}{2H} = \sqrt{\frac{45}{16\pi g_*}}\frac{M_P}{T^2} \sim \left(\frac{T}{MeV}\right)^{-2} seg, \tag{56}$$

$$T \propto \frac{1}{a}, \tag{57}$$

where it has been taken into account that $G \propto M_P^{-2}$, where $M_P = 1.2\times 10^{19}GeV$ is the Planck mass.

1.9 Conservation of entropy

In processes that are locally in thermal equilibrium, the entropy per comoving unit volume corresponding to the species that participate in them is conserved. Therefore, we find an expression for this entropy. To accomplish this end, we use the second law of thermodynamics

$$TdS = d\left[\left(\rho+p\right)V\right] - Vdp - \mu d\left(nV\right), \tag{58}$$

where $V = a^3$ is the comoving volume to which we apply this law.

Differentiating equation (46) with respect to T, we find

$$\frac{dp}{dT} = -\frac{gT}{6\pi^2}\int\limits_m^\infty \sqrt{\left(E^2-m^2\right)^3}\left[\frac{E}{T^2} + \frac{d}{dT}\left(\frac{\mu}{T}\right)\right]\frac{df\left(\left|\overrightarrow{p}\right|\right)}{dE}dE, \tag{59}$$

where $f\left(\left|\overrightarrow{p}\right|\right)$ is the corresponding distribution function. Integrating the above equation by parts, we arrive at

$$\frac{dp}{dT}=\frac{1}{T}\left[\frac{g}{2\pi^2}\int\limits_m^\infty f\left(|\overrightarrow{p}|\right)E^2\sqrt{E^2-m^2}dE+\frac{g}{6\pi^2}\int\limits_m^\infty f\left(|\overrightarrow{p}|\right)\sqrt{\left(E^2-m^2\right)^3}dE\right]$$

$$+T\frac{d}{dT}\left(\frac{\mu}{T}\right)\frac{g}{2\pi^2}\int\limits_m^\infty f\left(|\overrightarrow{p}|\right)E\sqrt{E^2-m^2}dE,$$

or; using equations (44), (45) and (46):

$$\frac{dp}{dT}=\frac{\rho+p}{T}+nT\frac{d}{dT}\left(\frac{\mu}{T}\right). \tag{60}$$

Now, from equation (51), and using (53), we find

$$dS=\frac{1}{T}d\left[\left(\rho+p\right)V\right]-\frac{\mu}{T}d\left(nV\right)-\frac{V}{T}\frac{\rho+p}{T}dT-\frac{V}{T}nTd\left(\frac{\mu}{T}\right),$$

and rearranging, we have:

$$S=\frac{\rho+p-\mu n}{T}V. \tag{61}$$

Recall that the conservation of the energy-momentum tensor, that is, $\frac{d\rho}{dt}=-3H\left(\rho+p\right)$, brings us to the relationship

$$d\left(\rho a^3\right)=-pd\left(a^3\right), \tag{62}$$

where by subtracting and adding the term a^3dp, we get the equation

$$\frac{d}{dT}\left[\left(\rho+p\right)a^3\right]=a^3\frac{dp}{dT}. \tag{63}$$

From equation (60), equation (63) takes the form

$$\frac{d}{dT}\left[\left(\rho+p\right)a^3\right]=a^3\frac{\rho+p}{T}+a^3nT\frac{d}{dT}\left(\frac{\mu}{T}\right), \tag{64}$$

then dividing by T, plus adding and subtracting $\left(\mu/T\right)\frac{d}{dT}\left(a^3n\right)$, equation (64) takes the form

$$\frac{d}{dT}\left[\left(\frac{\rho+p-\mu n}{T}\right)a^3\right]=-\frac{\mu}{T}\frac{d}{dT}\left(a^3n\right),$$

or, with the help of equation (61):

$$\frac{dS}{dT}=-\frac{\mu}{T}\frac{d}{dT}\left(a^3n\right). \tag{65}$$

From (65), therefore, we see that the entropy in a comoving volume is conserved during the expansion if $|\mu|\ll T$, or if $a^3n=constant$. The last condition tells us that there is no net creation or annihilation of particles in that volume.

The quantity that will be useful to work with is the entropy density, s:

$$s\equiv\frac{S}{V}=\frac{\rho+p}{T}, \tag{66}$$

where we have already assumed that $|\mu|\ll T$.

Recalling that in a Universe dominated by radiation, the energy and pressure will be for the relativistic species, and by virtue of equations (47) and (49), we find the entropy density associated with each of these species:

$$s_i=\frac{\rho_i+p_i}{T_i}=\frac{4}{3}\frac{\rho_i}{T_i}=\begin{cases}\frac{2\pi^2}{45}g_iT_i^3 & (BE),\\ \frac{7}{8}\frac{2\pi^2}{45}gT_i^3 & (FD).\end{cases} \tag{67}$$

When calculating the entropy density, it is convenient to express the density of each species as a function of the photon temperature, T,

$$s=T^3\left[\sum_{i=bosons}\frac{2\pi^2}{45}g_i\left(\frac{T_i}{T}\right)^3+\frac{7}{8}\sum_{i=fermions}\frac{2\pi^2}{45}g_i\left(\frac{T_i}{T}\right)^3\right],$$

where the parameter g_* (which counts the effective relativistic degrees of freedom) is defined

$$g_{*S}=\sum_{i=bosons} g_i\left(\frac{T_i}{T}\right)^3+\frac{7}{8}\sum_{i=fermions} g_i\left(\frac{T_i}{T}\right)^3,$$

getting the equation

$$s=\frac{2\pi^2}{45}g_{*S}T^3. \tag{68}$$

Now, we have the entropy, it allows us to easily relate the entropy density of the Universe with the density of photons. From (68) and (48) we find that

$$s=\frac{\pi^4}{45\varsigma\left(3\right)}g_{*S}n_\gamma=1.8g_{*S}n_\gamma, \tag{69}$$

relationship that will change as the number of relativistic species changes.

The conservation of entropy in a comoving volume, S, implies that the entropy density decreases with the variation of the volume, that is, $s\propto a^{-3}$. This variation is the same as that suffered by the density of particles n, if the number of particles in the comoving volume remains constant. This allows us to define the amount

$$Y\equiv\frac{n}{s}\sim Constant, \tag{70}$$

which is useful, since it remains constant if the number of relativistic species does not change.

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2 Relic Particles

If the Universe had always been in a state of thermal equilibrium, then what it would look like today would be very simple, i. e., a gas in thermal equilibrium at 2.75 ºK. Thanks to processes out of equilibrium we can now observe all the richness of the Universe. Examples of non-equilibrium processes are the decoupling of neutrinos and photons [37] [38], Primordial Nucleosynthesis [38–45], Bariogenesis [46–51], the inflationary period [52–57], etc. It is of utmost importance to understand these processes, if we want to treat the evolution of species that decouple from the thermal reservoir. Such species are the background of relic particles in the early Universe and play a very important role in questions of dark matter.

Primordial nucleosynthesis (Big Bang Nucleosynthesis, or BBN) describes the processes that occurred in the Universe in the period between 10^{-2} and 100 seconds. In these moments when were formed the lightest nuclei of the periodic table of chemical elements, which are basically ^4He, ^3He, D and ^7Li.

The BBN model is supported by the standard model of particle physics and the cosmological standard model. The model presented below allows predictions to be made about the number of light nuclei that formed in the period between 10^{-2} and 100 seconds. Primordial nucleosynthesis (BBN) is one of the great pillars of the cosmological standard model. Therefore, the predictions of this model agree with the observations, within the range of error. However, BBN predictions depend on a large number of parameters, one of the most important being the expansion rate of the Universe. The expansion of the Universe of the most important predictions of general relativity, and its rate is given, from the Friedmann equation, that is, equation (16), when taking into account that the Universe is homogeneous and isotropic.

2.1 Relic particles

One of the most notable consequences of the gradual decrease in the temperature of the Universe is the evolution suffered by the distributions of particle species.

Therefore, it is necessary to determine the distribution functions f_i of each species as a function of $|\overrightarrow{p}|$ and t. The spatial dependence disappears due to the homogeneity of the Universe. As a first approximation, a qualitative approach is used. This criterion can be verified numerically with the general formalism that will be presented in the Boltzmann equation. The criterion consists in comparing the rate of expansion of the Universe, $H\left(t\right)$, with the total rate of interaction $\Gamma_i\left(t\right)$, which takes into account all the interactions of the species i-th with any other particle. This comparison allows for two regimens:

1. When $\Gamma_i\left(t\right)\gg H\left(t\right)$, the interactions allow to maintain the thermodynamic equilibrium between the interacting species at a certain temperature T, therefore, we have a plasma. Furthermore, all relevant interactions are short-range. For this reason we can assume that the interactions are limited to making thermalization possible and that the distribution function is simply that of an ideal gas of bosons or fermions, that is, equation (2.36). The function f evolves adiabatically.

2. From $\Gamma_i(t) \lesssim H(t)$, keeping $\Gamma(t)_{j\neq i} \gg H(t)$ for all other species, the rate of reactions that keep particle i in equilibrium is not fast enough to overcome the expansion of the Universe, and so the particle of species i decouples from the plasma. The other species remain in thermal equilibrium, with distribution functions (1.43). To determine the distribution function of the decoupled particles, f_D, we can follow the following argument. When species i decouples, it falls freely following a geodesic in spacetime. Suppose then that at a certain instant t, a comoving observer observes $dN = fd^3pd^3r$ particles in a proper volume d^3r and with an interval of moments $(\overrightarrow{p}, \overrightarrow{p} + d\overrightarrow{p})$. Since the particles cannot interact, at time $t + \delta t$, the observer will continue to observe dN particles, in a proper volume that has increased by a factor $[a(t+\delta t)/a(t)]^3$ and in a range of moments that has decreased by $[a(t+\delta t)/a(t)]^{-3}$. Since the volume of the phase space is conserved and dN is conserved, then f must be conserved along the geodesic.

As the distribution function is conserved, even after decoupling, this allows to determine the function f_D valid for moments after the decoupling time, t_D. From the equilibrium distribution function f_{equil}, valid for $t < t_D$ and given by equation (43), it is found that

$$f_D(\overrightarrow{p}, t) = f_{equil}\left(\frac{a(t)}{a_D}\overrightarrow{p}, t_D\right). \tag{71}$$

2.2 Uncoupled particle density

With what has been described so far, we will be able to describe the non-equilibrium behavior of particles that were relativistic or non-relativistic when decoupling. Whatever the class, it will depend on the relationship between the mass of the particle and the temperature at the moment of decoupling, T_D. For simplicity, in this section we will neglect the chemical potential of the particles, that is, $\mu \ll T_D$.

In the case of relativistic particles, let us consider those particles that at the moment of decoupling fulfill the condition $T_D \gg m$. For these species, $E \simeq |\overrightarrow{p}|$ and therefore, according to equation (71) we have that

$$f_D(\overrightarrow{p}) = f_{equil}\left(\frac{a(t)}{a_D}\overrightarrow{p}, T_D\right) \simeq \frac{1}{\exp\left[E\left(a(t)/a_D\right)T_D^{-1}\right] \pm 1}, \tag{72}$$

even though the particles are no longer in equilibrium, it corresponds to an f_{equil} distribution with a temperature $T(t) = [a_D/a(t)]\,T_D$. This decrease in temperature, $T \propto a^{-1}$, is slightly faster than at equilibrium, i.e. than the equation $T \propto g_{*S}^{-1/3}a^{-1}$.

If we take into account equations (47), (48) and (49), then we obtain the energy densities of decoupled particles and the pressure

$$\rho = \begin{cases} \frac{\pi^2}{30} g T_D^4 \left[\frac{a_D}{a(t)}\right]^4 & (BE), \\ \frac{7}{8}\frac{\pi^2}{30} g T_D^4 \left[\frac{a_D}{a(t)}\right]^4 & (FD), \end{cases} \tag{73}$$

$$n = \begin{cases} \frac{1}{\pi^2}\varsigma(3)\, g T_D^3 \left[\frac{a_D}{a(t)}\right]^3 & (BE), \\ \frac{3}{4\pi^2}\varsigma(3)\, g T_D^3 \left[\frac{a_D}{a(t)}\right]^3 & (FD), \end{cases} \tag{74}$$

$$p = \frac{\rho}{3}. \tag{75}$$

If a species decouples when it is already in the nonrelativistic regime, that is, when $T_D \ll m$, the distribution function is

$$f_D(\overrightarrow{p}) = f_{equil}\left(\frac{a(t)}{a_D}\overrightarrow{p}, T_D\right) \simeq \exp\left(-\frac{m}{T_D}\right)\exp\left[-\frac{p^2}{2m}\frac{1}{T_D}\left(\frac{a}{a_D}\right)^2\right], \tag{76}$$

where the Maxwell-Boltzmann distribution function has been used, as it corresponds to a non-relativistic gas. In this case $E \simeq m + p^2/2m$, so the dependence of the distribution function on the kinetic energy corresponds to a function in equilibrium with an effective temperature $T(t) = [a_D/a(t)]^2\,T_D$. We see that it decreases faster than in the relativistic case, being $T \propto a^{-2}$.

From of equations (50), (51) and (52) we can find the energy, particle and pressure densities:

$$n = g\sqrt{\left(\frac{mT_D}{2\pi}\right)^3}\left(\frac{a_D}{a}\right)^3 \exp\left[-\frac{m}{T_D}\right], \tag{77}$$

$$\rho = g\sqrt{\left(\frac{mT_D}{2\pi}\right)^3}\left(\frac{a_D}{a}\right)^3 m\exp\left[-\frac{m}{T_D}\right], \tag{78}$$

$$p = g\sqrt{\left(\frac{mT_D}{2\pi}\right)^3}\left(\frac{a_D}{a}\right)^5 T_D\exp\left[-\frac{m}{T_D}\right] \ll \rho. \tag{79}$$

As expected, according to equation (77), we find $n \propto a^{-3}$, just as in the relativistic case.

2.3 Boltzmann equation

The Boltzmann equation is the partial differential equation that governs the evolution of a given distribution function of a species A in phase space. This equation, in its non-relativistic version, is written as follows [58] [59]

$$\widehat{D}_{NR}f = \left(\frac{\partial}{\partial t} + \overrightarrow{\mathbf{v}}\cdot\nabla + \overrightarrow{\mathbf{F}}\cdot\nabla_{\overrightarrow{p}}\right) f(\overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{p}}, t) = \widehat{C}_{NR}[f]. \tag{80}$$

The relativistic generalization [59–64] of equation (80) is expressed by

$$\widehat{D}f = p^{\rho}\frac{\partial f}{\partial x^{\rho}} - \Gamma^{\lambda}_{\mu\nu}p^{\mu}p^{\nu}\frac{\partial f}{\partial p^{\lambda}} = \widehat{C}\left[f\right], \tag{81}$$

where $\widehat{C}[f]$ is the collision term. For the Friedmann case, where the distribution function depends on energy and time, $f = f(E,t)$, the Boltzmann equation; that is, equation (81), takes the form:

$$E\frac{\partial f}{\partial t} - \frac{1}{a(t)}\frac{da(t)}{dt}\left|\overrightarrow{p}\right|^{2}\frac{\partial f}{\partial E} = \widehat{C}\left[f\right]. \tag{82}$$

Integrating this equation over $\overrightarrow{p}_A$, and recalling from equation (40) that

$$n_A = \frac{g_A}{(2\pi)^3}\int f(E,t)\, d^3p_A, \tag{83}$$

we obtain

$$\frac{dn_A}{dt} + 3\frac{1}{a(t)}\frac{da(t)}{dt}n_A = \frac{g_A}{(2\pi)^3}\int \widehat{C}\left[f\right]\frac{d^3p_A}{E_A}. \tag{84}$$

The collision term for an arbitrary process P: $A+a+b+ ... \longrightarrow i+j+...$ will be given by

$$\frac{g}{(2\pi)^3}\int \widehat{C}\left[f\right]\frac{d^3p_A}{E_A} = -\int d\Pi_A d\Pi_a d\Pi_b ... d\Pi_i d\Pi_j ... (2\pi)^4\,\delta^4\left(p_A + p_a + p ... - p_i - ...\right)$$
$$\times\left[\left|\mathcal{M}\right|^2_P f_A f_a f_b ... \left(1\pm f_i\right)\left(1\pm f_j\right) ... - \left|\mathcal{M}\right|^2_Q f_i f_j ... \left(1\pm f_A\right)\left(1\pm f_a\right) ...\right], \tag{85}$$

where P and Q represent the processes $A+a+b+... \longrightarrow i+j+...$ and $i+j+... \longrightarrow A+a+b+...$, also the different f correspond to the different functions of distribution in the phase space of the different species. The factors $(1\pm f)$ are the factors of amplification (+, for bosons) or inhibition (−, for fermions). On the other hand, we have defined

$$d\Pi \equiv \frac{g d^3p}{(2\pi)^3\, 2E}. \tag{86}$$

Using the invariance T, we can be stated that

$$\left|\mathcal{M}\right|^2_{A+a+b+...\longrightarrow i+j+...} = \left|\mathcal{M}\right|^2_{i+j+...\longrightarrow A+a+b+...}, \tag{87}$$

which we will simply denote as $|\mathcal{M}|^2$. Finally, if we consider that we normally do not have Bose-Einstein degenerate or condensed fermions, we can approximate the Bose-Einstein and Fermi-Dirac statistics to the Maxwell-Boltzmann one, without having to introduce relevant modifications. This is because, where $T \gtrsim \mu_i$, the three distribution functions are very similar around the maximum. Also, if they are much less than 1, then $1 \pm f \simeq 1$. Taking into account these considerations, and equation (87), we obtain a simpler differential equation

$$\frac{dn_A}{dt} + 3Hn_A = -\int d\Pi_A d\Pi_a d\Pi_b ... d\Pi_i d\Pi_j ... (2\pi)^4\,\delta^4\left(p_A + p_a + ... - p_i - ...\right)\left|\mathcal{M}\right|^2\left[f_A f_a f_b ... - f_i f_j ...\right], \tag{88}$$

which will be applied below in two types of two-body reactions, which are generally the most relevant.

2.3.1 Processes $A\overline{A} \leftrightarrow X\overline{X}$

Consider an annihilation process $A\overline{A} \leftrightarrow X\overline{X}$, where X represents any of the particles into which A can annihilate. We will assume that X and $\overline{X}$ have thermal distributions and zero chemical potential, in order to apply equation (88). So, the collision term of the integrand is $f_A f_{\overline{A}} - f_X f_{\overline{X}} = f_A f_{\overline{A}} - \exp\left(-\frac{E_X+E_{\overline{X}}}{T}\right)$. On the other hand, conservation of energy implies that $E_x + E_{\overline{x}} = E_A + E_{\overline{A}}$, so the collision term takes the form, $f_A f_{\overline{A}} - f_X f_{\overline{X}} = f_A f_{\overline{A}} - \exp\left(-\frac{E_A+E_{\overline{A}}}{T}\right)$, that is, $f_A f_{\overline{A}} - f_X f_{\overline{X}} = f_A f_{\overline{A}} - f_A^{equilibrio} f_{\overline{A}}^{equilibrio}$. If there is no asymmetry between A and $\overline{A}$, by virtue of equation (83), we can transform equation (88) into the differential equation

$$\frac{dn_A}{dt} + 3Hn_A = -\left\langle \sigma_{A\overline{A}\rightarrow X\overline{X}}\left|v\right|\right\rangle\left[n^2 - \left(n_A^{equil}\right)^2\right], \tag{89}$$

where

$$\left(n_A^{equil}\right)^2\left\langle \sigma_{A\overline{A}\rightarrow X\overline{X}}\left|v\right|\right\rangle = -\int d\Pi_A d\Pi_{\overline{A}} d\Pi_X d\Pi_{\overline{X}}\,(2\pi)^4\,\delta^4\left(p_A + p_{\overline{A}} - p_X - p_{\overline{X}}\right)\left|\mathcal{M}\right|^2$$

$$\times \exp\left(-\frac{E_A + E_{\overline{A}}}{T}\right).$$

The second term of the differential equation (89), due to the expansion, can be eliminated by working with the ratio between the particle density and the entropy density. This quantity is denoted by Y and was defined in equation (70). However, we know that $s \propto a^{-3}$ along the expansion, $\frac{d}{dt}s = -3Hs$, so we have

$$\frac{dY}{dt} = \frac{1}{s}\left(\frac{dn}{dt} + 3Hn\right).$$

On the other hand, we would be interested in working with temperature, rather than with time. To do this, we introduce the independent variable $x \equiv m/T$, using the relationship $\frac{1}{T}\frac{dT}{dt} = -\frac{1}{a}\frac{da}{dt} = -H$ we arrive at the expression

$$\frac{dY}{dt} = Hx\frac{dY}{dx}. \tag{90}$$

Finally, if we consider the possibility that there are other annihilation channels for A, we will simply have to add the different distributions σ_j. Then, with the help of equation (90), the Boltzmann equation (89) takes the form

$$\frac{dY}{dx} = -\frac{\langle\sigma\,|v|\rangle\, s}{xH\,(x)}\left(Y^2 - Y_{equil}^2\right) = -\frac{x\,\langle\sigma\,|v|\rangle\, s}{H\,(m)}\left(Y^2 - Y_{equil}^2\right). \tag{91}$$

In the second equality of expression (91), (55) has been used to establish the change of variable $x = m/T$

$$H \simeq 1.66\sqrt{g_*}\left(\frac{T^2}{M_P}\right) = 1.66\sqrt{g_*}\left(\frac{m^2}{M_P}\right)x^{-2} \equiv H\,(m)\,x^{-2}. \tag{92}$$

By working a bit on equation (91) it is possible to make analytically visible the approximate decoupling criterion, $\frac{\Gamma}{H} \sim 1$, exposed above. If we define

$$\Gamma = n_{equil}\,\langle\sigma\,|v|\rangle\,, \tag{93}$$

then, we can write equation (91) in the form

$$\frac{x}{Y_{equil}}\frac{dY}{dx} = -\frac{\Gamma}{H}\left[\left(\frac{Y}{Y_{equil}}\right)^2 - 1\right], \tag{94}$$

which tells us that the change of particles A per volume comovil is controlled by the factor Γ/H, which gives an idea of the effectiveness of the annihilations and the deviation of Y from equilibrium.

From equation (94) we can deduce a qualitative behavior of the rate of interaction and the rate of expansion of the Universe. For high temperatures, the value of x approaches zero. Under these conditions, $\Gamma \gg H$ is usually the case, so for equation (94) to hold and the second member term to be very small, the approximation $Y \simeq Y_{equil}$ must hold. For very large values of x; that is, for very small values of temperature, when $\Gamma \ll H$ holds, the term on the first member of Boltzmann's equation (94) must be very small, so that, in such a case, $Y \sim Cte$. Thus, since $\Gamma \propto Y_{equil}$ and, therefore, decreases as T decreases, whatever the regime, there must be a more or less localized interval, centered on x_D, from which annihilations are inefficient.

2.3.2 Processes $A + 1 \leftrightarrow 2 + 3$

Let us now consider a process other than pair annihilation, such as the type $A + 1 \leftrightarrow 2 + 3$, where species 1, 2 and 3 represent any of the plasma particles that interact with A. The procedure for finding the equation Boltzmann, which gives us the particle density A, is analogous to the one followed in the previous subsection, so we will not repeat it. Which implies that we will find a differential equation different from equation (94). As in the previous section, species 1, 2, and 3 will be assumed to be in thermal equilibrium and to have zero chemical potential. So, the collision term of the integrand is $f_A f_1 - f_2 f_3 = f_A \exp\left(-\frac{E_1}{T}\right) - \exp\left(-\frac{E_2+E_3}{T}\right)$. As in the previous case, we assume that the particles are non-relativistic, therefore they obey the Maxwell-Boltzmann distribution function. On the other hand, conservation of energy implies that $E_A + E_1 = E_2 + E_3$, so the term of the collisions is of the form, $f_A f_1 - f_2 f_3 = \left[f_A - \exp\left(\frac{E_A}{T}\right)\right]\exp\left(-\frac{E_1}{T}\right)$, that is, $f_A f_1 - f_2 f_3 = \left[f_A - f_A^{equil}\right]\exp\left(-\frac{E_1}{T}\right)$. This leads us to write equation (88) in the form

$$\frac{dn_A}{dt} + 3Hn_A = -\langle \sigma_{A+1\to 2+3} |v| \rangle n_A^{equil} \left[n - n_A^{equil} \right], \quad (95)$$

where

$$\left(n_A^{equil} \right)^2 \langle \sigma_{A+1\to 2+3} |v| \rangle = -\int d\Pi_A d\Pi_1 d\Pi_2 d\Pi_3 \left(2\pi \right)^4 \delta^4 \left(p_A + p_1 - p_2 - p_3 \right) |\mathcal{M}|^2 \exp\left(-\frac{E_A + E_1}{T} \right).$$

Analogously to the development of the previous section, we use the definitions (70), (92), (93) and the equation $x \equiv m/T$ to arrive at the differential equation

$$\frac{x}{Y_{equil}} \frac{dY}{dx} = -\frac{\Gamma}{H} \left[\frac{Y}{Y_{equil}} - 1 \right]. \quad (96)$$

We have obtained an equation slightly different from equation (94), but with a completely equivalent behavior related to the number of particles of species A.

2.4 Relic Particle Density

One of the most important features of relic particles is that they may be an important component of the dark matter present in the Universe. As we know, dark matter contributes 23% to the total energy density of the Universe. Therefore, to know if dark matter is composed of one relic species or another, one must know how to calculate the contribution to the energy density of the Universe corresponding to a relic particle species. To calculate the current density of relic particles, it is done through the equation

$$n_{A0} = Y_\infty s_0, \quad (97)$$

where

$$s_0 = \frac{2\pi^2}{45} g_{*S_0} T_0^3, \quad (98)$$

with $T_0 \simeq 2.75^{\text{o}} K$ and $g_{*S_0} \simeq 3.91$.

Calculation of Y_∞

The value of Y_∞ is obtained by solving the differential equation (94), or (96), and looking for $Y(x \to \infty)$. To solve these differential equations, we must first find the behavior of Γ with temperature. Once known, it is not always possible to find analytical solutions to equations (94) and (95) and approximate methods must be used. To find the value of Y_∞, it is necessary to know the function $Y_{equil}(x)$, which will depend on whether the regime is relativistic, $x \ll 1$, or non-relativistic, $x \gg 1$. By virtue of equations (48), (50), and (68) and considering the limit $\mu \ll T$, we have

$$Y_{equil} = \begin{cases} \frac{45\zeta(3)}{2\pi^2} \left(g/g_{*S} \right) \simeq 0.28 \frac{g}{g_{*S}} & (BE), \\ \frac{45\zeta(3)}{2\pi^4} \frac{3}{4} \left(g/g_{*S} \right) \simeq 0.21 \frac{g}{g_{*S}} & (FD), \end{cases} \quad (99)$$

for relativistic particles, and

$$Y_{equil} = \frac{45}{2\pi^4} \sqrt{\frac{\pi}{8}} \frac{g}{g_{*S}} \left(\frac{m}{T} \right)^{3/2} \exp\left(-\frac{m}{T} \right) \simeq 0.15 \left(\frac{g}{g_{*S}} \right) x^{3/2} e^{-x}, \quad (100)$$

for non-relativists.

As can be seen, according to equation (99), when the particles are relativistic, the value of Y_{equil} remains constant. For this reason, when decoupling occurs in the relativistic regime, the value of Y_∞ does not depend on its details. The only dependence on the moment in which it occurs is found in the value of g_{*S}. On the other hand, when the decoupling occurs in the non-relativistic regime, the dependence of x_D is much greater.

The difference between the value of $Y_{equil}(x_D)$ and the value of Y_∞ found by solving the differential equation is quite small. As we have just seen, this difference is completely negligible when the decoupling occurs in the case that the particle is relativistic. This qualitative reasoning has been adopted in the calculations, since we have always been interested in the case of decoupling of relativistic particles. However, for the case of particles that decoupled being non-relativistic, if you want to obtain more precise values of Y_∞ than those obtained with $Y_{equil}(x_D)$, the procedure to follow to find them is standard [65].

As already mentioned, one of the reasons why it is important to know how to calculate the current density of relic particles is to elucidate whether any of these species could be dark matter. For this reason, we are interested in knowing what fraction of the energy density of the Universe would correspond to relic particles. Particles that can be dark matter are non-relativistic, at present, so their current energy density is

$$\rho_{A0} = n_{A0} m_A, \quad (101)$$

where n_{A0} is found from equation (97). The contribution to the density of the Universe corresponding to the relic particle A, will be

$$\Omega_A = \frac{\rho_{A0}}{\rho_c} = \frac{n_{A0} m_A}{\rho_c}, \tag{102}$$

where ρ_c is the value of the critical density of the Universe at present, estimated in equation (28).

The contribution of dark matter to the density of the Universe is bounded by $\Omega_{DM} \lesssim 0.23$. This limit obviously also has to be respected by the above equation for Ω_A, so we impose restrictions on the characteristics of the relic particles.

2.5 Nucleosintesis primordial

The first ideas about the BBN appeared in the 1950s, George Gamow [66–69] and his collaborators Alpher and Herman [70] developed great advances in the Big Bang theory, for example, the prediction of the cosmic microwave background.

It could be thought that this field is scientifically complete, which is not entirely true if you look at the number of publications that appear regularly on the subject. On the other hand, over the years, observational data on primordial abundances have improved significantly. Keep in mind that determining, through observations, the essential elements is not an easy task. Not only do we have to measure concentrations of elements in different stellar environments, but we also have to know what the evolution of these elements has been since they were created, to know if the concentrations we currently observe are higher or lower than the primordial ones.

Thus, the BBN, the Hubble expansion and the cosmic microwave background (CMB) are the pillars on which the Big Bang theory is based. Of the three, the BBN gives evidence of the Universe in its most remote times, so we can say that the BBN period is the oldest laboratory we have. It is a tool that allows us to test and constrain a large number of parameters, and any new theory that emerges has to pass the BBN test. This is another reason why the BBN is a model that will always be important, since any extension of the cosmological standard model has to be consistent with the predictions of the BBN.

2.6 Formation of light nuclei

In previous paragraphs it was mentioned that the period in which the BBN takes place is between 10^{-2} y 100 seconds. This is equivalent to a temperature range of 10 MeV $\lesssim T \lesssim 0.1$ MeV, the energy range characteristic of nuclear reactions.

In the initial conditions of the BBN period, the particles present in the Universe are photons (γ), neutrinos (ν) and antineutrinos ($\overline{\nu}$) of the three lepton families, electrons $\left(e^-\right)$, neutrons (n), and protons (p). The reactions between these particles are mediated by weak interactions

$$\begin{aligned} n &\longrightarrow p + e^- + \overline{\nu}, \\ \nu + n &\longrightarrow p + e^-, \\ e^+ + n &\longrightarrow p + \overline{\nu}. \end{aligned} \tag{103}$$

Under these conditions the rate of interaction of the reactions is much lower than the rate of expansion of the Universe. As stated at the beginning of this chapter, if $\Gamma_w \gg H$, all the particles involved in the weak interactions are in thermal and chemical equilibrium, so the number densities follow the distributions (47-52) . In our time the neutrons and protons are non-relativistic, the ratio of their densities when they are in equilibrium is:

$$\left(\frac{n}{p}\right)_{equil} = \exp\left[-\frac{m_n - m_p}{T}\right], \tag{104}$$

where it has been taken into account that $\frac{m_n}{m_p} \simeq 1$ and $\mu_n = \mu_p$. The last assumption follows from chemical equilibrium and taking the lepton chemical potentials to be negligible. If the case of non-zero neutrino lepton chemical potential is considered, then we have a phenomenon called degenerate BBN.

If the rate of interaction, Γ_w, becomes less than the rate of expansion of the Universe, H, thermal equilibrium is lost. To know when this deviation from equilibrium occurs, it is necessary to know the evolution equation of the rate of interaction with temperature. For electroweak reactions, we have $\Gamma_w \propto G_F^2$, where $G_F = 1.2 \times 10^{-6} GeV^{-2}$ is the Fermi constant. If we are in a temperature regime in which $T > m_e$, then $\Gamma_w \sim G_F^2 T^5$, taking into account that the rate of interaction has dimensions of energy.

By virtue of equation (55), we have that $H \sim T^2/M_P$, we can estimate the Fermi temperature, at which neutrons and protons decouple from the thermal plasma. This happens when $\Gamma_w \simeq H$, whence $G_F^2 T^5 = T^2/M_P$, and consequently

$$T_F \simeq \frac{1}{\sqrt[3]{G_F^2 M_P}} \simeq 0.8 \text{ MeV}. \tag{105}$$

Below the Fermi temperature, electroweak two-body reactions become ineffective, and if neutron decay did not exist, the ratio of neutrons to protons would be constant, taking the value

$$\left(\frac{n}{p}\right)_F = \exp\left[-\frac{m_n - m_p}{T_F}\right] \simeq \frac{1}{6}. \tag{106}$$

The first nuclear reaction necessary for nuclei to begin to form is the union of a proton with a neutron to form a deuteron, i. e., $p + n \longrightarrow D + B_D$, where B_D represents a photon with an energy corresponding to the binding energy of Deuteron, $B_D = 2.2$ MeV. Once the

Deuterium reaction is effective to the right, it opens a series of nuclear channels, initiated with Deuterium. These channels are:

$$\begin{aligned} D+n &\longrightarrow^3 H + B_{^3H} \quad con \quad B_{^3H} = 6.9 \text{ MeV}, \\ D+p &\longrightarrow^3 He + B_{^3He} \quad con \quad B_{^3He} = 7.7 \text{ MeV}, \\ D+D &\longrightarrow^4 He + B_{^4He} \quad con \quad B_{^4He} = 28.3 \text{ MeV}, \end{aligned} \tag{107}$$

the third of these is the most effective, since ^4He is the most strongly bound nucleus of all those formed. Note that all these nuclear reactions occur when the temperature is less than that of deuterium, that is, $T \simeq 0.1$ MeV and they take place in one direction. In nature, there are no stable nuclei with mass numbers $A = 5,\ 8$ and this is what explains why heavier nuclei are not formed in appreciable concentrations in Primordial Nucleosynthesis. For example, the reaction $^4He +^4 He \longrightarrow^8 Be + B_{^8Be}$ is not effective, since ^8Be is unstable and immediately decays into ^4He. The rest of the heavier nuclei will form much later in stellar nucleosynthesis, because in stars the densities are much higher, allowing three-body reactions to occur.

However, not all neutrons end up forming ^4He nuclei. There comes a time when the temperature decreases to values where it is not possible to break the Coulomb barriers, that is, electrostatic repulsion prevents the reaction between deuteron and deuteron from taking place, so the remaining neutrons form deuterium, although at very low densities. The order of magnitude of the ratio of deuterium nuclei to protons is $\frac{D}{H} \sim 10^{-5}$.

Apart from ^4He and ^2H, nuclei of ^3He and ^7Li are also formed in the BBN, but residually. Their abundances are of the order $\frac{^3He}{H} \sim 10^{-6}$and $\frac{^7Li}{H} \sim 10^{-10}$, which are very difficult to determine. measure, so the information that comes from these atomic nuclei is very poor.

Everything exposed in this section is a semi-qualitative development of the relevant processes that occur in promondial nucleosynthesis. Actually, to obtain precise data, a numerical code is used that simulates a network of 88 nuclear reactions between 26 atomic nuclei, in an expanding box [71]. The primordial abundances predicted by the BBN depend on many parameters. Any parameter that affects the expansion of the Universe, the rate of nuclear reactions, the weak interactions, the binding energies, the rate of expansion, etc., will be important in the final concentrations of the light nuclei. Some of the parameters that introduce variations in the predictions are the rate of expansion of the Universe, H, the weak interactions, density of Baryons, Ω_B, and the asymmetry between neutrinos and antineutrinos. No more are mentioned, since it is impossible to list them all. We only mention the most relevant ones.

2.7 Primordial Abundances

The BBN predictions for ^2H, ^3He, ^4He and ^7Li correspond to the concentrations present when $t \sim 200s$. However, the observed abundances correspond to much later times, after the start of stellar nucleosynthesis. For this reason, the greatest difficulty in experimentally determining the primordial abundances lies in the fact that the different synthesized elements have undergone a large number of chemical processes, with which the abundances observed in recent years differ significantly from the primordial concentrations. Nuclei of ^2H, ^3He, ^4He and ^7Li are very brittle and burn up rapidly inside stars, at a relatively low temperature, around 10^6K. Generally, the gas inside a star usually exceeds the critical temperature. However, although stellar processes can alter the abundances of light nuclei from their primordial values, they also produce other heavy elements, such as C, N, O and Fe, that is, metals. These metals are the traces of stellar activity, so if astrophysical regions with low abundances of metals are sought, the abundances of light elements measured there will initially approach their primordial values. These regions are usually stellar environments as far away as possible (high gravitational redshift), which is equivalent to measuring in remote times when there had not been much stellar activity yet. However, even if the metallicity is low, it is not easy to show that the measured abundances are really the primordials.

Helium is observed in hot, ionized gas, $T \gg 10^4K$, found near young, luminous stars. The zones where this gas is found are called H II and are formed mainly by hydrogen and helium, with an O abundance of 0.02 to 0.2 times greater than of the Sun.

Instead, the evolution of Helium after BBN is simpler. Deuterium is destroyed in stars, producing ^3He. Because it is a very loosely bound nucleus, there is no astrophysical process that can produce significant amounts of Deuterium. For this reason, any measure of deuterium provides a lower bound on its primordial abundance. In recent decades, thanks to high-resolution spectra, it has been seen that in quasar systems with high redshift and low metallicity, there is a significant presence of deuterium. It is believed that these systems are not contaminated by stellar processes, so the abundance of deuterium that is observed must be very close to the primordial one. These deuterium measurements are the first of light elements at cosmological distances and exhibit a large number of systematic errors.

^7Li measurements are very problematic. The most suitable systems for the observation of ^7Li are the atmospheres of stars of poor metallicity located in the halo of our galaxy. However, different factors make it difficult to obtain primordial abundance. On the one hand, the detection of ^6Li in these stars suggests that ^6Li and ^7Li

were created prior to their formation. On the other hand, a small but significant correlation is observed between Li and Fe. All this indicates that the ^7Li observed in these stars is not the primordial one. However, the primordial abundance of ^3He is the most difficult to estimate, since this nucleus is created and destroyed in stellar environments. In addition, the only available observations correspond to the solar system and to H II regions of our galaxy, with high metallicity [72]. This makes estimation of primordial abundance very difficult. This problem is compounded because stellar nucleosynthesis models conflict with 3 He observations [73]. For these reasons 3 He and ^{7}Li cannot be used as cosmological indicators.

3 Concluding remarks

One of the cosmological parameters that we have mentioned is the dark matter density parameter. In astrophysics and cosmology, dark matter is called non-interacting matter with a composition that does not emit or reflect electromagnetic radiation to be observed directly with current technical means, but whose existence can be inferred from the gravitational effects it causes on visible matter (Visible matter is that which emits or reflects electromagnetic radiation). Despite the similar name, dark matter and dark energy are thought to be different phenomena, it is not known whether they are related or not. Dark matter is a form of matter that does not emit and absorb light, i. e., the only interaction to which it reacts is gravitation. Instead, dark energy is a new ingredient to explain the Universe, because the known forms of matter and dark matter explain only 30% of the Universe. The rest, that is, 70%, would be explained by dark energy, which is distinguished from dark matter by the fact that it is gravitationally repulsive, leading the Universe to accelerated expansion.

One of the most notable consequences of the gradual decrease in the temperature of the Universe is the evolution suffered by the distributions of particle species. Therefore, it is necessary to determine the distribution functions f_i of each species as a function of $|\overrightarrow{p}|$ and t. The spatial dependence disappears due to the homogeneity of the Universe. As a first approximation, a qualitative approach is used. This criterion can be verified numerically with the general formalism that was presented in the Boltzmann equation. The criterion consists in comparing the rate of expansion of the Universe, $H(t)$, with the total rate of interaction $\Gamma_i(t)$, which takes into account all the interactions of the species i-th with any other particle. This comparison allows for two regimens:

1. When $\Gamma_i(t) \gg H(t)$, the interactions allow to maintain the thermodynamic equilibrium between the interacting species at a certain temperature T, therefore, we have a plasma. Furthermore, all relevant interactions are short-range.

2. From $\Gamma_i(t) \lesssim H(t)$, keeping $\Gamma(t)_{j\neq i} \gg H(t)$ for all other species, the rate of reactions that keep particle i in equilibrium is not fast enough to overcome the expansion of the Universe, and so the particle of species i decouples from the plasma. The other species remain in thermal equilibrium, with distribution functions (1.43).

The BBN, the Hubble expansion and the cosmic microwave background (CMB) are the pillars on which the Big Bang theory is based. Of the three, the BBN gives evidence of the Universe in its most remote times, so we can say that the BBN period is the oldest laboratory we have. It is a tool that allows us to test and constrain a large number of parameters, and any new theory that emerges has to pass the BBN test. This is another reason why the BBN is a model that will always be important, since any extension of the cosmological standard model has to be consistent with the predictions of the BBN.

So far, it has been described that the cosmological standard model is based on the general theory of relativity and the standard model of elementary particles. However, it is known that one theory like the other has its limitations. Consequently, currently, there are some open problems that do not find a solution within the framework of the cosmological standard model. Some of these open problems are, for example, explaining the nature of dark matter, the cosmological constant, explaining why the total energy density is so close to critical, etc.

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