


## Gravitation, cosmology and dark matter

 Mississippi Valenzuela <sup>\*1,2</sup>

<sup>1</sup> *Universidad Nacional Autónoma de México, Instituto de Física, Departamento de Física Teórica, Apartado Postal 20-364, Cd. México 01000, México*

<sup>2</sup> *Unidad Académica de Ciencia y Tecnología de la Luz y la Materia, Universidad Autónoma de Zacatecas, Circuito Marie Curie s/n, Parque Científico y Tecnológico QUANTUM Ciudad del Conocimiento, Zacatecas 98160, Zacatecas México*

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### Abstract

In this work, we present the fundamentals of the general theory of relativity, the standard cosmology and a description of current and interesting research topics about our current Universe, in addition the emergence of the dark matter problem has been raised, trying to outline possible dark matter candidates. This assumption seems realistic, however the analysis, from the point of view of standard cosmology, which is also studied in this work, turns out to be more complex than if dark matter were considered with a single species of elementary particle.

**Keywords:** Curvature Tensor, field equations, Friedmann equations, dark matter candidates..

### Resumen

### Gravitación, cosmología y materia oscura

En este trabajo, presentamos los fundamentos de la teoría general de la relatividad, la cosmología estándar y una descripción de temas actuales e interesantes de investigación acerca de nuestro Universo actual, además se ha planteado el surgimiento del problema de materia oscura, tratando de bosquejar posibles candidatos a materia oscura. Esta suposición parece realista, sin embargo el análisis, desde el punto de vista de la cosmología estándar, la cual también se estudia en el presente trabajo, resulta ser más complejo que si se considerará a la materia oscura con una sola especie de partícula elemental.

**Palabras clave:** Tensor de curvatura, ecuaciones de campo, ecuaciones de Friedmann, candidatos a materia oscura..

## 1 Introduction

The theory of relativity originates from the problems of the field. The theory of relativity was developed in two stages at the beginning of the 20th century. The first leads to the so-called special theory of relativity that is applied to inertial coordinate systems, that is, where the principle of inertia formulated by Isaac Newton in the 17th century is fulfilled [1].

The second stage, that is, the general relativity [2]

[3] [4] is no longer restricted to only inertial coordinate systems. He attacks the problem of gravitation and formulates new laws that give the structure of gravitation. Consider the equivalence between inertial mass and gravitational mass, which supports the equivalence principle. Replaces the concept of gravitational force in Newtonian theory with the concept of curvature of space-time. The fundamental principles on which the general relativity is based are the principle of equivalence; which supports that acceleration and gravity are two different aspects of

\*mvalenzuelalumat@uaz.edu.mx

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the same reality, the notion of curvature of space-time and the principle of general covariance; this principle argues that the laws of physics behave in the same way in all coordinate systems.

Variational principles [5] have always played a very important role in physical theories and the general relativity is no exception. For example, in classical mechanics, Hamilton's principle. It is through this principle that the Euler-Lagrange equations of motion can be obtained, these equations describe the dynamics of the system when solving them. In the relativistic theory of gravitation, the variational principle is the principle of gravitational action, from which the field equations are obtained. Once the field equations have been obtained by applying the variational calculus to the principle of gravitational action, the gravitational equations can be solved. The general relativity is the best existing theory in the description of certain phenomena of gravitation.

The Lagrangian formulation of a field theory begins with the principle of minimal action; in the relativistic theory of gravitation, the Einstein-Hilbert action, which is defined as the integral of a Lagrangian density over a region of space-time. The fundamental function of the Lagrangian formulation is the Lagrangian function  $L(t)$ , but in the case of the general relativity; since it is a field theory [6].

The last decade has seen an explosive increase in the amount and precision of data obtained from cosmological observations. The number of techniques for data analysis and verification has proliferated in recent years.

Theoretical cosmologists have not yet finished interpreting this amount of data. Likewise, the wonderful ideas we have for the beginning of the Universe have not yet been connected with concrete models of elementary particles. One of our hopes, in addition to providing an overview of the possible candidates that may constitute the subject, is to give an updated overview of the theoretical developments necessary to make this connection.

Understanding the nature of dark matter (DM) is one of the major problems of contemporary cosmology, astrophysics and particle physics. Since 1930s, when the problem of "missing matter" first arose in the context of galaxy rotation curves, despite major advances in studies of the universe and the role of dark matter in its formation, evolution and present behaviour, we are still unable to find out how dark matter incorporates into the particle physics framework.

Numerous theories beyond the Standard Model (SM) [7–11] with sensible dark matter candidates have been proposed. Among them are: supersymmetric particles, additional scalars in the extensions of the SM Higgs sector, axions, dark matter in technicolour or extra dimensions theories, to mention only the most intensely stud-

ied. Nonetheless no traces of new physics were found, neither at the LHC, nor at other particle colliders, while scalar particle, which fits the Standard Model predictions for Higgs boson was discovered. Furthermore experimental efforts leading to direct detection of dark matter particles are still inconclusive.

Our intention is to offer a brief introduction to general theory of relativity and standard cosmology [12] [13] [14]; having emphasis on the relativistic theory of gravitation, as well as a description of current and interesting research topics about our current Universe respect to dark matter. In section 2, we present the general relativity, as well as a consistent derivation of the field equations for relativistic gravitation. In section 3, we show the foundations of standard cosmology. In section 4, we present a description of dark matter. We will use units in which  $c = 1$ .

## 2 General relativity

### 2.1 Principles of General Relativity

We are ready to face our task, that is, to extend ourselves to a theory that incorporates gravitation. In this paper, we will undertake a detailed consideration of the physical principles that guided Einstein in his investigation of the general theory. Indeed, we could adopt the same attitude with general relativity. Furthermore, if we discover limitations in the theory, then there are more salvage changes by investigating the physical basis of the theory rather than playing with the mathematics. It is perhaps significant that Einstein spent much of his life trying to unify general relativity and electromagnetism [2, 3, 3, 15–27, 29–37, 40–56] by various mathematical tricks, but without success<sup>1</sup>.

There are five principles that, explicitly or implicitly, guided Einstein in his investigations. His names are:

1. Mach's principle
2. principle of equivalence
3. covariance principle
4. gravitational minimum coupling principle
5. principle of correspondence

#### Mach's principle

Here we make precise the statements of Mach's principle that are relevant in the formulation of general relativity. The first postulate tries to incorporate the essential part of Mach's ideas.

<sup>1</sup>see other theories of gravitation and electroagnetism [57–71] and quantum gravity [72–75]

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M1. The distribution of matter determines the geometry.

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The next postulate is that it is impossible to talk about motion or geometry in an empty universe, so there would be no correspondence in an empty universe.

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M2. If there is no matter then there is no geometry.

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The final postulate refers to a universe that contains a body, so since there is nothing interacting with it, it does not possess any inertial properties.

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M3. A body in an empty universe does not possess inertial properties.

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### Equivalence principle

Galileo Galilei did some experiments that were revolutionary for his time. Galileo found from his experiments two concepts that are cornerstones in the theory of gravitation. First, Galileo showed that the rate at which an object falls under gravity does not depend on its weight. Second, Galileo measured the rate at which objects fall and found that their acceleration is constant, that is, independent of time. In other words, the rate of fall of an object in free fall, that is, an object that only falls under the influence of gravity, is independent of its mass.

In contrast to the electromagnetic field, the gravitational field shows an extremely curious property; which we will present below. The bodies that move under the action of the gravitational field, experience an acceleration that does not depend in the least on the material or the physical state of the body.

In classical mechanics, the ratio of the masses of two bodies are defined in two ways that differ fundamentally from each other. The first of these ways of defining it; it is what makes said ratio equal to the inverse of the ratio of the accelerations that the same force imparts to moving objects, that is, inertial mass. The second way to define the mass ratio is to make it equal to the ratio of the forces acting on them in the same gravitational field, that is, active gravitational mass. The equality of both masses, defined in such a radically different way, is a fact confirmed by experimental physics with great precision and accuracy, that is, the facts are confirmed by the Eötvös experiment [76]. Classical mechanics does not justify this equality of masses.

By virtue of Newton's equation of motion for a gravitational field

$$(\text{inertial mass})(\text{Acceleration}) = (\text{gravitatory field intensity})(\text{gravitatory mass}).$$

According to the Eötvös experiment, which supports the strength of the equality between gravitational and inertial mass, it turns out that from Newton's equation of motion we deduce that acceleration is independent of the nature of the body.

Now let  $K_0$  be an inertial frame. Material masses that are far enough from each other and from other masses are devoid of acceleration with respect to  $K_0$ . Let us also refer these masses to a coordinate reference system  $K_1$  endowed with uniform acceleration with respect to  $K_0$ . With respect to  $K_1$  all the masses have accelerations equal and parallel to each other, in other words, that with respect to said system, all the masses behave exactly as if a gravitational field were present and  $K_1$  were not accelerated. We can consider this gravitational field as real. We call the physical equivalence between the two reference systems  $K_0$  and  $K_1$  the "principle of equivalence" [77].

### Principle of minimal gravitational coupling

The principles we have discussed so far do not show us how to obtain the field equations of the systems in General Relativity when the corresponding equations are known in special relativity. The principle of minimal gravitational coupling is a simple principle that essentially says that we do not add unnecessary terms to make the transition from the special to the general theory.

### General covariance principle

The general covariance principle ensures the validity of any physical law in a gravitational field if:

1. The mathematical equation that represents the physical law is valid in the absence of gravitation, that is, the metric tensor for curved spaces  $g_{\mu\nu}$  tends to the Minkowski tensor  $\eta_{\mu\nu}$  and the affine connection vanishes.
2. The physical equation is generally covariant, that is, it preserves its form under any coordinate transformation.

The general covariance principle is not a principle of invariance but information about the effects of gravitation.

### The principle of correspondence

The correspondence principle argues that some new theory, with its range of validity, must be consistent with some acceptable theories prior to the new one at some limiting value. General relativity agrees on the one hand with special relativity in the absence of gravitation and on the other hand is consistent with the Newtonian theory of gravitation in the limit of weak gravitational fields

and in the limit of low velocities (compared to the velocity of the light). This gives rise to the principle of correspondence.

As we know, special relativity is consistent with classical Newtonian mechanics at low speeds (compared to the speed of light), and the Newtonian theory of gravitation is consistent with classical mechanics in the absence of gravitation.

## 2.2 Some aspects of the metric tensor $g_{\mu\nu}$

### The covariant metric tensor

In the invariant expression for the square of the line element,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (2.1)$$

the part represented by the  $dx^\mu$  is that a contravariant vector could be changed to a covariant one. Besides,  $g_{\mu\nu} = g_{\nu\mu}$ , it follows from the expression for the square of the line element, that  $g_{\mu\nu}$  is a covariant tensor of second rank. We will call it the "metric tensor". In what follows, we derive the properties of this tensor.

### The contravariant metric tensor

We take the cofactor of each of the elements  $g_{\mu\nu}$  and divide by its determinant  $g = |g_{\mu\nu}|$ , thus we calculate the inverse matrix for the metric tensor, thus having a contravariant tensor. As a consequence, we are only interested in non-singular symmetric metric tensors, such tensors have an inverse given by the equation

$$g_{\mu\sigma} g^{\nu\sigma} = \delta_\mu^\nu, \quad (2.2)$$

where the kronecker symbol  $\delta_\mu^\nu$  denotes

$$\delta_\mu^\nu = \begin{cases} 1, & \text{if } \nu = \mu, \\ 0, & \text{if } \nu \neq \mu. \end{cases} \quad (2.3)$$

### Scalar volume

We look first for the transformation law of the determinant  $g = |g_{\mu\nu}|$ . The transformation law for the metric tensor is given by

$$g'_{\sigma\tau} = \frac{\partial x^\mu}{\partial x'^\sigma} \frac{\partial x^\nu}{\partial x'^\tau} g_{\mu\nu},$$

consequently

$$g' = \left| \frac{\partial x^\mu}{\partial x'^\sigma} \frac{\partial x^\nu}{\partial x'^\tau} g_{\mu\nu} \right|.$$

Therefore, by the rule for the multiplication of determinants, we find

$$g' = \left| \frac{\partial x^\mu}{\partial x'^\sigma} \right| \left| \frac{\partial x^\nu}{\partial x'^\tau} \right| |g_{\mu\nu}| = \left| \frac{\partial x^\mu}{\partial x'^\sigma} \right|^2 g,$$

or

$$\sqrt{-g'} = \left| \frac{\partial x^\mu}{\partial x'^\sigma} \right| \sqrt{-g}.$$

On the other hand, the law of transformation of the volume element

$$d\tau = dx_1 dx_2 dx_3 dx_4,$$

is in accordance with Jacobi's theorem

$$d\tau' = \left| \frac{\partial x'^\sigma}{\partial x^\mu} \right| d\tau.$$

By multiplying the last two equations, we get

$$\sqrt{-g'} d\tau' = \sqrt{-g} d\tau. \quad (2.4)$$

Instead of  $\sqrt{g}$ , we introduce the quantity  $\sqrt{-g}$ , which is always real since only Pseudo-Riemannian metrics will be considered.

## Forming new tensors by using the metric tensor

The internal, external and mixed multiplication of a tensor by means of the metric tensor generates tensors of different character and rank. We can now use  $g_{\mu\nu}$  and  $g^{\mu\nu}$  to raise and lower indices, that is,

$$\begin{aligned} T_{\mu_1 \dots \mu_n \nu_n}^{\nu_1 \dots \nu_{n-1}} &= g_{\nu_n \beta} T_{\mu_1 \dots \mu_{n-1} \mu_n}^{\nu_1 \dots \nu_{n-1} \beta}, \\ T_{\mu_1 \dots \mu_{n-1} \mu_n}^{\nu_1 \dots \nu_{n-1} \nu_n} &= g^{\mu_n \beta} T_{\mu_1 \dots \mu_{n-1} \beta}^{\nu_1 \dots \nu_{n-1} \nu_n}. \end{aligned} \quad (2.5)$$

## 2.3 Christoffel symbols

Consider a particle that moves freely under the influence of purely gravitational forces. According to the equivalence principle, there is a coordinate system in free fall in which the particle's equation of motion is that of a straight line in space-time, i.e.,

$$\frac{d^2 \xi^\alpha}{d\tau^2} = 0, \quad (2.6)$$

where  $d\tau$  denotes the proper time interval

$$d\tau^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta.$$

Suppose that now, we use another coordinate system  $x^\mu$  (a Cartesian system at rest with respect to the laboratory, or a curvilinear system, or an accelerating one, or rotating, or otherwise). The coordinates  $\xi^\alpha$  are functions of the  $x^\mu$ , that is,

$$\xi^\alpha = \xi^\alpha(x^\mu(\tau)),$$

so equation (2.6) takes the form

$$\frac{\partial \xi^\alpha}{\partial x^\mu} \frac{d^2 \xi^\alpha}{d\tau^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0,$$

which by multiplying it by  $\frac{\partial x^\lambda}{\partial \xi^\alpha}$  and using the relation  $\frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial x^\lambda}{\partial \xi^\alpha} = \delta_\mu^\lambda$ , we obtain as result

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma_{\mu\nu}^\lambda \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0, \quad (2.7)$$

where  $\Gamma_{\mu\nu}^\lambda$  is the affine connection defined by

$$\Gamma_{\mu\nu}^\lambda = \frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}. \quad (2.8)$$

The proper time can also be expressed in an arbitrary coordinate system

$$d\tau = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} dx^\mu \frac{\partial \xi^\beta}{\partial x^\nu} dx^\nu,$$

or

$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (2.9)$$

where the metric tensor is defined by

$$g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta}. \quad (2.10)$$

### Relationship between $g_{\mu\nu}$ and $\Gamma_{\mu\nu}^\lambda$

We have just seen that the field that determines the gravitational field acting on a free particle in a non-inertial frame is  $\Gamma_{\mu\nu}^\lambda$  (2.7) and that the proper time interval between two infinitesimally separated events is determined by  $g_{\mu\nu}$ . Next we will see that  $g_{\mu\nu}$  is also the gravitational potential, that is, that its derivatives determine the field  $\Gamma_{\mu\nu}^\lambda$ . Differentiating equation (2.10) with respect to  $x^\lambda$ , we get:

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta} + \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial^2 \xi^\beta}{\partial x^\lambda \partial x^\nu} \eta_{\alpha\beta},$$

equation that we can transform using equation (2.8)

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma_{\lambda\mu}^\rho \frac{\partial \xi^\alpha}{\partial x^\rho} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta} + \Gamma_{\lambda\nu}^\rho \frac{\partial \xi^\beta}{\partial x^\rho} \frac{\partial \xi^\alpha}{\partial x^\mu} \eta_{\alpha\beta},$$

and with the help of (2.10), we can write

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma_{\lambda\mu}^\rho g_{\rho\nu} + \Gamma_{\lambda\nu}^\rho g_{\rho\mu}. \quad (2.11)$$

If we add to equation (2.11) the result of exchanging  $\mu$  and  $\nu$  in it and subtract the same equation exchanging  $\nu$  with  $\lambda$ , we have:

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} = 2\Gamma_{\lambda\mu}^\rho g_{\rho\nu},$$

since  $\Gamma_{\mu\nu}^\rho$  and  $g_{\mu\nu}$  are symmetric under the exchange of  $\mu$  with  $\nu$ .

Defining the matrix  $g^{\nu\sigma}$  as the inverse of  $g_{\nu\sigma}$ , that is,  $g^{\nu\sigma} g_{\rho\nu} = \delta_\rho^\sigma$ ; and multiplying the above equation by  $g^{\nu\sigma}$ :

$$\Gamma_{\lambda\mu}^\sigma = \frac{1}{2} g^{\nu\sigma} \left( \frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} \right). \quad (2.12)$$

As a comment, the relationship between  $\Gamma_{\mu\nu}^\lambda$  and  $g_{\mu\nu}$  has two important consequences: (1) the equation of motion of a particle in free fall automatically maintains the shape of the proper interval,  $d\tau$ , and (2) the law of motion of bodies in free fall can be stated as a variational principle.

### 2.4 Covariant differentiation

Let  $A^\mu$  be a contravariant vector whose components are given with respect to the coordinate system  $x_\nu$ . Let  $P_1$  and  $P_2$  also be two infinitely close points on the continuum. For the infinitesimal region surrounding the point  $P_1$  there exists a coordinate system of  $x_\nu$  for which the manifold is Euclidean. Let us imagine a vector drawn at the point  $P_2$  using the local system of  $x_\nu$  with the same coordinates (parallel vector passing through  $P_2$ ). This parallel vector is therefore determined by the vector through  $P_1$  and the offset. This operation is called parallel displacement of the vector  $A^\mu$  from  $P_1$  to the infinitely close point  $P_2$ . If we make the vector difference between the vector  $A^\mu$  at the point  $P_2$  and the vector obtained by the parallel displacement from  $P_1$  to  $P_2$  we obtain a vector that can be considered as a differential.

If  $A^\nu$  are the coordinates of the vector at  $P_1$ , and  $A^\nu + \delta A^\nu$  the coordinates of the vector shifted to  $P_2$  along the interval  $dx^\nu$ , the  $\delta A^\nu$  (infinitesimal quantity indicating how much the vector field under consideration has shifted) do not cancel out in this case. Consequently, we can write the equation used to express parallel transport [78]:

$$\delta A^\nu = -\Gamma_{\alpha\beta}^\nu A^\alpha dx^\beta. \quad (2.13)$$

If we consider the invariant of the vector  $A^\nu$ , that is, the norm of  $A^\nu$ , defined as  $g_{\mu\nu} A^\mu A^\nu$ , this quantity is an invariant, it cannot vary in a parallel displacement.

In a general manifold, the intuitive concept of parallelism fails. If we transport a vector from one point to another through two different curves, we will obtain two different vectors. However, if we transport a vector from one point to some other and the resulting vector is independent of the path, then, for the usual concept of parallelism, space-time must have zero curvature, that is, the Riemann tensor-Christoffel is null ( $R_{\mu\nu\kappa}^\lambda = 0$ ).

We can obtain the law of parallel displacement of the covariant vector  $B_\mu$  by stipulating that the parallel displacement will be carried out in such a way that the scalar  $\Phi = A^\mu B_\mu$  remains invariant. We then get

$$\delta B_\mu = \Gamma_{\mu\sigma}^\alpha B_\alpha dx^\sigma. \quad (2.14)$$

We know that the transformation law of the functions  $A_\sigma^\alpha$  for a change of variables is given by the invariance of the form

$$F = A_\nu^\mu \xi^\nu u_\mu,$$

where  $A_\nu^\mu$  is an arbitrary second-rank tensor,  $\xi^\nu$  a contravariant vector, and  $u_\mu$  a covariant vector.

Differentiating the above equation with the  $\delta$  operator, we have

$$\delta F = \delta A_\nu^\mu \xi^\nu u_\mu + A_\nu^\mu \delta \xi^\nu u_\mu + A_\nu^\mu \xi^\nu \delta u_\mu.$$

By convention, the differentials  $d\xi^\nu$  and  $\delta u_\mu$  can be calculated by equations (2.13) and (2.14), while  $\delta A_\nu^\mu$  is given by the usual differentiation rule

$$\delta A_\nu^\mu = \frac{\partial A_\nu^\mu}{\partial x^\rho} \delta x^\rho. \quad (2.15)$$

Using these results we have

$$\delta F = \frac{\partial A_\nu^\mu}{\partial x^\beta} \xi^\nu u_\mu \delta x^\beta - A_\nu^\mu \Gamma_{\alpha\beta}^\nu \xi^\alpha u_\mu \delta x^\beta + A_\nu^\mu \Gamma_{\mu\beta}^\alpha \xi^\nu u_\alpha \delta x^\beta.$$

Interchanging  $\nu$  with  $\alpha$  in the second term and  $\alpha$  with  $\mu$  in the third term, we obtain the factor  $\xi^\nu u_\mu \delta x^\beta$ , and consequently we can write

$$\delta F = \left( \frac{\partial A_\nu^\mu}{\partial x^\beta} - \Gamma_{\nu\beta}^\alpha A_\alpha^\mu + \Gamma_{\alpha\beta}^\mu A_\nu^\alpha \right) \xi^\nu u_\mu \delta x^\beta.$$

The right-hand side of this equation is invariant (zero rank tensor), while  $\xi^\nu$ ,  $u_\mu$  and  $\delta x^\beta$  are arbitrary covariant or contravariant system, so we can put

$$\nabla_\beta A_\nu^\mu = \frac{\partial A_\nu^\mu}{\partial x^\beta} - \Gamma_{\nu\beta}^\alpha A_\alpha^\mu + \Gamma_{\alpha\beta}^\mu A_\nu^\alpha, \quad (2.16)$$

equation that is called the covariant derivative of the tensor  $A_\nu^\mu$ . If we denote  $\mu$  by the indices  $\mu_1, \mu_2, \dots, \mu_n$  and  $\nu$  by the indices  $\nu_1, \nu_2, \dots, \nu_m$

$$\begin{aligned} \nabla_\beta A_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_n} &= \frac{\partial}{\partial x^\beta} A_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_n} - \sum_{i=1}^m \Gamma_{\nu_i \beta}^\alpha A_{\nu_1 \dots \nu_{i-1} \alpha \nu_{i+1} \dots \nu_m}^{\mu_1 \dots \mu_n} \\ &\quad + \sum_{j=1}^n \Gamma_{\alpha \beta}^{\mu_j} A_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_{j-1} \alpha \mu_{j+1} \dots \mu_n} \end{aligned} \quad (2.17)$$

then we have the covariant derivative of the tensor  $A_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_n}$ .

Note that if the connection  $\Gamma_{\mu\nu}^\lambda$  is zero in the covariant derivative of some tensor, then we have the ordinary partial derivative as the equivalent of the covariant derivative.

## 2.5 Some cases of special importance

### The metric tensor

By the rule of differentiation of determinant of the metric tensor

$$dg = g^{\mu\nu} g dg_{\mu\nu} = -g g_{\mu\nu} dg^{\mu\nu}. \quad (2.18)$$

If we keep in mind that  $g_{\mu\nu} g^{\mu\nu} = \delta_\mu^\mu$ , and consequently

$$g_{\mu\nu} dg^{\mu\nu} + g^{\mu\nu} dg_{\mu\nu} = 0.$$

From equation (2.18), it follows that

$$\frac{\partial \log \sqrt{-g}}{\partial x^\sigma} = \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial x^\sigma} = \frac{1}{2} g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^\sigma} = -\frac{1}{2} g_{\mu\nu} \frac{\partial g^{\mu\nu}}{\partial x^\sigma}. \quad (2.19)$$

In addition to  $g_{\mu\sigma} g^{\nu\sigma} = \delta_\mu^\nu$ , it is found by differentiation that

$$\begin{aligned} g_{\mu\sigma} dg^{\nu\sigma} &= -g^{\nu\sigma} dg_{\mu\sigma}, \\ g_{\mu\sigma} \frac{\partial g^{\nu\sigma}}{\partial x^\lambda} &= -g^{\nu\sigma} \frac{\partial g_{\mu\sigma}}{\partial x^\lambda}. \end{aligned} \quad (2.20)$$

From equations (2.20), if we multiply both equations by  $g^{\mu\tau}$  respectively, and then take equation (2.2) into account, we get

$$\begin{aligned} dg^{\mu\nu} &= -g^{\mu\alpha} g^{\nu\beta} dg_{\alpha\beta}, \\ \frac{\partial g^{\mu\nu}}{\partial x^\lambda} &= -g^{\mu\alpha} g^{\nu\beta} \frac{\partial g_{\alpha\beta}}{\partial x^\lambda}. \end{aligned} \quad (2.21)$$

If we multiply equations (2.20) by  $g_{\nu\tau}$  respectively, and then take equation (2.2) into account, we obtain

$$\begin{aligned} dg_{\mu\nu} &= -g_{\mu\alpha} g_{\nu\beta} dg^{\alpha\beta}, \\ \frac{\partial g_{\mu\nu}}{\partial x^\lambda} &= -g_{\mu\alpha} g_{\nu\beta} \frac{\partial g^{\alpha\beta}}{\partial x^\lambda}. \end{aligned} \quad (2.22)$$

Considering the result

$$\frac{\partial g_{\alpha\beta}}{\partial x^\lambda} + \frac{\partial g_{\lambda\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\lambda}}{\partial x^\beta} = 2\Gamma_{\lambda\alpha}^\rho g_{\rho\beta},$$

and interchanging  $\mu$  with  $\nu$  we can write

$$\frac{\partial g_{\beta\alpha}}{\partial x^\lambda} + \frac{\partial g_{\lambda\alpha}}{\partial x^\beta} - \frac{\partial g_{\beta\lambda}}{\partial x^\alpha} = 2\Gamma_{\lambda\beta}^\rho g_{\rho\alpha},$$

we can make the sum of these two equations obtaining the equation

$$\frac{\partial g_{\alpha\beta}}{\partial x^\lambda} = \Gamma_{\lambda\alpha}^\rho g_{\rho\beta} + \Gamma_{\lambda\beta}^\rho g_{\rho\alpha},$$

or equivalently

$$\frac{\partial g_{\alpha\beta}}{\partial x^\lambda} - \Gamma_{\lambda\alpha}^\rho g_{\rho\beta} - \Gamma_{\lambda\beta}^\rho g_{\rho\alpha} = \nabla_\lambda g_{\alpha\beta} \equiv 0, \quad (2.23)$$

where use has been made of equation (2.17). Inserting the left side of equation (2.23) into the second equation of equations (2.21) we find the mathematical expression

$$\frac{\partial g^{\mu\nu}}{\partial x^\sigma} = -g^{\mu\tau}\Gamma_{\tau\sigma}^\nu - g^{\nu\tau}\Gamma_{\tau\sigma}^\mu. \quad (2.24)$$

Substituting the right-hand side of equation (2.24) into (2.19), we have

$$\frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial x^\sigma} = \Gamma_{\sigma\mu}^\mu. \quad (2.25)$$

### Isometries in the metric tensor

The metric tensor  $g_{\mu\nu}$  invariant under the transformation  $x^\lambda \rightarrow x'^\lambda$  if

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x), \forall x^\lambda. \quad (2.26)$$

So a metric tensor that ceases to be invariant is called an isometry. Since  $g_{\mu\nu}$  is a covariant tensor, we can write the transformation law

$$g_{\mu\nu}(x) = \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} g'_{\rho\sigma}(x').$$

Then, using equation (2.26),  $x^\lambda \rightarrow x'^\lambda$  will be an isometry if

$$g_{\mu\nu}(x) = \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} g'_{\rho\sigma}(x'). \quad (2.27)$$

In general, condition (2.27) is very complicated, but it can be significantly simplified if we consider the special case of an infinitesimal coordinate transformation

$$x^\mu \rightarrow x'^\mu = x^\mu + \varepsilon X^\mu(x), \quad (2.28)$$

where  $\varepsilon$  is small and arbitrary and  $X^a$  is a vector field. Differentiating equation (2.28) we find that

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta_\nu^\mu + \varepsilon \frac{\partial X^\mu}{\partial x^\nu},$$

and consequently, substituting in equation (2.27) we find

$$\begin{aligned} g_{\mu\nu}(x) \cong & \delta_\mu^\rho \delta_\nu^\sigma g_{\rho\sigma}(x) + \varepsilon \delta_\mu^\rho \delta_\nu^\sigma X^\beta \frac{\partial g_{\rho\sigma}(x)}{\partial x^\beta} + \varepsilon \delta_\mu^\rho g_{\rho\sigma}(x) \frac{\partial X^\sigma}{\partial x^\nu} + \varepsilon^2 \delta_\mu^\rho X^\beta \frac{\partial X^\sigma}{\partial x^\nu} \frac{\partial g_{\rho\sigma}(x)}{\partial x^\beta} + \varepsilon \delta_\nu^\sigma g_{\rho\sigma} \frac{\partial X^\rho}{\partial x^\mu} \\ & + \varepsilon^2 \delta_\nu^\sigma X^e \frac{\partial X^\rho}{\partial x^\mu} \frac{\partial g_{\rho\sigma}(x)}{\partial x^e} + \varepsilon^2 g_{\rho\sigma}(x) \frac{\partial X^\rho}{\partial x^\mu} \frac{\partial X^\sigma}{\partial x^\nu} + \varepsilon^3 \frac{\partial X^\rho}{\partial x^\mu} \frac{\partial X^\sigma}{\partial x^\nu} X^\beta \frac{\partial g_{\rho\sigma}(x)}{\partial x^\beta}, \end{aligned}$$

rearranging terms

$$g_{\mu\nu}(x) \cong g_{\mu\nu}(x) + \varepsilon \left[ g_{\mu\rho} \frac{\partial X^\rho}{\partial x^\nu} + g_{\nu\rho} \frac{\partial X^\rho}{\partial x^\mu} + X^\rho \frac{\partial g_{\mu\nu}}{\partial x^\rho} \right] + \mathcal{O}(\varepsilon^2).$$

Working to first order in  $\varepsilon$  and subtracting  $g_{\mu\nu}(x)$  from each side, it follows that the quantity in square brackets vanishes

$$0 \cong \varepsilon \left[ X^\rho \frac{\partial g_{\mu\nu}}{\partial x^\rho} + g_{\mu\rho} \frac{\partial X^\rho}{\partial x^\nu} + g_{\nu\rho} \frac{\partial X^\rho}{\partial x^\mu} \right].$$

This quantity is simply the Lie derivative of  $g_{\mu\nu}$  with respect to  $X$ , that is, using the Lie derivative:

$$\mathfrak{L}_X T_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n} = u^\alpha \partial_\alpha T_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n} + \sum_{j=1}^n T_{\nu_1 \dots \alpha \dots \nu_n}^{\mu_1 \dots \mu_m} \partial_{\nu_j} u^\alpha - \sum_{i=1}^m T_{\nu_1 \dots \nu_n}^{\mu_1 \dots \alpha \dots \mu_m} \partial_\alpha u^{\mu_i},$$

we can write

$$\mathfrak{L}_X g_{\mu\nu} = X^\rho \frac{\partial g_{\mu\nu}}{\partial x^\rho} + g_{\mu\rho} \frac{\partial X^\rho}{\partial x^\nu} + g_{\nu\rho} \frac{\partial X^\rho}{\partial x^\mu}.$$

We can now replace the ordinary derivatives by covariant derivatives in some expression for a Lie derivative

$$\mathfrak{L}_X g_{\mu\nu} = X^\rho \nabla_\rho g_{\mu\nu} + g_{\mu\rho} \nabla_\nu X^\rho + g_{\nu\rho} \nabla_\mu X^\rho,$$

and consequently, using equations (2.23), the condition for an infinitesimal isometry is

$$\mathfrak{L}_X g_{\mu\nu} = \nabla_\nu X_\mu + \nabla_\mu X_\nu = 0. \quad (2.29)$$

These are the Killing equations and some solution of them is called a Killing vector field  $X^\mu$  [78–84].

## 2.6 The Riemann–Christoffel tensor

We can make use of equation (2.8), to have the resulting equation

$$\Gamma_{\mu\nu}^{\lambda} = \frac{\partial x^{\lambda}}{\partial x'^{\tau}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \Gamma'_{\rho\sigma} + \frac{\partial x^{\lambda}}{\partial x'^{\tau}} \frac{\partial^2 x'^{\tau}}{\partial x^{\mu} \partial x^{\nu}}, \quad (2.30)$$

this equation is called transformation of the affine connection.

We can isolate the inhomogeneous term, that is, we multiply equation (2.30) by  $\frac{\partial x'^{\xi}}{\partial x^{\lambda}}$  to obtain

$$\frac{\partial^2 x'^{\tau}}{\partial x^{\mu} \partial x^{\nu}} = \frac{\partial x'^{\tau}}{\partial x^{\lambda}} \Gamma_{\mu\nu}^{\lambda} - \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \Gamma'_{\rho\sigma}. \quad (2.31)$$

We differentiate equation (2.31) with respect to  $x^{\kappa}$ , to arrive at the equation

$$\frac{\partial^3 x'^{\tau}}{\partial x^{\kappa} \partial x^{\mu} \partial x^{\nu}} = \frac{\partial^2 x'^{\tau}}{\partial x^{\kappa} \partial x^{\lambda}} \Gamma_{\mu\nu}^{\lambda} + \frac{\partial x'^{\tau}}{\partial x^{\lambda}} \frac{\partial \Gamma_{\mu\nu}^{\lambda}}{\partial x^{\kappa}} - \frac{\partial^2 x'^{\rho}}{\partial x^{\kappa} \partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \Gamma'_{\rho\sigma} - \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial^2 x'^{\sigma}}{\partial x^{\kappa} \partial x^{\nu}} \Gamma'_{\rho\sigma} - \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \frac{\partial \Gamma'_{\rho\sigma}}{\partial x^{\kappa}},$$

then we take into account equation (2.31), to rewrite the preceding equation in the form:

$$\begin{aligned} \frac{\partial^3 x'^{\tau}}{\partial x^{\kappa} \partial x^{\mu} \partial x^{\nu}} &= \Gamma_{\mu\nu}^{\lambda} \left( \frac{\partial x'^{\tau}}{\partial x^{\eta}} \Gamma_{\kappa\mu}^{\eta} - \frac{\partial x'^{\rho}}{\partial x^{\kappa}} \frac{\partial x'^{\sigma}}{\partial x^{\lambda}} \Gamma'_{\rho\sigma} \right) - \Gamma'_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \left( \frac{\partial x'^{\rho}}{\partial x^{\mu}} \Gamma_{\kappa\nu}^{\eta} - \frac{\partial x'^{\eta}}{\partial x^{\kappa}} \frac{\partial x'^{\lambda}}{\partial x^{\nu}} \Gamma'_{\rho\sigma} \right) \\ &\quad - \Gamma'_{\rho\sigma} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \left( \frac{\partial x'^{\rho}}{\partial x^{\eta}} \Gamma_{\kappa\mu}^{\eta} - \frac{\partial x'^{\eta}}{\partial x^{\kappa}} \frac{\partial x'^{\lambda}}{\partial x^{\mu}} \Gamma'_{\rho\sigma} \right) + \frac{\partial x'^{\tau}}{\partial x^{\lambda}} \frac{\partial \Gamma_{\mu\nu}^{\lambda}}{\partial x^{\kappa}} - \frac{\partial x'^{\rho}}{\partial x^{\lambda}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \frac{\partial x'^{\eta}}{\partial x^{\kappa}} \frac{\partial \Gamma'_{\rho\sigma}}{\partial x'^{\eta}} \end{aligned}$$

then after rearranging terms, we get

$$\begin{aligned} \frac{\partial^3 x'^{\tau}}{\partial x^{\kappa} \partial x^{\mu} \partial x^{\nu}} &= \frac{\partial x'^{\tau}}{\partial x^{\lambda}} \left( \frac{\partial \Gamma_{\mu\nu}^{\lambda}}{\partial x^{\kappa}} + \Gamma_{\mu\nu}^{\eta} \Gamma_{\kappa\eta}^{\lambda} \right) - \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\eta}}{\partial x^{\kappa}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \left( \frac{\partial \Gamma'_{\rho\sigma}}{\partial x'^{\eta}} - \Gamma'_{\rho\lambda} \Gamma'_{\eta\sigma} - \Gamma'_{\lambda\sigma} \Gamma'_{\eta\rho} \right) \\ &\quad - \Gamma'_{\rho\sigma} \frac{\partial x'^{\sigma}}{\partial x^{\lambda}} \left( \Gamma_{\mu\nu}^{\lambda} \frac{\partial x'^{\rho}}{\partial x^{\kappa}} + \Gamma_{\kappa\nu}^{\lambda} \frac{\partial x'^{\rho}}{\partial x^{\mu}} + \Gamma_{\kappa\mu}^{\lambda} \frac{\partial x'^{\rho}}{\partial x^{\nu}} \right). \end{aligned}$$

If in this equation we interchange  $\nu$  with  $\kappa$  and subtract the result from the original equation, all the products of the affine connections cancel, thus we find the tensor equation:

$$\frac{\partial x'^{\tau}}{\partial x^{\lambda}} \left( \frac{\partial \Gamma_{\mu\nu}^{\lambda}}{\partial x^{\kappa}} - \frac{\partial \Gamma_{\mu\kappa}^{\lambda}}{\partial x^{\nu}} + \Gamma_{\mu\nu}^{\eta} \Gamma_{\kappa\eta}^{\lambda} - \Gamma_{\mu\kappa}^{\eta} \Gamma_{\nu\eta}^{\lambda} \right) = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\eta}}{\partial x^{\kappa}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \left( \frac{\partial \Gamma'_{\rho\sigma}}{\partial x'^{\eta}} - \frac{\partial \Gamma'_{\rho\eta}}{\partial x'^{\sigma}} - \Gamma'_{\lambda\sigma} \Gamma'_{\eta\rho} + \Gamma'_{\lambda\eta} \Gamma'_{\rho\sigma} \right),$$

we can write this equation as a tensor transformation rule as follows:

$$R'_{\rho\sigma\eta}{}^{\tau} = \frac{\partial x'^{\tau}}{\partial x^{\lambda}} \frac{\partial x^{\mu}}{\partial x'^{\rho}} \frac{\partial x^{\nu}}{\partial x'^{\sigma}} \frac{\partial x^{\kappa}}{\partial x'^{\eta}} R_{\mu\nu\kappa}^{\lambda}.$$

On the other hand, we define the equation [85]

$$\nabla_{\sigma} \nabla_{\nu} A_{\mu} - \nabla_{\nu} \nabla_{\sigma} A_{\mu} = [\nabla_{\sigma}, \nabla_{\nu}] A_{\mu} = R_{\mu\nu\sigma}^{\rho} A_{\rho}, \quad (2.32)$$

where

$$R_{\mu\nu\sigma}^{\rho} = -\frac{\partial}{\partial x^{\sigma}} \Gamma_{\mu\nu}^{\rho} + \frac{\partial}{\partial x^{\nu}} \Gamma_{\mu\sigma}^{\rho} - \Gamma_{\mu\nu}^{\tau} \Gamma_{\tau\sigma}^{\rho} + \Gamma_{\sigma\mu}^{\tau} \Gamma_{\tau\nu}^{\rho}, \quad (2.33)$$

is called the Riemann-Christoffel tensor [86] [87].

From the tensor character of  $\nabla_{\sigma} \nabla_{\nu} A_{\mu} - \nabla_{\nu} \nabla_{\sigma} A_{\mu}$  together with the fact that  $A_{\rho}$  is an arbitrary vector and the transformation rule  $R'_{\rho\sigma\eta}{}^{\tau} = \frac{\partial x'^{\tau}}{\partial x^{\lambda}} \frac{\partial x^{\mu}}{\partial x'^{\rho}} \frac{\partial x^{\nu}}{\partial x'^{\sigma}} \frac{\partial x^{\kappa}}{\partial x'^{\eta}} R_{\mu\nu\kappa}^{\lambda}$ , it follows that  $R_{\mu\nu\sigma}^{\rho}$  is a tensor (the Riemann-Christoffel tensor).

The mathematical significance of this tensor is as follows: If the manifold is of such a nature that there is a coordinate system with reference for which the components of  $g_{\mu\nu}$  are constant, then all the components of  $R_{\mu\nu\sigma}^{\rho}$  cancel out. If we change to a new coordinate system instead of the original one, the referred components of  $g_{\mu\nu}$  will not be constant, but due to their tensor nature, the transformed components of  $R_{\mu\nu\sigma}^{\rho}$  would disappear in the new system.



A necessary and sufficient condition for a manifold to be flat is that the Riemann tensor be zero, as is the case with the special theory of relativity (Minkowski space).

Contracting equation (2.33) with respect to the indices  $\sigma$  and  $\rho$  we obtain the second rank covariant tensor

$$R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\nu\sigma} = R_{\mu\nu\rho}^{\rho} = -\frac{\partial}{\partial x^{\alpha}} \Gamma_{\mu\nu}^{\alpha} + \frac{\partial}{\partial x^{\nu}} \Gamma_{\mu\rho}^{\rho} - \Gamma_{\mu\nu}^{\alpha} \Gamma_{\alpha\rho}^{\rho} + \Gamma_{\mu\alpha}^{\beta} \Gamma_{\nu\beta}^{\alpha}, \quad (2.34)$$

which is often called the Ricci tensor.

From the Ricci tensor we can contract again to form the so-called scalar of curvature

$$R = g^{\mu\nu} R_{\mu\nu}. \quad (2.35)$$

## 2.7 Theory of the gravitational field

### The field equations in the absence of matter

Taking a certain coordinate system where the principle of equivalence is satisfied, where the components of  $g_{\mu\nu}$  are certain constant values. Relatively for this system all components for the Riemann tensor, it holds that  $R_{\mu\nu\sigma}^{\rho} = 0$ . For the space under consideration the components of the Riemann tensor also vanish for some other coordinate system, according to equation (2.33).

Therefore, it can be thought that the gravitational field in the absence of matter, that is, in a vacuum, requires that the Ricci tensor  $R_{\mu\kappa} = 0$  be derived from the Riemann tensor  $R_{\mu\nu\kappa}^{\lambda}$ , if and only if  $R_{\mu\nu\kappa}^{\lambda} = 0$ . Therefore, field equations of gravitation can be written as

$$R_{\mu\kappa} = 0. \quad (2.36)$$

### Field equations for momentum and energy in the absence of matter

In this section, we will show the equations corresponding to the laws of momentum and energy. For this we start from a Lagrangian density function defined by

$$\mathcal{L}_G = \sqrt{-g} g^{\mu\nu} \left( \Gamma_{\mu\lambda}^{\eta} \Gamma_{\nu\eta}^{\lambda} - \Gamma_{\mu\nu}^{\eta} \Gamma_{\lambda\eta}^{\lambda} \right). \quad (2.37)$$

Let us propose the Lagrangian density  $\mathcal{L}_G$  as a function of the term  $\sqrt{-g} g^{\mu\nu}$  and the quantity  $\partial_{\sigma} (\sqrt{-g} g^{\mu\nu}) = \partial (\sqrt{-g} g^{\mu\nu}) / \partial x^{\sigma}$ , that is, a function as

$$\mathcal{L}_G = \mathcal{L}_G \left( \sqrt{-g} g^{\mu\nu}, \frac{\partial}{\partial x^{\rho}} (\sqrt{-g} g^{\mu\nu}) \right). \quad (2.38)$$

First of all, let us take the covariant derivative of the quantity  $\sqrt{-g} g^{\mu\nu}$ , that is, the tensor equation

$$\nabla_{\rho} (\sqrt{-g} g^{\mu\nu}) = \frac{\partial}{\partial x^{\rho}} (\sqrt{-g} g^{\mu\nu}) + \Gamma_{\alpha\rho}^{\mu} \sqrt{-g} g^{\alpha\nu} + \Gamma_{\alpha\rho}^{\nu} \sqrt{-g} g^{\mu\alpha} - \Gamma_{\lambda\rho}^{\lambda} \sqrt{-g} g^{\mu\nu} \equiv 0, \quad (2.39)$$

from this covariant equation, since it is a tensor, we will obtain four equations that will help us find the field equations of gravitation in vacuum in a second form, in a different version of the Ricci tensor.

Now, to find the first equation, if we differentiate equation (2.39) with respect to  $\sqrt{-g} g^{\alpha\beta}$ ,

$$\begin{aligned} \frac{1}{2} \left( \delta_{\alpha}^{\lambda} \delta_{\beta}^{\nu} + \delta_{\alpha}^{\nu} \delta_{\beta}^{\lambda} \right) \Gamma_{\lambda\rho}^{\mu} + \sqrt{-g} g^{\lambda\nu} \frac{\partial \Gamma_{\lambda\rho}^{\mu}}{\partial (\sqrt{-g} g^{\alpha\beta})} + \frac{1}{2} \left( \delta_{\alpha}^{\mu} \delta_{\beta}^{\lambda} + \delta_{\alpha}^{\lambda} \delta_{\beta}^{\mu} \right) \Gamma_{\lambda\rho}^{\nu} + \sqrt{-g} g^{\mu\lambda} \frac{\partial \Gamma_{\lambda\rho}^{\nu}}{\partial (\sqrt{-g} g^{\alpha\beta})} \\ - \frac{1}{2} \left( \delta_{\alpha}^{\mu} \delta_{\beta}^{\nu} + \delta_{\alpha}^{\nu} \delta_{\beta}^{\mu} \right) \Gamma_{\lambda\rho}^{\lambda} - \sqrt{-g} g^{\mu\nu} \frac{\partial \Gamma_{\lambda\rho}^{\lambda}}{\partial (\sqrt{-g} g^{\alpha\beta})} = 0. \end{aligned}$$

then we multiply the resulting equation by  $\Gamma_{\mu\nu}^{\rho}$ , consequently, we get:

$$2\Gamma_{\alpha\rho}^{\mu} \Gamma_{\mu\beta}^{\rho} - \Gamma_{\lambda\rho}^{\lambda} \Gamma_{\mu\nu}^{\rho} + 2\sqrt{-g} g^{\lambda\nu} \Gamma_{\mu\nu}^{\rho} \frac{\partial \Gamma_{\lambda\rho}^{\mu}}{\partial (\sqrt{-g} g^{\alpha\beta})} - \sqrt{-g} g^{\mu\nu} \Gamma_{\mu\nu}^{\rho} \frac{\partial \Gamma_{\lambda\rho}^{\lambda}}{\partial (\sqrt{-g} g^{\alpha\beta})} = 0. \quad (2.40)$$

We can contract equation (2.39) with respect to  $\rho$  and  $\nu$ , to arrive at:

$$\nabla_\rho (\sqrt{-gg^{\mu\rho}}) = \frac{\partial}{\partial x^\rho} (\sqrt{-gg^{\mu\rho}}) + \Gamma_{\lambda\rho}^\mu \sqrt{-gg^{\lambda\rho}} = 0. \quad (2.41)$$

Usually; to find the second equation that will help us to find the field equations, we differentiate equation (2.41) with respect to  $\sqrt{-gg^{\alpha\beta}}$  and consequently we have:

$$\Gamma_{\alpha\beta}^\mu + \sqrt{-gg^{\lambda\nu}} \frac{\partial \Gamma_{\lambda\nu}^\mu}{\partial (\sqrt{-gg^{\alpha\beta}})} = 0. \quad (2.42)$$

The third equation is found by differentiating equation (2.40) with respect to  $\partial_\eta (\sqrt{-gg^{\alpha\beta}})$ , to then do algebraic manipulations and get:

$$\Gamma_{\alpha\beta}^\eta + 2\sqrt{-gg^{\lambda\nu}} \Gamma_{\mu\nu}^\rho \frac{\partial \Gamma_{\lambda\rho}^\mu}{\partial [\partial_\eta (\sqrt{-gg^{\alpha\beta}})]} - \sqrt{-gg^{\mu\nu}} \Gamma_{\mu\nu}^\rho \frac{\partial \Gamma_{\lambda\rho}^\lambda}{\partial [\partial_\eta (\sqrt{-gg^{\alpha\beta}})]} = 0. \quad (2.43)$$

Finally, to enter the treatment of the relativistic equations of gravity in a vacuum, let us differentiate equation (2.41) with respect to  $\partial_\eta (\sqrt{-gg^{\alpha\beta}})$ . Therefore, we have the result

$$\frac{1}{2} (\delta_\alpha^\eta \delta_\beta^\mu + \delta_\alpha^\mu \delta_\beta^\eta) + \sqrt{-gg^{\lambda\nu}} \frac{\partial \Gamma_{\lambda\nu}^\mu}{\partial [\partial_\eta (\sqrt{-gg^{\alpha\beta}})]} = 0. \quad (2.44)$$

Going back to equation (2.38), therefore, this leads us to compute the partial derivatives  $\frac{\partial \mathcal{L}_G}{\partial (\sqrt{-gg^{\alpha\beta}})}$  and  $\frac{\partial \mathcal{L}_G}{\partial (\partial_\rho (\sqrt{-gg^{\alpha\beta}}))}$ . Now, if we differentiate equation (2.37) with respect to  $\sqrt{-gg^{\alpha\beta}}$ ,

$$\begin{aligned} \frac{\partial \mathcal{L}_G}{\partial (\sqrt{-gg^{\alpha\beta}})} &= \left( \Gamma_{\alpha\lambda}^\eta \Gamma_{\beta\eta}^\lambda - \Gamma_{\alpha\beta}^\eta \Gamma_{\lambda\eta}^\lambda \right) + 2\sqrt{-gg^{\mu\nu}} \Gamma_{\mu\lambda}^\eta \frac{\partial \Gamma_{\nu\eta}^\lambda}{\partial (\sqrt{-gg^{\alpha\beta}})} - \sqrt{-gg^{\mu\nu}} \Gamma_{\mu\nu}^\eta \frac{\partial \Gamma_{\lambda\eta}^\lambda}{\partial (\sqrt{-gg^{\alpha\beta}})} \\ &\quad - \sqrt{-gg^{\mu\nu}} \Gamma_{\lambda\eta}^\lambda \frac{\partial \Gamma_{\mu\nu}^\eta}{\partial (\sqrt{-gg^{\alpha\beta}})}, \end{aligned}$$

then we substitute equations (2.40) and (2.42) into the calculated derivatives, and after doing algebraic manipulations we find

$$\frac{\partial \mathcal{L}_G}{\partial (\sqrt{-gg^{\alpha\beta}})} = \left( \Gamma_{\alpha\beta}^\eta \Gamma_{\lambda\eta}^\lambda - \Gamma_{\alpha\lambda}^\eta \Gamma_{\beta\eta}^\lambda \right). \quad (2.45)$$

We next differentiate equation (2.37) with respect to  $\partial_\rho (\sqrt{-gg^{\alpha\beta}})$ ,

$$\begin{aligned} \frac{\partial \mathcal{L}_G}{\partial [\partial_\rho (\sqrt{-gg^{\alpha\beta}})]} &= 2\sqrt{-gg^{\mu\nu}} \Gamma_{\mu\lambda}^\eta \frac{\partial \Gamma_{\nu\eta}^\lambda}{\partial [\partial_\rho (\sqrt{-gg^{\alpha\beta}})]} - \sqrt{-gg^{\mu\nu}} \Gamma_{\mu\nu}^\eta \frac{\partial \Gamma_{\lambda\eta}^\lambda}{\partial [\partial_\rho (\sqrt{-gg^{\alpha\beta}})]} \\ &\quad - \sqrt{-gg^{\mu\nu}} \Gamma_{\lambda\eta}^\lambda \frac{\partial \Gamma_{\mu\nu}^\eta}{\partial [\partial_\rho (\sqrt{-gg^{\alpha\beta}})]}, \end{aligned}$$

then substitute equations (2.43) and (2.44) into the resulting derivative of the Lagrangian density, and after making the necessary reductions we find; the desired partial derivative:

$$\frac{\partial \mathcal{L}_G}{\partial [\partial_\rho (\sqrt{-gg^{\alpha\beta}})]} = -\Gamma_{\alpha\beta}^\rho + \frac{1}{2} \delta_\alpha^\rho \Gamma_{\lambda\beta}^\lambda + \frac{1}{2} \delta_\beta^\rho \Gamma_{\lambda\alpha}^\lambda. \quad (2.46)$$

Carrying out the variation of the Lagrangian density, we then integrate by parts the second term resulting from the variation and use the divergence theorem  $\int_\Omega \partial_\rho \mathcal{F}^\rho d\tau = \int_{\partial\Omega} \mathcal{F} dS_\rho$ , so we find; the Euler derivative defined by:

$$-\frac{\delta \mathcal{L}_G}{\delta (\sqrt{-gg^{\mu\nu}})} = \frac{\partial}{\partial x^\sigma} \left[ \frac{\partial \mathcal{L}_G}{\partial (\partial_\sigma (\sqrt{-gg^{\mu\nu}}))} \right] - \frac{\partial \mathcal{L}_G}{\partial (\sqrt{-gg^{\mu\nu}})} = 0, \quad (2.47)$$

If we use equations (2.45) and (2.46), we find just the field equations of gravity in the absence of matter, that is,  $R_{\mu\nu} = 0$ . Therefore, Eqs. (2.47) represent a second way of writing the field equations.

If we multiply equation (2.47) by  $\partial_\rho (\sqrt{-gg^{\mu\nu}})$ , and transform the first term of the Euler derivative,

$$\frac{\partial}{\partial x^\sigma} \left[ \frac{\partial (\sqrt{-gg^{\mu\nu}})}{\partial x^\rho} \frac{\partial \mathcal{L}_G}{\partial (\partial_\sigma (\sqrt{-gg^{\mu\nu}}))} \right] - \frac{\partial \mathcal{L}_G}{\partial (\partial_\sigma (\sqrt{-gg^{\mu\nu}}))} \frac{\partial^2 (\sqrt{-gg^{\mu\nu}})}{\partial x^\rho \partial x^\sigma} - \frac{\partial \mathcal{L}_G}{\partial (\sqrt{-gg^{\mu\nu}})} \frac{\partial (\sqrt{-gg^{\mu\nu}})}{\partial x^\rho} = 0.$$

then we take fact that the second and third terms of the resulting equation reduce to  $\frac{\partial \mathcal{L}_G}{\partial x^\rho}$ , therefore the previous equation becomes

$$\frac{\partial}{\partial x^\sigma} \left[ \frac{\partial(\sqrt{-g}g^{\mu\nu})}{\partial x^\rho} \frac{\partial \mathcal{L}_G}{\partial(\partial_\sigma(\sqrt{-g}g^{\mu\nu}))} - \delta_\rho^\sigma \mathcal{L}_G \right] = 0.$$

or (the appearance of the  $-2\kappa$  factor will appear after),

$$\frac{\partial t_\rho^\sigma}{\partial x^\sigma} = 0, \tag{2.48}$$

where

$$-2\kappa t_\rho^\sigma = \frac{\partial(\sqrt{-g}g^{\mu\nu})}{\partial x^\rho} \frac{\partial \mathcal{L}_G}{\partial(\partial_\sigma(\sqrt{-g}g^{\mu\nu}))} - \delta_\rho^\sigma \mathcal{L}_G, \tag{2.49}$$

here  $\kappa$  is called the coupling constant in the general theory of relativity.

If we plug in equations (2.37), (2.45), and (2.46), then we rewrite equation (2.49) as follows:

$$\begin{aligned} \kappa t_\rho^\sigma = & -\sqrt{-g}g^{\alpha\nu}\Gamma_{\nu\mu}^\sigma\Gamma_{\alpha\rho}^\mu - \frac{1}{2}\sqrt{-g}(g^{\sigma\nu}\Gamma_{\lambda\rho}^\lambda\Gamma_{\lambda\nu}^\lambda - g^{\mu\nu}\Gamma_{\mu\nu}^\sigma\Gamma_{\lambda\rho}^\lambda - g^{\alpha\nu}\Gamma_{\alpha\rho}^\sigma\Gamma_{\lambda\nu}^\lambda - g^{\alpha\sigma}\Gamma_{\alpha\rho}^\mu\Gamma_{\lambda\mu}^\lambda) \\ & + \frac{1}{2}\delta_\rho^\sigma\sqrt{-g}g^{\mu\nu}(\Gamma_{\mu\lambda}^\eta\Gamma_{\nu\eta}^\lambda - \Gamma_{\mu\nu}^\eta\Gamma_{\lambda\eta}^\lambda). \end{aligned} \tag{2.50}$$

Note that  $t_\rho^\sigma$  is not a tensor. Equation (2.48) expresses the law of conservation of momentum and energy of gravitation.

We will express equation (2.36) in a third form, which will be used particularly to find the general form of the relativistic equations of the gravitational field, that is, with matter.

We can multiply (2.36) by  $\sqrt{-g}g^{\nu\alpha}$ ,

$$-\frac{\partial}{\partial x^\lambda}(\sqrt{-g}g^{\nu\alpha}\Gamma_{\mu\nu}^\lambda) + \frac{\partial(\sqrt{-g}g^{\nu\alpha})}{\partial x^\lambda}\Gamma_{\mu\nu}^\lambda + \sqrt{-g}g^{\nu\alpha}\frac{\partial\Gamma_{\mu\lambda}^\lambda}{\partial x^\nu} + \sqrt{-g}g^{\nu\alpha}\Gamma_{\mu\lambda}^\eta\Gamma_{\nu\eta}^\lambda - \sqrt{-g}g^{\nu\alpha}\Gamma_{\mu\nu}^\eta\Gamma_{\lambda\eta}^\lambda = 0.$$

then we transform the first term of the resulting equation by Leibniz's rule of differentiation, then by equation (2.44) and realizing appropriate index manipulations we find:

$$-\frac{\partial}{\partial x^\lambda}(\sqrt{-g}g^{\nu\alpha}\Gamma_{\mu\nu}^\lambda) - \sqrt{-g}g^{\nu\kappa}\Gamma_{\kappa\lambda}^\alpha\Gamma_{\mu\nu}^\lambda + \sqrt{-g}g^{\nu\alpha}\frac{\partial\Gamma_{\mu\lambda}^\lambda}{\partial x^\nu} = 0.$$

If we take into account equation (2.50) and replace the second term of the preceding equation

$$\begin{aligned} -\frac{\partial}{\partial x^\lambda}(\sqrt{-g}g^{\nu\alpha}\Gamma_{\mu\nu}^\lambda) + \kappa t_\mu^\alpha + \frac{1}{2}\sqrt{-g}(g^{\alpha\nu}\Gamma_{\lambda\mu}^\lambda\Gamma_{\lambda\nu}^\lambda - g^{\rho\nu}\Gamma_{\rho\nu}^\alpha\Gamma_{\lambda\mu}^\lambda - g^{\tau\nu}\Gamma_{\tau\mu}^\alpha\Gamma_{\lambda\nu}^\lambda - g^{\tau\alpha}\Gamma_{\tau\mu}^\rho\Gamma_{\lambda\rho}^\lambda) \\ - \frac{1}{2}\delta_\mu^\alpha\sqrt{-g}g^{\rho\nu}(\Gamma_{\rho\lambda}^\eta\Gamma_{\nu\eta}^\lambda - \Gamma_{\rho\nu}^\eta\Gamma_{\lambda\eta}^\lambda) + \sqrt{-g}g^{\nu\alpha}\frac{\partial\Gamma_{\mu\lambda}^\lambda}{\partial x^\nu} = 0. \end{aligned}$$

and if we also take into account the contraction of equation (2.50)

$$\kappa t = \sqrt{-g}g^{\mu\nu}(\Gamma_{\mu\lambda}^\eta\Gamma_{\nu\eta}^\lambda - \Gamma_{\mu\nu}^\eta\Gamma_{\lambda\eta}^\lambda), \tag{2.51}$$

that is, we use equation (2.51), we obtain the final third version of the field equations

$$-\frac{\partial}{\partial x^\lambda}(\sqrt{-g}g^{\nu\alpha}\Gamma_{\mu\nu}^\lambda) + \sqrt{-g}g^{\nu\alpha}\frac{\partial\Gamma_{\mu\lambda}^\lambda}{\partial x^\nu} + L_\mu^\alpha = -\kappa\left(t_\mu^\alpha - \frac{1}{2}\delta_\mu^\alpha t\right), \tag{2.52}$$

where we have defined the expression

$$L_\mu^\alpha = \frac{1}{2}\sqrt{-g}(g^{\alpha\nu}\Gamma_{\lambda\mu}^\lambda\Gamma_{\lambda\nu}^\lambda - g^{\rho\nu}\Gamma_{\rho\nu}^\alpha\Gamma_{\lambda\mu}^\lambda - g^{\tau\nu}\Gamma_{\tau\mu}^\alpha\Gamma_{\lambda\nu}^\lambda - g^{\tau\alpha}\Gamma_{\tau\mu}^\rho\Gamma_{\lambda\rho}^\lambda).$$

### Field equations in general form

But if we consider the solar system, the total mass of the system and its total gravitational action as well, will depend on the total energy of the system. Therefore, it is necessary to introduce the sum  $t_\mu^\sigma + \mathfrak{T}_\mu^\sigma$  of the components of the gravitational field and of the matter.

Consequently, instead of equations (2.52) we write the field equations

$$-\frac{\partial}{\partial x^\lambda} \left( \sqrt{-gg^{\nu\alpha}} \Gamma_{\mu\nu}^\lambda \right) + \sqrt{-gg^{\nu\alpha}} \frac{\partial \Gamma_{\lambda\mu}^\nu}{\partial x^\nu} + L_\mu^\alpha = -\kappa \left[ \left( t_\mu^\alpha - \frac{1}{2} \delta_\mu^\alpha t \right) + \left( \mathfrak{T}_\mu^\alpha - \frac{1}{2} \delta_\mu^\alpha \mathfrak{T} \right) \right]. \quad (2.53)$$

If we differentiate the first term of equation (2.53),

$$-\frac{\partial}{\partial x^\lambda} \left( \sqrt{-gg^{\nu\alpha}} \right) \Gamma_{\mu\nu}^\lambda - \sqrt{-gg^{\nu\alpha}} \frac{\partial \Gamma_{\mu\nu}^\lambda}{\partial x^\lambda} + \sqrt{-gg^{\nu\alpha}} \frac{\partial \Gamma_{\lambda\mu}^\nu}{\partial x^\nu} + L_\mu^\alpha = -\kappa \left[ \left( t_\mu^\alpha - \frac{1}{2} \delta_\mu^\alpha t \right) + \left( \mathfrak{T}_\mu^\alpha - \frac{1}{2} \delta_\mu^\alpha \mathfrak{T} \right) \right].$$

then we use equation (2.39) to substitute it in the first term of the differentiation of the first term of the first member of equation (2.53) and transform the first term in parentheses on the right side by means of equations (2.50) and (2.51), we arrive at:

$$\sqrt{-gg^{\nu\alpha}} \left( \frac{\partial \Gamma_{\lambda\mu}^\nu}{\partial x^\nu} - \frac{\partial \Gamma_{\mu\nu}^\lambda}{\partial x^\lambda} + \Gamma_{\nu\lambda}^\eta \Gamma_{\mu\eta}^\lambda - \Gamma_{\mu\nu}^\eta \Gamma_{\lambda\eta}^\lambda \right) = -\kappa \left( \mathfrak{T}_\mu^\alpha - \frac{1}{2} \delta_\mu^\alpha \mathfrak{T} \right),$$

or, if we use equation (2.34), then we write

$$\sqrt{-gg^{\nu\alpha}} R_{\mu\nu} = -\kappa \left( \mathfrak{T}_\mu^\alpha - \frac{1}{2} \delta_\mu^\alpha \mathfrak{T} \right).$$

In order for us to express the field equations in covariant form, we need to multiply the above field equations by  $g_{\nu\kappa}$  and using equation (2.2), consequently, we find

$$R_{\mu\kappa} = -\kappa \left( T_{\mu\kappa} - \frac{1}{2} g_{\mu\kappa} T \right). \quad (2.54)$$

If we manipulate equation (2.54) we find the scalar equation  $R = \kappa T$ , from which it follows that the relativistic field equations of gravitation are given by

$$R_{\mu\kappa} - \frac{1}{2} g_{\mu\kappa} R = -8\pi G T_{\mu\kappa} \quad (2.55)$$

where  $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$  is the Einstein's tensor,  $T_{\mu\nu}$  is the energy-momentum tensor and the coupling constant is expressed by  $\kappa = 8\pi G$ .

Field equations are the fundamental equations of the relativistic description of gravitation, which are part of the general theory of relativity. In the general theory of relativity, gravity is the effect of the existence of a curvature in space-time.

Einstein's field equations [88] [89] relate the presence of matter to the curvature of space-time. In the field equations, gravitational potentials are given in terms of a metric tensor, a quantity that describes the geometric properties of spacetime [90] [91].

### Laws of conservation of gravitation

Given a point  $x$ , certain differential identities for the curvature tensor are obtained simply by adopting a locally inertial frame in which  $\Gamma_{\mu\nu}^\lambda$  vanishes at  $x$ . From the equation

$$R_{\rho\mu\nu\sigma} = g_{\rho\tau} R_{\mu\nu\sigma}^\tau,$$

we obtain

$$\nabla_\lambda R_{\rho\mu\nu\sigma} = \frac{1}{2} \frac{\partial}{\partial x^\lambda} \left( \frac{\partial^2 g_{\rho\nu}}{\partial x^\sigma \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\sigma \partial x^\rho} - \frac{\partial^2 g_{\rho\sigma}}{\partial x^\mu \partial x^\nu} + \frac{\partial^2 g_{\mu\sigma}}{\partial x^\nu \partial x^\rho} \right),$$

since all other terms are first and second order in  $\Gamma_{\mu\nu}^\lambda$ . Cyclically permuting  $\nu, \sigma$  and  $\lambda$ , we get

$$\nabla_\lambda R_{\rho\mu\nu\sigma} + \nabla_\nu R_{\rho\mu\sigma\lambda} + \nabla_\sigma R_{\rho\mu\lambda\nu} \equiv 0, \quad (2.56)$$

which are called Bianchi differential identities. Equations that are generally covariant and therefore are generally valid.

Contracting  $\rho$  with  $\nu$ :

$$-\nabla_\lambda R_{\mu\sigma} + \nabla_\sigma R_{\mu\lambda} + \nabla_\nu R_{\mu\sigma\lambda}^\nu \equiv 0,$$

where equation (2.23) has been used, that is,  $\nabla_\lambda g^{\rho\nu} = 0$ . Contracting again  $\mu$  with  $\sigma$ , we have:

$$-\nabla_\lambda R + \nabla_\mu R_\lambda^\mu + \nabla_\nu R_\lambda^\nu \equiv 0,$$

or

$$\nabla_\mu \left( R_\lambda^\mu - \frac{1}{2} \delta_\lambda^\mu R \right) \equiv 0,$$

which can be rewritten as

$$\nabla_\mu \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \equiv 0, \quad (2.57)$$

equation representing the contracted Bianchi identities.

From equation (2.19), we find

$$\nabla_\rho A_\mu^\nu = \frac{\partial A_\mu^\nu}{\partial x^\rho} - \Gamma_{\mu\rho}^\alpha A_\alpha^\nu + \Gamma_{\alpha\rho}^\nu A_\mu^\alpha,$$

then contracting  $\nu$  with  $\rho$  and with the help of equation (2.25) we can write

$$\nabla_\nu (\sqrt{-g} A_\mu^\nu) = \frac{\partial}{\partial x^\nu} (\sqrt{-g} A_\mu^\nu) - \Gamma_{\mu\nu}^\alpha \sqrt{-g} A_\alpha^\nu. \quad (2.58)$$

If  $T_{\mu\nu}$  represents the covariant energy-momentum tensor of matter, we have, by virtue of equation (2.55), that the equality must be satisfied

$$\nabla_\nu (\sqrt{-g} T_\mu^\nu) = 0 = \frac{\partial}{\partial x^\nu} (\sqrt{-g} T_\mu^\nu) - \Gamma_{\mu\nu}^\alpha \sqrt{-g} T_\alpha^\nu. \quad (2.59)$$

It has already been shown that the divergence of the first term vanishes, therefore, the principle of conservation of energy of matter can be expressed by equation (2.58):

$$\frac{\partial}{\partial x^\nu} (\sqrt{-g} T_\mu^\nu) - \Gamma_{\mu\nu}^\alpha \sqrt{-g} T_\alpha^\nu = 0.$$

It can be seen that both operations lead to an identity.

## 2.8 Lagrangian Formulation of the gravitational theory

### The principle of gravitational action

The variational principle consists in finding the extremes of the Lagrangian density  $\mathcal{L}_G$ , for gravitation. The action must be expressed as an integral over space-time (with the volume-invariant element) of a scalar function. Together with the works in which Einstein disclosed his formulation, Hilbert postulated the variational principle for gravitational theory expressing the action in the form

$$S[g_{\mu\nu}] = \int \mathcal{L}_G[g_{\mu\nu}] d^4x, \quad (2.60)$$

where  $\mathcal{L}_G = R\sqrt{-g}$  is the scalar Lagrangian density and  $g$  is the determinant of the metric tensor  $g_{\mu\nu}$ . Making

small variations  $\delta g_{\mu\nu}$  in the metric tensor  $g_{\mu\nu}$  and keeping the tensor  $g_{\mu\nu}$  and its first derivatives constant on the boundary, in effect, we can find that  $\delta S = 0$  for  $\delta g_{\mu\nu}$  gives Einstein's equations in the absence of matter [92].

### Indirect derivation of the field equations

From the Lagrangian density

$$\mathcal{L}_G = \sqrt{-g}R,$$

we would think of this, a function dependent on the tensor  $g_{\mu\nu}$ , in addition to its first and second derivatives; namely,

$$\mathcal{L}_G = \mathcal{L}_G \left( g_{\mu\nu}, \frac{\partial g_{\mu\nu}}{\partial x^\rho}, \frac{\partial^2 g_{\mu\nu}}{\partial x^\sigma \partial x^\rho} \right).$$

To substantiate this statement. In the first place, we make use of equation (2.27) in the transformation law of the metric tensor in the case where there are no isometries, and proceeding similarly to the section on the isometries of the metric tensor, we find the equation:

$$\delta g_{\mu\nu} = g'_{\mu\nu}(x) - g_{\mu\nu}(x) = -\mathcal{L}_{\varepsilon X} g_{\mu\nu} = -\varepsilon (\nabla_\nu X_\mu + \nabla_\mu X_\nu).$$

Secondly, if we carry out the variation of equation (2.60) and making use of the principle of gravitational action we find

$$\delta S = \int \frac{\delta \mathcal{L}_G}{\delta g_{\mu\nu}} \delta g_{\mu\nu} d^4x = -2\varepsilon \int \frac{\delta \mathcal{L}_G}{\delta g_{\mu\nu}} \nabla_\mu X_\nu d^4x = 0.$$

Now, if we use the technique of integration by parts and also use Leibniz's rule, therefore, we find:

$$\int \nabla_\nu \left( \frac{\delta \mathcal{L}_G}{\delta g_{\mu\nu}} \right) X_\mu d^4x = \int \nabla_\nu \left[ \frac{\delta \mathcal{L}_G}{\delta g_{\mu\nu}} X_\mu \right] d^4x.$$

We can use the divergence theorem and apply it to the second member of the preceding equation, we find that the preceding equation becomes null. Consequently, from the first member we can find the differential identities

$$\nabla_\mu \left( \frac{\delta \mathcal{L}_G}{\delta g_{\mu\nu}} \right) = 0.$$

Comparing the newly obtained differential identities with the contracted Bianchi identities (considering that the Bianchi identities are first and second partial derivatives of the metric tensor), i.e. equation (2.57), we can note that the Euler-Lagrange derivative is equivalent to the field equations of gravitation in the absence of matter, therefore, we can conclude that the Lagrangian density of gravitation depends on the metric tensor, its first and second derivatives.

In the case where  $g_{\mu\nu}$  are the dynamic variables, the Euler-Lagrange derivative  $\delta \mathcal{L} / \delta g_{\mu\nu}$  is a generalization, we get the Euler-Lagrange derivative

$$\frac{\delta \mathcal{L}_G}{\delta g_{\mu\nu}} = \frac{\partial \mathcal{L}_G}{\partial g_{\mu\nu}} - \frac{\partial}{\partial x^\rho} \left[ \frac{\partial \mathcal{L}_G}{\partial (\partial_\rho g_{\mu\nu})} \right] + \frac{\partial^2}{\partial x^\sigma \partial x^\rho} \left[ \frac{\partial \mathcal{L}_G}{\partial (\partial_\sigma \partial_\rho g_{\mu\nu})} \right] = 0, \quad \delta S = - \int \sqrt{-g} \left( R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right) \delta g_{\alpha\beta} d^4 x,$$

which is equivalent to the equation

$$\mathcal{L}_G^{\mu\nu} = \frac{\delta \mathcal{L}_G}{\delta g_{\mu\nu}} = -\sqrt{-g} \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right),$$

and from this equation and from the Euler-Lagrange equations it is possible to write Einstein's law of gravity in a vacuum; that is, the equation

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 0. \quad (2.61)$$

### Palatini's proposal

Palatini's proposal [93] is very elegant and consists of the idea of treating the metric and the connection as independent fields in the Einstein Lagrangian. To be more specific, let's change  $\mathcal{L}_G$  as a functional of  $\mathfrak{g}^{\mu\nu}$  and the connection  $\Gamma_{\mu\nu}^\sigma$  and its derivatives only, i. e.

$$\mathcal{L}_G = \mathcal{L}_G \left( \mathfrak{g}^{\mu\nu}, \Gamma_{\mu\nu}^\sigma, \frac{\partial \Gamma_{\mu\nu}^\sigma}{\partial x^\rho} \right),$$

thus the Ricci tensor depends only on  $\Gamma_{\mu\nu}^\sigma$  and its derivatives only. So, if we carry out a variation with respect to  $\mathfrak{g}^{\mu\nu}$  and the principle of stationary action immediately gives the field equations in vacuum  $R_{\mu\nu}$ . We will now show the preceding at the beginning of this paragraph, starting from action (2.60) in the form

$$S = \int \mathfrak{g}^{\mu\nu} R_{\mu\nu} d^4 x,$$

we carry out a variation and use the Leibniz rule for products, to obtain

$$\delta S = \int (\delta \mathfrak{g}^{\mu\nu} R_{\mu\nu} + \mathfrak{g}^{\mu\nu} \delta R_{\mu\nu}) d^4 x,$$

where we have written  $\mathfrak{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$ .

Now we use the Palatini equation [66]

$$\delta R_{\mu\nu} = \nabla_\nu (\delta \Gamma_{\mu\sigma}^\sigma) - \nabla_\sigma (\delta \Gamma_{\mu\nu}^\sigma),$$

consequently

$$\delta S = \int \delta \mathfrak{g}^{\mu\nu} R_{\mu\nu} d^4 x + \int \mathfrak{g}^{\mu\nu} [-\nabla_\sigma (\delta \Gamma_{\mu\nu}^\sigma) + \nabla_\nu (\delta \Gamma_{\mu\sigma}^\sigma)] d^4 x,$$

the second term vanishes, since the covariant derivative of  $\mathfrak{g}^{\mu\nu}$  vanishes identically, in other words, by making use of the divergence theorem, this new quantity vanishes because the variations are assumed null at the frontier, to obtain

where we have used the equation

$$\delta \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{\alpha\beta} \delta g_{\alpha\beta}.$$

By virtue of the stationary gravitational action principle, we obtain the field equations

$$-\sqrt{-g} \left( R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right) = 0, \quad (2.62)$$

which are equivalent to equation (2.61) and the contracted Bianchi identities:

$$\nabla_\beta \left[ \sqrt{-g} \left( R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right) \right] = 0, \quad (2.63)$$

as the constraints of the field equations.

Next we do a variation with respect to  $\Gamma_{\mu\nu}^\sigma$ , so

$$\delta S = \int \mathfrak{g}^{\mu\nu} [\nabla_\nu (\delta \Gamma_{\mu\rho}^\rho) - \nabla_\rho (\delta \Gamma_{\mu\nu}^\rho)] d^4 x.$$

Integrating by parts and discarding the divergence term by the usual argument, we get

$$\delta S = \int [\delta_\rho^\nu \nabla_\sigma \mathfrak{g}^{\mu\sigma} - \nabla_\rho \mathfrak{g}^{\mu\nu}] \delta \Gamma_{\mu\nu}^\rho d^4 x.$$

Since  $\delta S$  vanishes for an arbitrary volume, the integrand must vanish, i.e.

$$\delta_\rho^\nu \nabla_\sigma \mathfrak{g}^{\mu\sigma} - \nabla_\rho \mathfrak{g}^{\mu\nu} = 0.$$

The variations  $\delta \Gamma_{\mu\nu}^\rho$  are arbitrary, but the symmetry in  $\mu$  and  $\nu$ , and therefore only the symmetrical part of the expression in square brackets vanishes, i.e.

$$\frac{1}{2} \delta_\rho^\nu \nabla_\sigma \mathfrak{g}^{\mu\sigma} + \frac{1}{2} \delta_\rho^\mu \nabla_\sigma \mathfrak{g}^{\nu\sigma} - \nabla_\rho \mathfrak{g}^{\mu\nu} = 0.$$

Finally, if

$$\frac{\partial g_{\mu\nu}}{\partial x^\rho} - \Gamma_{\mu\rho}^\alpha g_{\alpha\nu} - \Gamma_{\nu\rho}^\alpha g_{\mu\alpha} = \nabla_\rho g_{\mu\nu} \equiv 0,$$

then it follows that  $\Gamma_{\mu\nu}^\rho$  is necessarily the metric connection

$$\Gamma_{\mu\nu}^\rho = \frac{1}{2} g^{\rho\sigma} \left( \frac{\partial g_{\mu\sigma}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right).$$

In Palatini's proposal, the variation with respect to the metric leads to the field equations, and the variation with respect to  $\Gamma_{\mu\nu}^\rho$  reveals that the affin connection is necessarily the metric connection (see appendix A).

### Complete field equations

So far, we have been working with the field equations in a vacuum. To obtain the complete field equations, we assume that there are other fields present alongside the gravitational field, which can be described by an appropriate Lagrangian density  $\mathcal{L}_M$  (the Lagrangian density of matter). The action integral is then

$$I = \int (\mathcal{L}_G + \kappa \mathcal{L}_M) d^4x, \quad (2.64)$$

where  $\kappa$  is the coupling constant. Both Lagrangians are considered as functionals of the metric and its derivatives, and therefore, varying with respect to  $g_{\mu\nu}$ , we obtain

$$\delta I = \int \left( \frac{\delta \mathcal{L}_G}{\delta g_{\mu\nu}} + \kappa \frac{\delta \mathcal{L}_M}{\delta g_{\mu\nu}} \right) \delta g_{\mu\nu} d\Omega = \int (\mathcal{L}_G^{\mu\nu} + \kappa \mathcal{L}_M^{\mu\nu}) \delta g_{\mu\nu} d^4x,$$

but

$$\frac{\delta \mathcal{L}_G}{\delta g_{\mu\nu}} = -\sqrt{-g} G^{\mu\nu} \quad (2.65)$$

and

$$\frac{\delta \mathcal{L}_M}{\delta g_{\mu\nu}} = \sqrt{-g} T^{\mu\nu}, \quad (2.66)$$

where the last equation defines the energy-momentum tensor,  $T^{\mu\nu}$ , for the fields present. After dividing by  $\sqrt{-g}$ , the field equations are

$$\delta I = \int (\mathcal{L}_G^{\mu\nu} + \kappa \mathcal{L}_M^{\mu\nu}) \delta g_{\mu\nu} d^4x = \int (-\sqrt{-g} G^{\mu\nu} + \kappa \sqrt{-g} T^{\mu\nu}) \delta g_{\mu\nu} d^4x = 0,$$

therefore

$$G^{\mu\nu} = \kappa T^{\mu\nu}. \quad (2.67)$$

## 3 Cosmology

We can define cosmology as the branch of physics that studies the origin of the Universe on its largest scale. In 1687, Isaac Newton published his book entitled "Mathematical Principles of Natural Philosophy" [1], better known as "Principia", where he formulated the bases of classical mechanics through the laws that bear his name and his theory of gravitation, then, was born the analytic cosmology.

In the year 1916 Einstein, supported by his equivalence principle, tensor calculus, the general covariance principle and the notion of the curvature of space-time, published the general theory of relativity in its complete

and definitive version. Soon after, various solutions of the Einsteinian field equations of gravity, which are the structure of modern cosmology, were published. The hypothesis of isotropy and homogeneity applied to general relativity opened the field of modern cosmology with the construction of models that accept exact solutions, known as Friedmann-Lemaître-Robertson-Walker models. These types of models were developed by Alexander Friedmann [94] and later by Howard Percy Robertson [95] and Arthur Geoffrey Walker [96], among others [97]. Modern cosmology adheres to the cosmological principle, which helps to more easily solve Einstein's equations.

Isotropy refers to the observation that the Universe is the same in every direction and sense. Homogeneity refers to the observation that the Universe looks the same at all points. On the other hand, the cosmological principle establishes that on large scales the Universe is homogeneous and isotropic, that is, there are no privileged positions or directions in the Universe.

### 3.1 Friedmann-Lamaitre-Robertson-Walker cosmological model

According to the cosmological principle, we can approximate the Universe as a three-dimensionally homogeneous and isotropic space that can expand (or contract) as a function of time. The metric of this space is the Friedmann-Lamaitre-Robertson-Walker (FLRW) metric. The FLRW metric can be determined from the geometry of homogeneous spaces as shown below.

Let us consider the transformations in a four-dimensional Euclidean space with the coordinates  $(x, y, z, w)$

$$\begin{aligned} w &= a \cos \psi, \\ x &= a \sin \psi \cos \theta, \\ y &= a \sin \psi \sin \theta \cos \phi, \\ z &= a \sin \psi \sin \theta \sin \phi. \end{aligned} \quad (3.1)$$

The infinitesimal distance in this system is given by:

$$d\sigma^2 = dx^2 + dy^2 + dz^2 + dw^2. \quad (3.2)$$

If we differentiate equations (3.1), then we substitute the total differentials in equation (3.2) and after making the necessary simplifications we obtain

$$d\sigma^2 = a^2 [d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2)]. \quad (3.3)$$

Similarly, we consider a homogeneous hypersurface with coordinates  $(x, y, z, w)$  embedded in a flat Minkowski space. In other words, a hyperbolic space, with the transformations

$$\begin{aligned} w &= a \cosh \psi, \\ x &= a \sinh \psi \cos \theta, \\ y &= a \sinh \psi \sin \theta \cos \phi, \\ z &= a \sinh \psi \sin \theta \sin \phi. \end{aligned} \quad (3.4)$$

The line element for a homogeneous hypersurface in a hyperbolic space is expressed by the equation

$$d\sigma^2 = dx^2 + dy^2 + dz^2 - dw^2. \quad (3.5)$$

If we differentiate equations (3.4), then we substitute the total differentials in equation (3.5) and after making the necessary simplifications we obtain

$$d\sigma^2 = a^2 [d\psi^2 + \sinh^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2)]. \quad (3.6)$$

Let us consider a spherical surface, with Cartesian equation  $x^2 + y^2 + z^2 = a^2$ , in three-dimensional Euclidean space with the transformations

$$\begin{aligned} x &= a \sin \theta \cos \phi, \\ y &= a \sin \theta \sin \phi, \\ z &= a \cos \theta, \end{aligned} \quad (3.7)$$

which also has a line element given by  $d\sigma = \sqrt{dx^2 + dy^2 + dz^2}$ . If we proceed in a similar way as in the past occasions with the help of equations (3.7), the line element for the spherical surface is

$$d\sigma^2 = a^2 (t) [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)]. \quad (3.8)$$

Instead of the four coordinates for which the spatial isotropy of the universe is most evident, we will now choose different coordinates that are more convenient from the point of view of physical interpretation. In such a coordinate system the metric is of the form:

$$ds^2 = (dx_0)^2 - d\sigma^2 = (dx_0)^2 - g_{ik} dx^i dx^k, \quad (3.9)$$

where  $i, k = 1, 2, 3$ .

If we take the radial transformations,  $\sin \psi = r$  in equation (3.3) and  $\sinh \psi = r$  in equation (3.6), then we calculate from these transformations the total differentials expressed by  $dr = \left\{ \begin{smallmatrix} \cos \psi \\ \cosh \psi \end{smallmatrix} \right\} d\psi$ , we then take the radial transformations and rewrite the total differentials as  $dr^2 = (1 \mp r^2) d\psi^2$ , then equations (3.3) and (3.6) are given by

$$d\sigma^2 = a^2 (t) \left[ \frac{dr^2}{1 \mp r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \quad (3.10)$$

If we confine equations (3.6), (3.8) and (3.10) and use equation (3.9), we therefore obtain the FLRW metric expressed by:

$$ds^2 = dt^2 - a^2 (t) \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (3.11)$$

where  $k = 0, +1, -1$ . Therefore, the three types of geometries for space described in the FLRW model metric are classified as open if  $k = -1$  (i.e., hyperbolic space), flat if  $k = 0$  (i.e., Euclidean space), or closed if  $k = 1$  (i.e., spherical space).

### 3.2 Friedmann Equations

Suppose now that the Universe is filled with an ideal fluid; frictionless adiabatic fluid, ie fluid characterized by the fact that in a local coordinate system of a fluid element there is only one isotropic pressure. Therefore, the energy-momentum tensor according to the theory of general relativity can be represented by:

$$T^{\mu\nu} = (\rho + p) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - pg^{\mu\nu}, \quad (3.12)$$

or covariant

$$T_{\mu\nu} = (\rho + p) g_{\mu\rho} g_{\nu\sigma} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} - pg_{\mu\nu}.$$

With this simple description of matter, if we make use of equation (3.11) and introduce equation (3.12) into Einstein's gravitational field equations, that is, equation (2.59) of the preceding section, with constant cosmological  $R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R - \Lambda g^{\mu\nu} = -8\pi GT^{\mu\nu}$  [98], we get the Friedmann equations:

$$\frac{2}{a} \frac{da}{dt^2} + \frac{1}{a^2} \left( \frac{da}{dt} \right)^2 + \frac{k}{a^2} - \Lambda = -8\pi G\rho, \quad (3.13)$$

$$3 \left[ \frac{1}{a^2} \left( \frac{da}{dt} \right)^2 + \frac{k}{a^2} \right] - \Lambda = 8\pi G\rho. \quad (3.14)$$

A direct consequence of equations (3.13) and (3.14) is the continuity equation. If we differentiate equation (3.14), then the resulting differential equation is divided by  $a \frac{da}{dt}$ , and finally we subtract this resulting equation with equation (3.13), then we obtain the differential equation

$$\frac{d\rho}{dt} + 3H(\rho + p) = 0, \quad (3.15)$$

where we have defined the Hubble parameter

$$H(t) = \frac{1}{a} \frac{da}{dt}. \quad (3.16)$$

The Hubble parameter refers to the speed with which most distant galaxies are receding from us through Hubble's law [99]



$$v = Hd, \quad (3.17)$$

where  $v$  is the speed and tells us how an object moves away or approaches and  $d$  is the distance between the observer and the distant galaxy that moves away.

There is another physically feasible way to find equation (3.15). This way would be through the application of the equation of conservation of momentum and energy for matter in the relativistic theory of gravitation. The conservation of energy is expressed in General Relativity by nullifying the divergence of the energy-momentum tensor, that is, by the equation

$$0 = \frac{\partial(\sqrt{-g}T_{\sigma}^{\alpha})}{\partial x^{\alpha}} - \Gamma_{\sigma\beta}^{\alpha}\sqrt{-g}g^{\rho\beta}T_{\rho\alpha}. \quad (3.18)$$

Applying this conservation law to our assumption of the FLRW metric of equation (3.11) and the moment energy tensor for a perfect fluid, we obtain a single energy conservation equation, that is, the equation

$$\frac{d\rho}{dt} + 3H(\rho + p) = 0.$$

This equation is not actually independent of the Friedmann equations, but it is necessary for consistency. This equation implies that the expansion of the Universe (as specified by  $H$ ) can give rise to local changes in energy density. Note that there is no notion of "total energy" conservation, since energy can be exchanged between matter and geometric space.

### 3.3 Solution of the dynamical equations in a Euclidean universe ( $k = 0$ ) and without cosmological constant

In order to determine the solutions to the Friedmann differential equations, let us consider an equation of state for cosmological matter of the form  $p = \omega\rho$ , with  $\omega = \omega(a)$ , not necessarily constant. Integrating equation (3.15) we get

$$\rho(a) = \rho(a_0) \exp \left\{ -3 \int_{a_0}^a [1 + \omega(u)] \frac{du}{u} \right\}. \quad (3.19)$$

If  $\omega$  is constant, then we have

$$\rho(a) = \rho(a_0) \left( \frac{a}{a_0} \right)^{-3(1+\omega)}. \quad (3.20)$$

Equation (3.20) includes the three types of matter that play a very important role in cosmology. The first type of matter is dust (non-relativistic matter), for which  $\omega = 0$ , the second type of matter is radiation (relativistic particle gas), where  $\omega = \frac{1}{3}$ , and finally, the so-called vacuum energy, with the value  $\omega = -1$ .

Let us consider a flat Universe, for this, we substitute equation (3.20) in equation (3.14) obtaining the differential equation

$$\frac{1}{a^2} \left( \frac{da}{dt} \right)^2 = \frac{8\pi G}{3} \rho_0 \left( \frac{a}{a_0} \right)^{-3(1+\omega)}, \quad (3.21)$$

which implies the evolution of the scale factor. By virtue of the method of separation of variables, the solution to the differential equation (3.21), considering the initial condition  $a(t_0) = t_0$ , is given by

$$a(t) = a_0 \left( \frac{t}{t_0} \right)^{\frac{2}{3(1+\omega)}}, \quad (3.22)$$

where  $a_0$  is the current scale factor, unless  $\omega = -1$ , in which case we get  $a(t) \propto \exp(Ht)$ . It is important to note that the matter and global radiation in flat Universes start with  $a = 0$ , this is a singularity, known as the Big Bang [100] [101] [102] [103]. We can therefore calculate the age of the Universe in question. If we take into account equation (3.16), we can find the result

$$H(t) = \frac{1}{a(t)} \frac{da(t)}{dt} = \frac{2}{3(1+\omega)t}. \quad (3.23)$$

From equation (3.23), we calculate

$$t_0 = \int_0^1 \frac{da}{aH(a)} = \frac{2}{3(1+\omega)H_0}. \quad (3.24)$$

Unless  $\omega$  is close to  $-1$ , it is often convenient to approximate equation (3.24) to the quantity:

$$t_0 \sim H_0^{-1}. \quad (3.25)$$

This is why the quantity is known as the Hubble time.

If we consider the previous analysis and use equations (3.20) and (3.21) we can build the following table:

tipo de energia	$\rho(a)$	$a(t)$
dust	$a^{-3}$	$t^{\frac{2}{3}}$
radiation	$a^{-4}$	$t^{\frac{1}{2}}$
cosmological constant	const.	$\exp(Ht)$

**Table 1:** The behavior of the scale factor applies to the case of a flat Universe; with curvature  $k = 0$ , the behavior of the energy densities is perfectly general.

### 3.4 Cosmological parameters

The best known cosmological parameter is the Hubble parameter, defined in equation (3.16), whose value today is called the Hubble constant  $H_0 = \frac{1}{a(t_0)} \frac{da}{dt} \Big|_{t=t_0}$ . The Hubble constant is given by

$$H_0 = 100h \text{ (km/seg/Mpc)}. \quad (3.26)$$

The Hubble Telescope is one of the main instruments on extragalactic distances [104] [105]. From the observations and data collected by this telescope it is inferred that

$$h = 0.71 \pm 0.06. \quad (3.27)$$

From the Hubble parameter, the critical energy density is defined<sup>2</sup> for

$$\rho_0 = \frac{3H^2(t_0)}{8\pi G}, \quad (3.28)$$

and with the help of equations (3.26) and (3.27)

$$\rho_0 \simeq 1.88h^2 10^{-29} \frac{g}{cm^3} = 9.5 \times 10^{-30} \frac{g}{cm^3}, \quad (3.29)$$

where this amount is approximately equivalent to 6 protons per cubic meter.

In terms of the energy density of the Universe and the Hubble parameter it is possible to define the density parameter by the mathematical equation

$$\Omega = \frac{\rho}{\rho_0} = \frac{8\pi G}{3H^2(t_0)} \rho, \quad (3.30)$$

where the sign can be used to determine the spatial curvature. The classification of the Universe, according to the definition of equation (3.42), can be done in the following way:

- For a closed Universe ( $k = +1$ ) we have that  $\Omega > 1$ .
- Flat Universe ( $k = 0$ ), the density parameter is equal to zero, that is,  $\Omega = 0$ .
- Open Universe ( $k = -1$ ), It is true that  $\Omega < 1$ .

Cosmological observations allow us to estimate the different density parameters as follows:

- Baryonic matter (matter made up of electrons, neutrons and protons):  $\Omega_B \approx 0.04$  [106] [107].
- Dark matter:  $\Omega_{DM} = \Omega_{CDM} + \Omega_{HDM} \approx 0.26$  [105].
- Dark energy (compatible with a cosmological constant):  $\Omega_{DE} = \Omega_\Lambda = 0.7$  [108].
- Radiation:  $\Omega_R \approx 5 \times 10^{-5}$ .

<sup>2</sup>This energy is taken into account for a flat Universe.

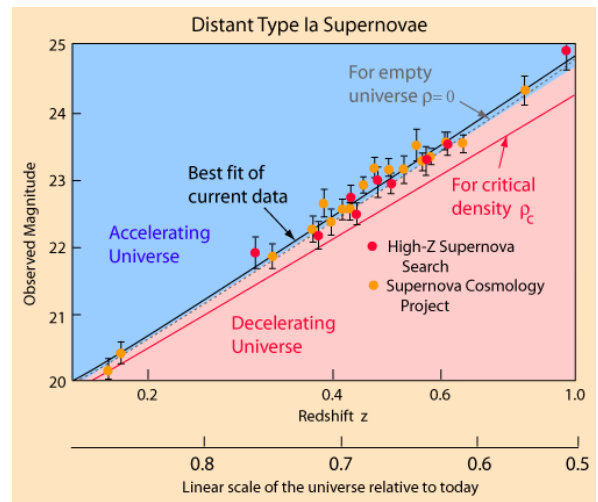
An important consequence of the density parameters is the consistency relationship between the cosmological parameters [108]:

$$\Omega_B + \Omega_{DM} + \Omega_{DE} = 1. \quad (3.31)$$

If we substitute equation (3.14) into equation (3.13), then we have the equation  $\frac{1}{a} \frac{d^2 a}{dt^2} = -\frac{8\pi G}{3} (\rho + 3p) + \frac{\Lambda}{3}$ . From this equation another basic parameter is defined, the deceleration parameter [109]:

$$q = -a \left( \frac{da}{dt} \right)^{-2} \frac{d^2 a}{dt^2} = \frac{4\pi G}{3H^2} (\rho + 3p) - \frac{\Lambda}{3H^2}. \quad (3.32)$$

The uniform expansion corresponds to  $q = 0$  and requires a cancellation between matter and vacuum energy. For matter we have  $q > 0$ , otherwise, for the vacuum energy domain,  $q < 0$ . According to the current density parameter, the presence of radiation is negligible, but in the past radiation was dominant. At present, the total energy content is dominated by dark energy, similar to a cosmological constant, and therefore the expansion of the Universe at present is accelerating (see figure 1).



**Figure 1:** Magnitude-Redshift diagram to study the expansion of the Universe [110].

### 3.5 Geometry and dark energy

In recent years it has become clear that the dominant element of energy density in the Universe today is neither dust nor radiation, but rather dark energy [111]. This component is characterized by a parametric equation of state for  $\omega < -\frac{1}{3}$ .

For simplicity, let's focus on what happens if the energy density in the Universe is just a cosmological constant  $\Lambda$ , with  $\omega = -1$ . In this case, the Friedmann equation (in relativistic units) can be solved for any value of the spatial curvature  $k$ . If  $\Lambda > 0$  and  $k = 0$ , then from the Friedmann equation (3.14) we have

$$a(t) = a_0 \exp\left(\sqrt{\frac{\Lambda}{3}}t\right),$$

Analogously for  $k = \pm 1$

$$\frac{1}{a(t)} \frac{da(t)}{dt} = \sqrt{\frac{\Lambda}{3} \mp \frac{1}{2a^2(t)}},$$

equation from which, if we integrate, we get

$$\frac{a(t)}{a_0} = \begin{cases} \cosh\left(\sqrt{\frac{\Lambda}{3}}t\right), & k = +1, \\ \sinh\left(\sqrt{\frac{\Lambda}{3}}t\right), & k = -1. \end{cases}$$

So the solutions are

$$\frac{a(t)}{a_0} = \begin{cases} \cosh\left(\sqrt{\frac{\Lambda}{3}}t\right), & k = +1, \\ \exp\left(\sqrt{\frac{\Lambda}{3}}t\right), & k = 0, \\ \sinh\left(\sqrt{\frac{\Lambda}{3}}t\right), & k = -1, \end{cases} \quad (3.33)$$

where they have encountered the case where  $k = 0$ , above. It is clear that, in the limit as  $t \rightarrow \infty$ , all solutions expand exponentially, regardless of the spatial curvature. In fact, these solutions are exactly the same space-time de Sitter space- [112] [113] [114] just in different coordinate systems.

Now, let us complete the description of the spaces with a cosmological constant considering the case  $\Lambda < 0$ . This space is called the anti-de Sitter space (AdS) [115] [116] and it should be clear from the Friedmann equation that such a space can only exist in a space of spatial curvature with  $k = -1$ . The solution for the corresponding scale factor is

$$a(t) = a_0 \sin\left(\sqrt{-\frac{\Lambda}{3}}t\right). \quad (3.34)$$

Once again, this solution does not cover all Anti-Sitter spaces (see [117]).

## 4 Dark Matter

In the previous section we have studied the tools needed to analyze the kinematics and dynamics of homogeneous and isotropic cosmologies in General Relativity. In this part, we refer to the real Universe in which we live, and we will discuss the properties, evidence and other topics about dark matter.

### 4.1 Matter: Ordinary and Dark

For "ordinary matter" is meant anything made of atoms and their components (protons, neutrons, and electrons), which would include all of the stars, planets, gas, and dust in the universe, immediately visible or otherwise. Usually such matter in question is called "baryonic matter", where "baryons" include protons, neutrons, and particles (strongly interacting particles with a conserved quantum number known as "baryon number").

Ordinary baryonic matter turns out not to be enough to account for the mass density. Our current best estimate for the baryon density [118] is

$$\Omega_b = 0.04 \pm 0.02 \quad (4.1)$$

where these uncertainties are preserved by most models. This determination comes from a variety of methods: direct counting of baryons (less precise method), consistency with the CMB power spectrum, and agreement with predictions of light element abundances for Big-Bang nucleosynthesis. Most of the density of matter, therefore, must be in the non-baryonic form, the so-called "Dark Matter", which we will simply abbreviate "Dark Matter".

### 4.2 Evidence of dark matter

The basic assumption of contemporary cosmology holds that the homogeneous, isotropic and flat Universe is governed by the laws of General Relativity. Numerous experiments give evidence that the Universe is filled not only with the known baryonic matter, but also with two other components: dark matter and dark energy. Dark energy is the form of energy with negative pressure, which homogeneously permeates space causing the accelerated expansion of the Universe. Its exact nature and origin are still unclear, however, it can be described by Einstein's equations with cosmological constant. On the other hand, observations indicate that dark matter behaves like ordinary matter. Such dark matter is gravitationally interacting and has a tendency to clump together, but as its name suggests, it does not emit or scatter light and is therefore difficult to detect using standard astronomical methods.

The commonly adopted cosmological model is the  $\Lambda$ CDM model, which includes dark energy in the form of

the cosmological constant and cold dark matter, which moves slowly compared to the speed of light. The LCDM is in good agreement with precise and contemporary observations. These indicate that only 5% of the energy in the Universe is in the form of known baryonic matter, the rest is described by the dark components. Currently, the best experimental value for the energy density of dark energy  $\Omega_\Lambda$ , dark matter  $\Omega_c$ , and baryonic  $\Omega_b$  (calculated with respect to  $\rho_c$ - the current critical density,  $\Omega = \rho/\rho_c$ ) comes from data produced by Planck and WMAP for the study of the cosmic microwave background (CMB) [119]

$$\Omega_\Lambda = 0.683, \quad \Omega_c = 0.268, \quad \Omega_b = 0.049. \quad (4.2)$$

Evidence for dark matter extends from the scale of galaxies to the Universe as a whole. From various observations and theoretical models, the essential properties of dark matter are inferred, namely, that it is electrically neutral (thus not luminous), massive, non-baryonic, non-relativistic (cold), and weakly interacts with ordinary matter. Furthermore, dark matter must be stable or decay with a lifetime much greater than the age of the Universe. These requirements cannot be met by any known particle. Many reasonable candidates beyond the Standard Model were proposed. To become one of them, a particle must pass a ten-point test<sup>3</sup> [120].

The "missing matter" problem was first noted by Franck Zwicky in 1933 [121]. He was looking at the Coma cluster and, using the virial theorem, found that it contains an insufficient amount of luminous matter to explain the distribution of galaxies' velocities. Similar conclusions can be reached by examining the rotation curves of individual galaxies. Measurements show (figure 2) that the rotation curves of galaxies at great distances remain flat, while Newton's gravitational law gives the following expression for the circular velocity  $v(r)$ :

$$\frac{mv^2}{r} = \frac{GMm}{r^2} \rightarrow v = \sqrt{\frac{GM}{r}} r^{-1/2}, \quad (4.3)$$

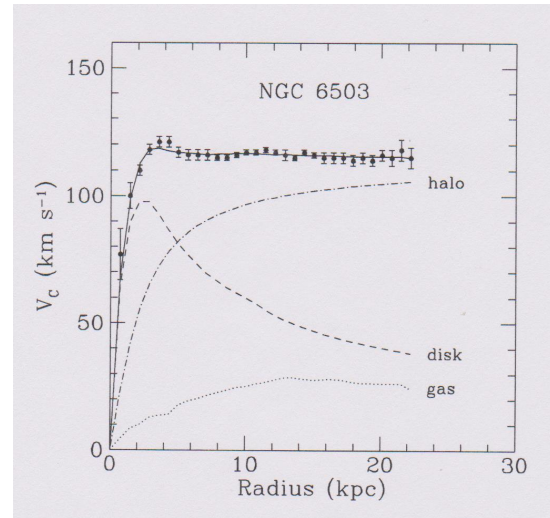
<sup>3</sup>The 10 essential points are:

1. Relic density.
2. Is dark matter hot or cold?
3. Is dark matter neutral?
4. Consistency with the BBN.
5. Stellar evolution.
6. Support for self-interaction restrictions.
7. Compatibility with constraints on gamma-ray observations.
8. Indirect detection.
9. Direct detection.
10. Can it be verified experimentally?

where the mass of matter at radius  $r$  is given by

$$M(r) = \int_0^r 4\pi\rho(r)r^2 dr. \quad (4.4)$$

This means that for large  $r$ , where the density of luminous matter is negligible and  $M(r)$  is nearly constant, the rotation curves should decrease as  $r^{-1/2}$ , but the observed shape advocates by the existence of halos of dark matter galaxies with  $\rho r^{-2}$ , that gives  $M(r)r$ .



**Figure 2:** Rotation curve for the spiral galaxy NGC 6503 [122]. The points are equivalent to stars in the galaxy. The dashed line shows the expected rotation curve if only luminous matter is taken into account. The dashed and dotted line would be the contribution of the dark matter to the halo. The dotted line shows the gas rotation curve of the galaxy.

It turns out that the observed rotation curves could be described by modifying Newtonian gravity [123], however, the observation of cluster collisions is hardly explainable without the existence of dark matter. The clearest evidence is given by the famous "Bullet Cluster" (figure 3) consisting of perceptible matter in the vis-

ible and X-ray spectrum and weakly interacting dark halos detected by gravitational lensing, which did not slow down during the collision and were separated. of the baryons.



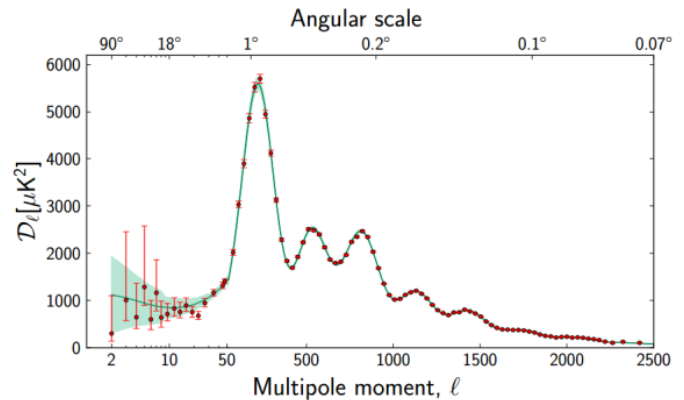
**Figure 3:** Source: NASA. Bullet cluster. The image is composed of an optical image and an X-ray image taken by the Chandra telescope. The hot gas contributions depicted in pink are clearly separated from the dark matter component depicted in blue [124].

We can also infer the properties of dark matter from larger-scale observations of the Universe. In the formation of the scale structure not only the density but also the type of dark matter particle is important [125]. In the case of light particles, which move with relativistic speeds, dark matter would escape from regions the size of overdense galaxies, preventing the formation of smaller structures. This results in a top-down hierarchy of structures with the smaller ones formed by the fragmentation of the larger ones. Such behavior is in contradiction to observed galaxy distributions at very high redshift. It favors the existence of cold dark matter with a mass of order GeV or higher. On the other hand, particles moving with non-relativistic speeds can be concentrated on smaller scales. It leads to the formation of opposite structures with the smaller ones merging with the larger ones. This point of view is also supported by N-body simulations like the Millennium project [126] or Illustris [127].

We can derive information about the role of dark matter in the early Universe from relics, the most important of which is the cosmic microwave background (CMB) radiation. It originates at the time of recombination, when electrons were bonded to protons to form neutral atoms. From that moment on, photons were able to flow freely through the Universe, which is now observed as radiation with a perfect blackbody spectrum of temperature 2.7252 K. The CMB is almost homogeneous. Although the fluctuations are of the order  $10^{-5}$ , they can be used to derive the distribution of baryons and matter in the

Universe. The power spectrum of the CMB in angular decay (figure 4) includes the number of peaks that come from the acoustic oscillations of the primordial plasma. These oscillations were driven by the gravitational attraction of matter and radiation pressure. Their behavior was imprinted on the CMB at the time of recombination. Odd peaks refer to plasma compression and even peaks to rarefaction, while baryon density affects only the latter [128]. Therefore, the discrepancy between the number of baryons and the distribution of total matter can be measured. By fitting the power spectrum to the LCDM model, the precise values (4.2) of dark matter, dark energy and baryon density are obtained.

To summarize, there is clear evidence for the existence of dark matter on a wide range of cosmic scales. Many cosmological observations are in remarkable mutual agreement and create a self-consistent model called "Concordance Cosmology", where dark matter is the essential ingredient. However, cosmology alone cannot explain the role of dark matter in the context of particle physics.



**Figure 4:** The power spectrum measured by Planck, showing temperature fluctuations over a range of size scales in the sky. The three main peaks show the relative contributions of dark energy, baryonic matter, and dark matter [129].

### 4.3 Detection of dark matter

Today, there is compelling evidence that provides a strong scenario for the existence of a new type of cold, collisionless, nonluminous matter, called dark matter, that gravitationally interacts with ordinary matter. In this section, the different astrophysical and cosmological evidences for dark matter and the candidates for dark matter particles that emerge from various theoretical models are presented. Finally, the great variety of experimental techniques that have been developed in recent decades to detect, directly or indirectly, or produce these dark matter particles in accelerators are shown.

There are basically three different methods that can be used to identify dark matter particles, all of them are based on the assumption that the DM particles interact with the SM also through forces other than gravity. Dark matter particles are expected to be produced in colliders [130–133], because they allow to control the initial conditions and obtain accurate results, however, no dark matter signals were found in the Large Hadron Collider (LHC) [134]. In addition to collider experiments, there are also numerous efforts to detect dark matter directly in laboratories or indirectly in its annihilation products [135] [136].

The goal of direct detection experiments is to measure the back energies of the scattering of nuclei by DM particles with unknown mass  $m_\chi$ . It can be estimated that the DM particles present in our galactic halo flow through the Earth with a flux of  $10^5 (100\text{GeV}/m_\chi) \text{cm}^{-2}\text{s}^{-1}$ . The speed and energy gained from nuclear recoils can be used to reconstruct the properties of dark matter. There are two leading technologies in current detectors [137]:

1. Cryogenic detectors that measure the heat produced when the particle interacts with the solid state or superfluid  $^3\text{He}$  absorbed at a temperature below  $100\text{mK}$ .
2. Scintillation detectors that use the light emitted during a collision of dark matter with liquefied noble gas, such as argon or xenon. All types of detectors are placed underground the earth to ensure detection of cosmic rays and ambient radioactivity.

There has been an impressive improvement in detector accuracy in recent years, mainly in the mass range below 100 GeV, where direct detection methods complement the search for the collider. For weakly interacting particles and typical galactic velocities of order  $v/c \sim 10^{-3}$ , nuclei scattering should take place at  $10^{-8}\text{pb}$  or less, which is in the range of sensitivity of current detectors. In recent years there have been numerous reports of dark matter detection. The most promising is from the DAMA/LIBRA experiment, which announced the observation of the annual modulation of the nuclear recoil rate, a possible effect that comes from the movement of the earth around the Sun. Although the signal is strong, its interpretation it is problematic because it was not confirmed by any other experiments. An excess of events, related to the interaction of dark matter particles, was also reported by COGENT, CRESST, EGRET and others [138]. However, direct detection methods are still inconclusive, due to a large amount of uncertainty in the elements of nucleon matrix or nuclear shape factors.

Indirect detection aims to find the secondary particles produced by annihilation or decay of dark matter particles, mainly in the center of our galaxy. Recently most studies focus on  $\gamma$ -rays, because they travel fairly un-

absorbed through the galaxy in a straight line and indicate the point where the annihilation of dark matter took place. Due to these properties, one can separate the distribution of energy from the signal [129]. An interesting case is the two-photon annihilation considered in [139]. Due to conservation of energy and non-relativistic motion, the particle will annihilate into photons with energy nearly equal to its rest mass  $E_\gamma = m_\chi$ . Also, the motion of a particle and the Doppler shift result in a spectral line broadening by only  $10^{-3}$ . Unfortunately, these processes are still difficult to observe, due to suppressed annihilation rates (the interaction between photons and neutral dark matter must be induced by a loop). Indirect search is not as competitive as direct search, but there are many preliminary results. There is an excess of positrons observed by the PAMELA satellite [140] and the gamma line found by FERMI-LAT [141]. The last signal is difficult to explain by astrophysical sources and corresponds to a DM particle if it is confirmed with a mass greater than 130 GeV.

#### 4.4 Dark matter candidates

Dark matter needs to have a nature, such that it meets certain requirements in order to explain the observational evidence of it. Its properties help to identify possible candidates from very diverse theories.

##### 4.4.1 Supersymmetric dark matter

WIMPs ( $\chi$ ) are particles with masses between 10 GeV and a few TeV and with low intensity cross sections. Its relic density is calculated in the case that the WIMPs are in equilibrium with the particles of standard model. The current relic particle density  $\chi$  is expressed by

$$\Omega_\chi h^2 \simeq Cte. \frac{T_0^3}{M_P \langle \sigma v \rangle}, \quad (4.5)$$

where  $\sigma$  is the cross section corresponding to the annihilation of pairs of two WIMPs in standard model particles,  $v$  is the relative velocity between the two WIMPs in the center of mass system,  $T_0$  is the current bottom temperature of radiation,  $M_P$  is the Planck mass and  $\langle \dots \rangle$  indicates thermal average.

The best motivated WIMP candidate is the Lightest Superparticle-LSP, of the supersymmetric models [142] [143]. The different possibilities of LSPs commonly considered are sneutrinos and neutralinos. Sneutrinos have large cross sections, so to be dark matter they would have to have unacceptably large masses. The most popular WIMP is the neutralino [144] [145], since it can have the appropriate relic density in different parts of space and the expected masses for the neutralino are of the order of about 150 GeV.

A popular candidate for cold dark matter is offered by the supersymmetric theory (SUSY) [146]. Supersymmetry is a symmetry between fermions and bosons. It was proposed to attack the hierarchy problem by doubling the number of particles so that each Standard Model fermion has a supersymmetric boson partner and each boson a fermionic partner. Also in the framework of SUSY the three gauge couplings, strong, weak and electromagnetic, can be unified to a value on scales of  $\sim 10^{16}$  GeV. The lightest supersymmetric particle, the neutralino, ( $X$ ) is the most viable candidate in this framework: In the supersymmetric model, the superpartners of the four neu-

tral Standard Model bosons (the bino, the wino, and two neutral Higgsinos) mix into 4 physical particles known as neutralinos. This mixture can be written as the superposition

$$X = \alpha B + \beta W^0 + \gamma H_1 + H_2 \quad (4.6)$$

where the binos ( $B$ ) and the winos  $W^0$  are known as gauginos and  $H_1$  and  $H_2$  are Higgsinos.

To obtain the lightest eigenstate that gives us the neutralino of the previous superposition equation, we must diagonalize the mass matrix (in the basis  $\{B, W^0, H_1, H_2\}$ )

$$\begin{pmatrix} M_2 & 0 & -M_Z \cos \beta \sin \theta_W & M_Z \sin \beta \sin \theta_W \\ 0 & M_1 & M_Z \cos \beta \cos \theta_W & -M_Z \sin \beta \cos \theta_W \\ -M_Z \cos \beta \sin \theta_W & M_Z \cos \beta \cos \theta_W & 0 & -\mu \\ M_Z \sin \beta \sin \theta_W & M_Z \sin \beta \cos \theta_W & -\mu & 0 \end{pmatrix}. \quad (4.7)$$

#### 4.4.2 Dark matter in Kaluza-Klein theory

Extra dimensional theories may provide viable candidates for dark matter. Let us consider only one extra dimension (4+1 dimensions) “ $y$ ”, and denote the Lagrangian density  $\mathcal{L}$  for a non-massive scalar field  $\Phi \equiv \Phi(x_\mu, y)$  where  $\mu = 0, 1, 2, 3$  as

$$\mathcal{L} = -\frac{1}{2} \partial_A \Phi \partial^A \Phi, \quad (4.8)$$

where  $A = 0, 1, 2, 3, 4$ .

The fifth dimension “ $y$ ” is compacted (it must be in a "compacted" form since we don't usually see manifestations of this dimension in our 4D universe) into a circle of radius  $R$  called the radius of compactification. At a scale  $M \gg R$ , the effect of this extra dimension is not apparent. Since compactification is on a circle, “ $y$ ” is periodic such that  $y \rightarrow y + 2\pi R$  ( $\Phi(x, y) = \Phi(x, y + 2\pi R)$ ). Therefore, we have

$$\Phi = \sum_{n=-\infty}^{\infty} \phi_n(x) \exp\left(\frac{iny}{R}\right), \quad (4.9)$$

(with  $\phi_*(x) = \phi_{-n}(x)$ ). So from (4.8) and (4.9) we get

$$\mathcal{L} = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \left[ \partial_\mu \phi_n \partial^\mu \phi_m - \frac{nm}{R^2} \phi_n \phi_m \right] \exp\left[\frac{i(n+m)y}{R}\right]. \quad (4.10)$$

The action is given by

$$S = \int d^4x \int_0^{2\pi R} dy \mathcal{L}. \quad (4.11)$$

Integrating over the extra dimension (4.10) takes the form

$$S = -\frac{1}{2} \int d^4x \partial_\mu \psi_0 \partial^\mu \psi_0 - \int d^4x \sum_{k=1}^{\infty} \left( \partial_\mu \psi_k \partial^\mu \psi_k + \frac{k^2}{R^2} \psi_k \psi_k^* \right), \quad (4.12)$$

where  $\psi_n = \sqrt{2\pi R} \phi_n$ . From (4.12), we obtain, for a non-massive 5D scalar field (after compactification of the extra dimension), a zero mode  $\psi_0$  as a real scalar field and an infinite number of complex massive scalar fields. The masses of these modes are given by  $m_k = k/R$ . These modes are known as Kaluza-Klein modes (or Kaluza-Klein towers). The quantum number  $k$  is called the Kaluza-Klein number and corresponds to the quantized momentum  $p_5$  in the compactified dimension. From Lorentz invariance in 5D, we have the relation

$$E^2 = \mathbf{p}^2 + p_5^2 = \mathbf{p}^2 + m_k^2. \quad (4.13)$$

The number  $k$  is conserved. This could give us a light and stable Kaluza-Klein particle (LKP).

According to the universal model of extra dimensions (UED) all Standard Model particles can propagate into an extra dimension and every SM particle has a Kaluza-Klein tower. The proposed dark matter candidate may be the LKP, which is the first KK partner of the gauge boson of hypercharge [147].

#### 4.4.3 The scalar singlet as dark matter

This model was first proposed by Silveira and Zee in [148] and explored in more detail in [149] [150]. The most general form for the potential of the scalar sector adding a real scalar singlet to the Standard Model is as follows:

$$V = \frac{m^2}{2} H^\dagger H + \frac{\lambda}{4} (H^\dagger H)^2 + \frac{\delta_1}{2} H^\dagger H S + \frac{\delta_2}{2} H^\dagger H S^2 + \frac{\delta_2 m^2}{2\lambda} S + \frac{\kappa_2}{2} S^2 + \frac{\kappa_3}{3} S^3 + \frac{\kappa_4}{4} S^4, \quad (4.14)$$

and the Lagrangian density of the model is given by

$$\mathcal{L} = \mathcal{L}_{SM} + \frac{1}{2} \partial_\mu S \partial^\mu S - \frac{\delta_1}{2} H^\dagger H S - \frac{\delta_2}{2} H^\dagger H S^2 - \frac{\delta_2 m^2}{2\lambda} S - \frac{\kappa_2}{2} S^2 - \frac{\kappa_3}{3} S^3 - \frac{\kappa_4}{4} S^4, \quad (4.15)$$

where  $\mathcal{L}_{SM}$  is the Lagrangian density of the Standard Model [7–12],  $H$  is the Higgs doublet SM and  $S$  is the real scalar singlet. For  $S$  to be a candidate for dark matter, it must be stable and have no interaction with fermions. For this purpose, a  $Z_2$  symmetry is imposed on the potential, so that  $S$  and  $L$  become:  $S \rightarrow -S$  and  $L \rightarrow -L$ . The imposition of this symmetry ensures that the coefficients of the odd powers of  $S$  in (4.14) are zero ( $\delta_1 = \kappa_3 = 0$ ), which means that there are no vertices where a number intervenes odd number of singlet fields. Considering also that  $s$  does not generate any vev, it is ensured that  $S$  can be a dark matter candidate if  $Z_2$  is a good symmetry. Adding  $Z_2$  to the potential we have:

$$V = \frac{m^2}{2} H^\dagger H + \frac{\lambda}{4} (H^\dagger H)^2 + \frac{\delta_2}{2} H^\dagger H S^2 + \frac{\delta_2 m^2}{2\lambda} S + \frac{\kappa_2}{2} S^2 + \frac{\kappa_4}{4} S^4, \quad (4.16)$$

Now with

$$H = \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix}, \quad (4.17)$$

where  $h$  is the physical Higgs field and  $v$ , the vev of the scalar field  $H$  is defined by  $m$ ,  $\lambda$  and  $\sqrt{-\frac{2m^2}{\lambda}}$ . After electroweak symmetry breaking, the term  $H^\dagger H S^2$  becomes

$$\frac{\delta_2}{2} H^\dagger H S^2 = \frac{\delta_2}{2} \begin{pmatrix} 0 & \frac{v+h}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} S^2 = \frac{\delta_2}{2} \left( \frac{v^2 S^2}{2} + v h S^2 + \frac{h^2 S^2}{2} \right). \quad (4.18)$$

After electroweak symmetry breaking the scalar mass terms can be written as

$$V_{masa} = -\frac{1}{2} (m_h^2 + m_s^2), \quad (4.19)$$

where

$$m_h^2 = -m^2 + \frac{\lambda v^2}{2} \quad (4.20)$$

and

$$m_s^2 = \kappa_2 + \frac{\delta_2 v^2}{2}. \quad (4.21)$$

The term  $\delta_2 H^\dagger H S^2$  shows us the interaction between the two scalar fields and the physical Higgs field and  $\lambda = \delta_2 v/2$  is the coupling between the two scalars and the Higgs. With this coupling it is possible to calculate the scattering cross section  $\sigma$  of the scalar  $S$  of a nucleon and annihilation cross section  $\Gamma$  of  $S$ , where the two scalars annihilate via the Higgs to a fermion-antifermion pair  $f\bar{f}$  or to  $W^+W^-$ ,  $ZZ$  or  $hh$ . We require  $\sigma$  to calculate direct detection range and  $\Gamma$  for relic density calculations.

### Higgs portals

Dark matter can couple with particles of Standard Model through a Higgs boson [151] [152]. The scalar singlet case is the simplest version of this proposal. Abdullah et. to the. provides in [153] the following classification, in addition to a more extensive description.

- The dark matter particle is a scalar singlet under the gauge group of the Standard Model, and is coupled through a quartic interaction with the Higgs. His phenomenology has been extensively studied.
- The dark matter particle is a fermionic singlet under the gauge symmetries of the Standard Model, which couples to a scalar boson which mixes with the Higgs.
- The dark matter candidate can be a mixture of an electroweak singlet and doublet, just like in MSSM where it has a bino and a Higgsino component.

### 4.4.4 Dark matter from an inert doublet

In the Inert Doublet model, proposed by Ma et. to the. [154] [155], an additional scalar is added to the Standard Model. In the resulting frame with two Higgs doublets ( $H_1$  y  $H_2$ ), the additional scalar doublet does not generate any vev and does not couple to the Standard Model fermions. A  $Z_2$  symmetry is imposed, so under this symmetry,  $SM \rightarrow SM$  and the additional scalar doublet  $H_2 \rightarrow -H_2$ . All SM fields are even under  $Z_2$ . Since the  $Z_2$  symmetry ensures that the  $H_2$  doublet does not couple with matter, the neutral components of this doublet are stable and the lighter component may be a candidate for dark matter. After electroweak symmetry breaking

$$\langle H_1 \rangle = \begin{pmatrix} 0 \\ \langle H_1^0 \rangle \end{pmatrix}, \quad (4.22)$$

and since  $Z_2$  must be preserved in the ground state, it does not generate any vev for  $H_2$ . we have therefore

$$\langle H_2 \rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (4.23)$$



At the minimum  $H_1$  and  $H_2$  are written in terms of the physical scalar fields  $h_1, h_2^0, A^0$  and  $h^\pm$  as

$$H_1 = \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} \quad (4.24)$$

and

$$H_2 = \begin{pmatrix} h_2^\pm \\ \frac{h_2^0 + iA^0}{\sqrt{2}} \end{pmatrix} \quad (4.25)$$

where  $v = 246$  GeV is the Higgs vev. In this model we have four new particles, two charged scalars ( $h_2^\pm$ ) and two neutral scalars ( $h_2^0, A^0$ ), where any of these last two can be the dark matter candidate. The Lagrangian density of this model can be written as:

$$\mathcal{L} = \mathcal{L}_{SM} + \mathcal{L}_{DM}, \quad (4.26)$$

where

$$\begin{aligned} \mathcal{L}_{DM} = & (D_\mu H_2)^\dagger D_\mu H_2 - \mu_2 (H_2^\dagger H_2) - \rho_2 (H_2^\dagger H_2)^2 - \\ & \lambda_1 (H_1^\dagger H_1) (H_2^\dagger H_2) \\ & - \lambda_2 (H_2^\dagger H_1) (H_1^\dagger H_2) - \lambda_3 \left[ (H_2^\dagger H_1)^2 + h.c. \right]. \end{aligned} \quad (4.27)$$

After the spontaneous breakup of  $SU(2) \times U_Y(1)$ , the masses of the new particles are:

$$m_{h_2^\pm} = \sqrt{\mu_2^2 + \frac{1}{2}\lambda_1 v^2} \quad (4.28)$$

$$m_{h_2^0} = \sqrt{\mu_2^2 + \lambda_{L_1} v^2} \quad (4.29)$$

$$m_{A^0} = \sqrt{\mu_2^2 + \lambda_{L_2} v^2}, \quad (4.30)$$

where

$$\lambda_{L_1, L_2} = \frac{1}{2} (\lambda_1 + \lambda_2 \pm 2\lambda_3). \quad (4.31)$$

The mass of the Higgs is given by

$$m_h^2 = 2\rho_1 v^2, \quad (4.32)$$

where  $\rho_1$  is the quantum coupling coefficient of SM Higgs. Bounds can be imposed on the parameters  $m_{h_2^\pm}, m_{h_2^0}, m_{A^0}, \lambda_{L_1}, \lambda_{L_2}, \rho_2$ , etc. taking into account the following:

1. Vacuum stability. The potential at  $\mathcal{L}_{DM}$  must have a lower bound so that its minimum is not infinitely negative. This condition requires

$$\rho_1, \rho_2 > 0, \quad (4.33)$$

$$\lambda_{L_1}, \lambda_{L_2} > -2\sqrt{\rho_1 \rho_2}, \quad (4.34)$$

$$\lambda_1 > -2\sqrt{\rho_1 \rho_2}. \quad (4.35)$$

2. Unitarity. In order for the model to remain within the perturbation limits, the value of the parameters must be  $< 4\pi$ .

3. LEP bound on the decay width of the Z boson. The LEP bound requires that

$$m_{h_2^0} + m_{A^0} > m_Z. \quad (4.36)$$

Furthermore, since the scalars ( $h_2^0$  or  $A_0$ ) are dark matter candidates, the parameter space can be bounded by comparing the relic densities with the data obtained by WMAP and Planck and the effective sections scattering with the results obtained from experiments of direct detection for dark matter.

#### 4.4.5 Axions

The existence of the axion was first postulated to solve the CP problem of strong interactions [156] [157]. These particles are pseudo Goldstone bosons associated with the spontaneous breaking of the Peccei-Quinn (PQ)  $U(1)$  symmetry, which occurs on a scale  $f_a$  [158] [159]. Axions are mostly produced by non-thermal mechanisms, even though they are extremely light and their nature is non-relativistic. For this reason they are candidates to contribute to cold dark matter. One of the production mechanisms is through coherent oscillations of the axion field. This mechanism produces the following relic axion density [160]

$$\Omega_a h^2 = C_a \left( \frac{f_a}{10^{12} \text{GeV}} \right)^{1.175} \theta_i^2, \quad (4.37)$$

where  $C_a$  is a constant between 0.5 and 10 and  $\theta_i \sim \mathcal{O}(1)$  is the initial angle of the oscillations. In addition to non-thermal mechanisms, axions can also be produced thermally [161].

#### 4.4.6 Hot dark matter

The most popular candidates for hot dark matter are neutrinos [162]. Neutrinos can be produced or destroyed in the early Universe by the reaction

$$\gamma + \gamma \rightleftharpoons \nu + \bar{\nu} \rightleftharpoons e^+ + e^-, \quad (4.38)$$

in thermal equilibrium, neutrinos also interacted with matter through the reactions

$$\nu + n \rightleftharpoons e^- + p, \quad (4.39)$$

$$\bar{\nu} + p \rightleftharpoons e^+ + n. \quad (4.40)$$

These interactions are weak, so the effective section is of the order of

$$\sigma \sim G_F^2 T^5, \quad (4.41)$$

where  $G_F$  is the Fermi constant, which satisfies  $G_F/(\hbar c)^3 = 1.16 \times 10^{-5}$  GeV and  $E_\nu$  is the energy of

the neutrino. At very high temperatures, the density number  $n_\nu \sim T^3$  so the annihilation rate is given by:

$$n_\nu \langle \sigma \nu \rangle \sim G_F^2 T^5. \quad (4.42)$$

The expansion rate of the universe at high temperatures can be estimated from the Hubble parameter  $H = \dot{a}/a$ , where  $a$  is the cosmological scale factor, as:

$$\frac{1}{a} \frac{da}{dt} = \sqrt{\frac{8\pi G}{3} g(T) \frac{\pi^2 (kT)^4}{30 (\hbar c)^3}} \sim T^2, \quad (4.43)$$

where,  $g(T)$  is the effective number of degrees of freedom of the spin in thermal equilibrium. Therefore, the rate of annihilation is greater than the rate of expansion in thermal equilibrium.

Neutrinos with a mass greater than 1 MeV will begin to annihilate before decoupling, being in equilibrium, so their density will be exponentially suppressed. Because of this, calculations of the number density for light ( $m_\nu \lesssim 1$  MeV) and heavy ( $m \gtrsim 1$  MeV) neutrinos differ greatly.

The light neutrino number density can be expressed as:

$$\rho_\nu = m_\nu Y_\nu n_\gamma, \quad (4.44)$$

where  $Y = n_\nu/n_\gamma$  is the density of  $\nu$ 's relative to the photon density, which is currently 411 photons per  $cm^3$ . In a universe in adiabatic expansion  $Y_\nu = 3/11$ . This suppression is a result of the  $e^+e^-$  annihilation that occurs after neutrino decoupling. The matter component is required to be bounded by

$$\Omega h^2 < 0.3. \quad (4.45)$$

From here we obtain a strong bound for neutrinos,

$$m_{tot} = \sum_{\nu} m_\nu \lesssim 28eV, \quad (4.46)$$

where the sum runs over the mass eigenstates of the neutrino. It is possible to put tighter limits on the light neutrino mass density  $\Omega_\nu h^2$ , and thus on the neutrino mass, based on the spectrum of Lyman- $\alpha$ ,  $m_{tot} < 5.5$  eV. Adding observational bounds of CMB (cosmic background radiation) and galaxy clusters it is possible to lower this limit to  $< 2.4$  eV. Focusing on the favored  $\Lambda$ CDM (Cold Dark Matter Model- $\Lambda$ ), the bound for the neutrino mass is  $m_{tot} < 1.8eV$  for  $\Omega < 0.5$ . Including more bounds such as the ‘‘HST Key project’’, type Ia supernova data, and big bang nucleosynthesis (BBN) the limit may be  $m_e < 10.3$  eV [163].

The calculation of the relic density for heavy neutrinos is determined by the cooling of the neutrino annihilations, which occurs at  $T \lesssim m_\nu$ , after the annihilations

have begun to reduce the neutrino density. The annihilation range is given by [164]:

$$\langle \sigma \nu \rangle n_\nu \sim \frac{m_\nu^2}{m_Z^4} (m_\nu T)^{3/2} \exp\left(-\frac{m_\nu}{T}\right), \quad (4.47)$$

where it is assumed that annihilation is dominated by  $\nu\bar{\nu} \rightarrow f\bar{f}$  via  $Z$  boson exchange and that  $\langle \sigma \nu \rangle \sim \frac{m_\nu^2}{m_Z^4}$ . When the annihilation rate becomes slower than the expansion rate of the universe, the relative abundance of neutrinos becomes fixed.

Based on the invisible lepton width of the  $Z$  boson, experiments at LEP (Large Electron-Positron Collider) have determined that the number of neutrinos is  $N_\nu = 2.9841 \pm 0.0083$  and therefore LEP excludes additional neutrinos with masses of  $m_\nu \lesssim 45$  GeV. Combining this with the limits provided by calculations for relic Dirac neutrino densities, we get that the mass density for ordinary heavy neutrinos is small,  $\Omega_\nu h^2 < 0.001$  for masses between  $m_\nu \sim 45$  GeV and  $\sim 100$  TeV. Laboratory bounds exclude neutrinos with masses between 10 GeV and 4.7 TeV. This rules out the possibility of neutrinos as dark matter based on an asymmetry between  $\nu$  and  $\bar{\nu}$ . Majorana neutrinos are excluded as dark matter since  $\Omega_\nu h_0^2 < 0.001$  for  $m_\nu > 45$  GeV.

Cosmological and observational data imply that the cosmological energy density for all light, weakly interacting neutrinos is restricted to the range

$$0.0005 \lesssim \Omega_\nu h^2 \lesssim 0.09. \quad (4.48)$$

This restriction indicates that ordinary weakly interacting neutrinos are not a dominant component of dark matter. Considering upright neutrinos gives us new possibilities. These must have decoupled before the left neutrinos, since their interactions are weaker. For a suitable large scale of right neutrino interactions, their masses may be of the order of up to a few KeV [165]. These neutrinos may be good candidates for dark matter, however, the viable mass range for galaxy formation is very restricted [166].

#### 4.4.7 MACHOs

The baryons that make up dark matter could be found in different forms. The most plausible candidates for baryonic dark matter could be Jupiter-type planets or brown dwarfs (which are stars with masses less than  $0.08M_\odot$ ). These objects are called massive compact halo objects (MACHOs). Their pressure is not enough for them to support the combustion of hydrogen, so their only source of light radiation is the gravitational energy that they lose during their slow combustion. If a MACHO passes in front of a distant star, it will act as a gravitational lens. However, MACHOs have already been detected between Earth and the Large Magellanic Cloud [167]. Recent

results from the MACHO Collaboration indicate the existence of many of these objects with masses around 0.5 solar masses, which may correspond to 20% of the section of the halo made up of dark matter [168].

#### 4.4.8 Asymmetric dark matter

The models of Asymmetric dark matter (ADM) [169] are based on the hypothesis that the current dark matter abundance has the same origin as ordinary matter, an asymmetry between particle and antiparticle density.

The simplest model of ADM consists of a stable particle and its antiparticle, with one of them constituting the dark matter present. Importantly, like ordinary dark matter, the dark sector may be made up of a more complex set of particles. The annihilation of dark matter, converted to radiation, could be discovered through BBN or CMB observations, or perhaps the dark particles annihilated into Standard Model particles through intermediate states.

The potential richness of the dark sector suggests a great phenomenology in cosmology, astrophysics, direct sensing, and accelerator experiments. Although not required, most ADM models have dark matter particles in the few GeV regime, suggesting that the microphysics behind the origin of the dark sector mass is related to the Standard Model scales such as the electroweak scale or the QCD scale. For example, the mirror model, where the mass of dark matter is exactly the same as the QCD scale that fixes the mass of the proton.

#### 4.4.9 Inelastic dark matter

An annual modulation in the range of events consistent with a relic of WIMPs has been reported by the DAMA/Nal direct dark matter detection experiment [170]. However, CDMS (Cryogenic Dark Matter Search) Ge excludes most of the region preferred by DAMA. It has been proposed that if the dark matter disperses by making an energy transition to a slightly heavier state ( $\delta m = 100$  KeV), the experiments no longer conflict.

The inelastic dark matter model assumes that a dark excited state  $\chi^*$  exists together with a dark matter  $\chi$  with a mass  $\delta$ . The inelastic scattering of dark matter by the nucleus can be expressed as  $N_\chi \leftarrow N_{\chi^*}$ . The dark matter particle  $\chi$  scatters inelastically from the nucleus of mass  $m_N$ , satisfying

$$\delta < \frac{\beta^2 m_\chi m_N}{2(m_\chi + m_N)}. \quad (4.49)$$

Inelastic dark matter causes changes in the kinematics. For CDMS, a greater suppression of events would occur than would occur in DAMA. Dark matter models such as the supersymmetric neutralino candidate or the complex scalar sneutrino can provide inelastic dark matter candidates.

## 5 Conclusions

A review of the concepts of mathematics is made, as well as the basic principles that support the gravitational theory. Subsequently, we present an exhaustive derivation of the Einstein field equations from a Lagrangian density  $\mathcal{L}_G$ , showing the introduction of the tensor  $T_{\mu\nu}$  for any material field in the equations of field considering a complete system, that is, where the energetic part of the matter  $T_G^\rho$  is also taken into account.

We present an alternative justification of the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, as well as the tensor equation that represents the Universe as a perfect fluid, in order to obtain the dynamic Friedmann equations and their solutions to the different models of the Universe. Universe with the different components of matter considered here.

We have studied from different perspectives the cosmological parameters related to the ordinary matter, dark matter and dark energy of the Universe.

There is compelling evidence for the existence of dark matter. Although our understanding of its nature and distribution is still incomplete, many independent observations suggest that about 30% of the total energy density of the Universe is made of some sort of non-baryonic dark matter.

The dark matter problem is not only relevant to astrophysicists but also to the particle and high-energy physics community. In fact, some of the best dark matter candidates come from possible extensions of the Standard Model of particle physics. There is certainly no shortage of particle dark matter candidates found in such models. Among those proposed in literature, we have focused on the dark matter particles found in models of supersymmetry (the lightest neutralino) and models with Universal Extra Dimensions (Kaluza-Klein dark matter). Although many simple models of supersymmetry, extra dimensions or other scenarios are widely discussed by the particle and astroparticle communities, the phenomenology of the actual physical theory could be more rich and complex. Collider experiments are probing significant regions of the parameter space of these hypothetical particles. Conversely, a positive astrophysical detection of dark matter would provide invaluable information regarding the physics “beyond the Standard Model”.

## Appendix A: Transformation of the affine connection

From equation (2.8), passing to any other system  $x'^\mu$ , we will find that:

$$\Gamma'_{\mu\nu}{}^\lambda = \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\tau}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma_{\tau\sigma}^\rho + \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}. \quad (5.1)$$

The first term on the right is what would be expected if  $\Gamma_{\mu\nu}^\lambda$  were a tensor, however, the second term indicates that  $\Gamma_{\mu\nu}^\lambda$  it is not a tensor.

On the other hand, from the transformation law for the metric tensor  $g'_{\mu\nu} = g_{\rho\sigma} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu}$ , note that:

$$\frac{\partial g'_{\mu\nu}}{\partial x'^\kappa} = \frac{\partial g_{\rho\sigma}}{\partial x^\tau} \frac{\partial x^\tau}{\partial x'^\kappa} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} + g_{\rho\sigma} \frac{\partial^2 x^\rho}{\partial x'^\kappa \partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} + g_{\rho\sigma} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial^2 x^\sigma}{\partial x'^\kappa \partial x'^\nu},$$

which implies that

$$\frac{\partial g'_{\kappa\nu}}{\partial x'^\mu} + \frac{\partial g'_{\kappa\mu}}{\partial x'^\nu} - \frac{\partial g'_{\mu\nu}}{\partial x'^\kappa} = \frac{\partial x^\tau}{\partial x'^\kappa} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \left( \frac{\partial g_{\sigma\tau}}{\partial x^\rho} + \frac{\partial g_{\rho\tau}}{\partial x^\sigma} - \frac{\partial g_{\rho\sigma}}{\partial x^\tau} \right) + 2g_{\rho\sigma} \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} \frac{\partial x^\sigma}{\partial x'^\kappa},$$

or

$$\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}' = \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\tau}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \left\{ \begin{matrix} \rho \\ \tau\sigma \end{matrix} \right\} + \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}, \quad (5.2)$$

with

$$\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} = \frac{1}{2} g^{\lambda\kappa} \left( \frac{\partial g_{\kappa\nu}}{\partial x^\mu} + \frac{\partial g_{\kappa\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\kappa} \right).$$

Subtracting equation (5.1) from (5.2), we see that  $\Gamma_{\mu\nu}^\lambda - \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}$  is a tensor, i. e.,

$$\left[ \Gamma_{\mu\nu}^\lambda - \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} \right]' = \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\tau}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \left[ \Gamma_{\tau\sigma}^\rho - \left\{ \begin{matrix} \rho \\ \tau\sigma \end{matrix} \right\} \right]. \quad (5.3)$$

Since  $\Gamma_{\mu\nu}^\lambda - \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}$  fades into a locally inertial coordinate system; that is, where there is no gravitational field, and since it is a tensor, then it must be annulled in all coordinate systems, i. e.,  $\Gamma_{\mu\nu}^\lambda - \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}$ .

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