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The Boltzmann equation and cosmic microvawe background

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Abstract

Observations on the anisotropies of the Cosmic Microwave Background (CMB) have become a fundamental tool in Cosmology. We present a brief description of the formalism necessary to understand the evolution of the anisotropies, and how their power spectrum gives us information about the evolution and composition of the universe.

Keywords: Cosmology, cosmic microwave background, Boltzmann equation..

La ecuación de Boltzmann y el fondo cósmico de microondas

Resumen

Las observaciones de las anisotropías de la Radiación del Fondo Cósmico se han convertido en una herramienta fundamental en cosmología. Aquí presentamos brevemente el formalismo necesario para entender la evolucion de las anisotropías, y como su espectro de potencias nos permite conocer la evolucion y composición del universo.

Palabras clave: Cosmología, radiacion del fondo cósmico, ecuación de Boltzmann..

1 Introduction

A brief treatment of anisotropies in the Cosmic Microwave Background (CMB) is presented in this article [1–5]. They can be completely described by the distribution function in the phase space of the photons. Ignoring the polarization of the cosmic microwave background (an effect of less than 5%), all the information is included in the distribution function $f\left(x^{\mu}, p^{\mu}\right)$.

The conjugate momentum is related to the proper momentum measured by a comoving observer, p^{μ}/a, such that $\mathbf{p} = cte$ along the photon paths, in the absence of metric perturbations.

2 Boltzmann equation for the Cosmic Microwave Background

Despite the metric perturbations and scattering produced by free electrons, the phase space distribution function of the CMB photons remains blackbody to high precision:

$$f\left(x^{\mu}, p^{\mu}\right) = f_{Planck}\left(\frac{E}{kT}\right) = f_{Planck}\left(\frac{p}{kT_0\left(1+\delta T\right)}\right), \tag{1}$$

where $T_0 = 2.728$ K is the current temperature of the CMB and $\delta T\left(x^{\mu}, \widehat{\mathbf{n}}\right)$ is the temperature variation at the x^{μ} position of the photons traveling in the direction $\widehat{\mathbf{n}}$.

The density in phase space can be calculated from the initial conditions in the early Universe through the

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Boltzmann equation [6–8]

$$\widehat{D}\left[f\right]=\frac{\partial f}{\partial x^{\mu}}\frac{\partial x^{\mu}}{\partial t}+\frac{\partial f}{\partial p^{\mu}}\frac{\partial p^{\mu}}{\partial t}=\widehat{C}\left[f\right], \tag{2}$$

where $\widehat{D}\left[f\right]$ is the Liouville operator and $\widehat{C}\left[f\right]$ is the collisional term containing the effects of non-relativistic elastic scattering between photons and electrons . Under the sole action of gravity, the collisional term vanishes, that is, the density of photons in phase space is conserved.

Since the photons obey the geodesic equation, it follows that

$$\frac{dp^{\lambda}}{d\lambda}+\Gamma^{\lambda}_{\mu\nu}p^{\mu}p^{\nu}=0. \tag{3}$$

In this case, the affine parameter λ can be chosen so that $p^{0}=\frac{dx^{0}}{d\lambda}$, so the moment will be given by $p^{i}/p^{0}=\frac{dx^{i}}{dt}$, and the geodesic equation can be expressed as

$$\frac{dp^{i}}{dt}=g^{i\nu}\left(\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}-\frac{\partial g_{\nu\alpha}}{\partial x^{\beta}}\right)\frac{p^{\alpha}p^{\beta}}{p^{0}}. \tag{4}$$

To solve this equation we must know the perturbations that affect the metric. We will introduce scalar perturbations into the metric and use Newton's conformal norm, where the metric takes the form

$$g_{00}=-\left[1+2\Psi\left(x^{i},t\right)\right], \tag{5}$$

$$g_{ij}=a^{2}\left[1+2\Phi\left(x^{i},t\right)\right]\gamma_{ij}, \tag{6}$$

where γ_{ij} is the spatial part to perturb the FLRW metric, Ψ can be interpreted as the Newtonian potential. Φ is the fractional perturbation in spatial curvature. If p can be neglected, it holds that

$$\Phi=-\Psi. \tag{7}$$

3 Gravitational redshift and time dilation

For the moment we will analyze the a-collision Boltzman equation, that is, considering $\widehat{C}\left[f\right]=0$. It is possible to rewrite this equation in terms of the energy p and the direction of propagation of the photons ς^{i} in a reference system that is locally orthogonal on the surfaces of $t=cte$. This will be just

$$\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x^{i}}\frac{dx^{i}}{dt}+\frac{\partial f}{\partial p}\frac{dp}{dt}+\frac{\partial f}{\partial\varsigma^{i}}\frac{d\varsigma^{i}}{dt}=0. \tag{8}$$

It should be noted that $d\varsigma^{i}/dt\neq0$ only in the presence of curvature K or perturbations, since otherwise the geodesic equations are straight lines. Therefore, since the factor $\partial f/\partial\varsigma^{i}$ is already first order in the fluctuations, this term can be neglected if the global curvature is $K=0$, which we assume from now on. The redshift term, dp/dt, is always important. We will now develop an expression for this factor. The energy and direction of propagation of photons can be expressed explicitly by

$$p^{2}=p^{i}p_{i}, \tag{9}$$

$$\varsigma^{i}=a\frac{p^{i}}{p}\left(1+\Phi\right), \tag{10}$$

it follows that

$$p^{0}=\left(1+\Psi\right)p. \tag{11}$$

By virtue of equations (9), (10), (11) and the components of the metric given by (6) and (7), the geodesic equation takes the form, to the first order in the fluctuations

$$\frac{1}{p}\frac{dp^{0}}{dt}=-\left[\frac{\partial\Psi}{\partial t}+\frac{1}{a}\frac{da}{dt}\left(1-\Psi\right)+\frac{\partial\Phi}{\partial t}+\frac{2\varsigma^{i}}{a}\frac{\partial\Psi}{\partial x^{i}}\right]. \tag{12}$$

From this relation we obtain an expression for the redshift of the photons

$$\frac{1}{p}\frac{dp}{dt}=-\left[\frac{1}{a}\frac{da}{dt}+\frac{\partial\Phi}{\partial t}+\frac{\varsigma^{i}}{a}\frac{\partial\Psi}{\partial x^{i}}\right] \tag{13}$$

or

$$\frac{1}{p}\frac{dp}{dt}=\frac{1}{p}\frac{dp^{0}}{dt}\left(1+\Psi\right)+\frac{\partial\Psi}{\partial t}+\frac{\partial\Psi}{\partial x^{i}}\frac{dx^{i}}{dt}. \tag{14}$$

The physical interpretation of this equation is as follows. The change in p will be given by

$$\frac{1}{p}\frac{dp}{dt}=\frac{1}{p}\frac{\partial p}{\partial t}+\frac{1}{p}\frac{\partial p}{\partial x^{i}}\frac{dx^{i}}{dt}. \tag{15}$$

Secondly

$$\frac{1}{p}\frac{\partial p}{\partial t}=-\frac{1}{a}\frac{da}{dt}-\frac{\partial\Phi}{\partial t}, \tag{16}$$

the first term of this equation corresponds to the cosmological redshift. But the presence of a spatial curvature perturbation also stretches space. The actual redshift factor corresponds to the coefficient of the spatial part of the metric, that is, $a\left(1+\Phi\right)$. Thus, the second term is due to the redshift caused by perturbations in the curvature.

As for the last term of equation (15), it can be associated with

$$\frac{1}{p}\frac{\partial p}{\partial x^{i}}\frac{dx^{i}}{dt}=\frac{\varsigma^{i}}{a}\frac{\partial\Psi}{\partial x^{i}}, \tag{17}$$

which corresponds to the gravitational shift of the photons falling and rising from the potential wells. The redshift depends on the gradient of the potential along the direction of propagation.

4 A-collisional brightness equation

Let us define the brightness function, Θ, by

$$4\Theta = \frac{1}{\pi^2\rho_\gamma}\int dpp^3 f - 1 = \frac{\delta\rho_\gamma}{\rho_\gamma}, \tag{18}$$

wheree ρ_γ is the photon energy density averaged over space and directions. So $\rho_\gamma \propto T^4$, $\delta\rho_\gamma \propto 4T^3\delta T$, therefore

$$\Theta\left(t, \overrightarrow{\mathbf{x}}, \varsigma\right) = \frac{\delta T}{T} \tag{19}$$

is the fractional change in blackbody temperature.

If equation (13) is used in the acollision Boltzmann expression (8) and multiplied by $p^3/\pi^2\rho_\gamma$, integration over p gives the equation of acollisional brightness

$$\frac{\partial\Theta}{\partial\eta} + \varsigma^i\frac{\partial}{\partial x^i}\left(\Theta+\Psi\right) + \frac{\partial\varsigma^i}{\partial\eta}\frac{\partial\Theta}{\partial\varsigma^i} + \frac{\partial\Phi}{\partial\eta} = 0. \tag{20}$$

Since the potential $\Psi\left(\eta, \overrightarrow{\mathbf{x}}\right)$ is not an explicit function of the angle ς and $\dot{x}^i = \varsigma^i$, this equation can be written as

$$\frac{d}{d\eta}\left[\Theta\left(\eta, \overrightarrow{\mathbf{x}}, \varsigma\right) + \Psi\left(\eta, \overrightarrow{\mathbf{x}}, \varsigma\right)\right] = \frac{\partial\Psi}{\partial\eta} - \frac{\partial\Phi}{\partial\eta}, \tag{21}$$

which shows that in a static potential, the quantity in brackets is conserved. In this way, the fluctuations in temperature are given simply by the differences in the potential

$$\Theta\left(\eta_0, \overrightarrow{\mathbf{x}}_0, \varsigma_0\right) = \Theta\left(\eta_1, \overrightarrow{\mathbf{x}}_1, \varsigma_1\right) + \left[\Psi\left(\eta_1, \overrightarrow{\mathbf{x}}_1\right) - \Psi\left(\eta_0, \overrightarrow{\mathbf{x}}_0\right)\right]. \tag{22}$$

5 The collision term

The collision term $\widehat{C}\left[f\right]$ must be considered, which will be given by the Compton scattering of photons by free electrons. This mechanism is primarily responsible for the thermalization of the cosmic microwave background (CMB) and governs the mutual evolution of these fluids up to the moment of decoupling.

We will not derive the expression for the collision integral here, we will only see the result and analyze its implications. In its derivation, the following hypotheses are assumed:

- We apply the limit of the Thomson scattering $\delta p/p \ll 1$ in the radiation rest system.
- The radiation is not polarized and stays that way.
- The density of e^- is low enough that the Pauli suppression terms are ignored.
- The electron velocity distribution is thermal around a given ensemble velocity $\overrightarrow{\mathbf{v}_b}$ determined by the baryons.

These approximations are generally valid in cosmology. As the anisotropies are small, the fluctuations induced in the polarization are of the order of 10% with respect to the temperature fluctuations. The polarization produces an effect of 5% in $\delta T/T$ so it can be neglected in a first approximation.

With these hypotheses, the equation of collisional brightness given by

$$\frac{\partial\Theta}{\partial\eta} + \varsigma^i\frac{\partial}{\partial x^i}\left(\Theta+\Psi\right) + \frac{\partial\varsigma^i}{\partial\eta}\frac{\partial\Theta}{\partial\varsigma^i} + \frac{\partial\Phi}{\partial\eta} = \frac{\partial\tau}{\partial\eta}\left(\Theta_0 - \Theta - \varsigma_i v_b^i + \frac{1}{16}\varsigma_i\varsigma_j\Pi_\varsigma^{ij}\right) \tag{23}$$

where

$$\Pi_\varsigma^{ij} = \frac{4}{\pi^2\rho_\gamma}\int dpp^3 f^{ij}\left(\eta, \overrightarrow{\mathbf{x}}, \varsigma\right),$$

with $f^{ij} = \frac{1}{4\pi}\int d\Omega\left(3\varsigma^i\varsigma^j - \delta^{ij}\right) f\left(\eta, \overrightarrow{\mathbf{x}}, \eta\right)$. Or, we also write

$$\Pi^{ij} = \frac{1}{4\pi}\int d\Omega\left(3\varsigma^i\varsigma^j - \delta^{ij}\right) 4\Theta\left(\eta, \overrightarrow{\mathbf{x}}, \varsigma\right), \tag{24}$$

where are the quadrupole moments of the energy distribution, and τ is the optical depth of the Thompson scattering defined by the scattering rate

$$\frac{\partial\tau}{\partial\eta} = a\chi_e n_e\sigma_T, \tag{25}$$

where

$$\sigma_T = \frac{8\pi\alpha^2}{3m_e^2} \tag{26}$$

is the effective section of Thompson.

In this way, we can summarize the main effects that tend to produce anisotropies in the CMB, which are:

1. Pure gravitational effects due to changes in Ψ and Φ, which are present in th acollision Boltzmann's equation.
2. Thompson scattering induces a Doppler effect in the overall system, represented by the $\overrightarrow{\mathbf{v}}_b$ dependent term in equation (24).
3. Thompson scattering isotropes the rest-system distribution of electrons, which induces anisotropies in the radiation system, which are given by the Π_ς^{ij} dependent term in equation (24).

6 The angular power spectrum

A map of the fluctuations in the temperature of the CMB allows numerous statistical analyses, the most important of which is the angular power spectrum. If the fluctuations in temperature are Gaussian with random phases, the angular power spectrum gives a complete description of the statistical properties of the CMB. Observations so far have clearly shown that fluctuations in the CMB do not strongly deviate from these assumptions. Thus, the observed power spectrum provides the primary point of contact between the observations and the cosmological parameters.

A map of the fluctuations in temperature $\Delta T\left(\widehat{\mathbf{n}}\right)/T$, where $\widehat{\mathbf{n}}$ is a unit vector, defined over the entire sky can be decomposed into spherical harmonics of the form

$$\frac{\Delta T}{T}\left(\widehat{\mathbf{n}}\right)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}a_{lm}Y_{lm}\left(\widehat{\mathbf{n}}\right) \tag{27}$$

with

$$a_{lm}=\int d\Omega\frac{\Delta T}{T}\left(\widehat{\mathbf{n}}\right)Y_{lm}^{*}\left(\widehat{\mathbf{n}}\right). \tag{28}$$

If the anisotropies in the CMB are Gaussian and with random phases, then the a_{lm} have a Gaussian distribution with $\langle a_{lm}\rangle=0$ and

$$\langle a_{lm}a_{l'm'}^{*}\rangle=\delta_{ll'}\delta_{mm'}C_{l}, \tag{29}$$

where δ is the Kronecker delta function and C_l is the power spectrum. C_l is the mean variance times l that would be found over a hypothetical ensemble of observers distributed in the Universe. The actual power spectrum calculated based on our sky, for a typical observer assumption is

$$C_{l}^{sky}=\frac{1}{2l+1}\sum_{m=-l}^{m=l}\left|a_{lm}\right|^{2}. \tag{30}$$

If we have CMB data without noise, and over the entire sky, then equation (28) can be evaluated exactly and (30) is an unbiased estimator of the true power spectrum, in the sense that $\left\langle C_{l}^{sky}\right\rangle=C_{l}$, when we average over the ensemble. Since we only measure $2l+1$ modes for each l, the estimator (30) has an intrinsic uncertainty (or cosmic variance) given by

$$\sigma_{l}=\sqrt{\frac{2}{2l+1}}C_{l}. \tag{31}$$

The real data, on the other hand, have noise and other sources of error, in addition, the data close to the galactic plane must be excluded from the analysis, so a complete sky map is not available. For these reasons, another method must be found to determine the power spectrum.

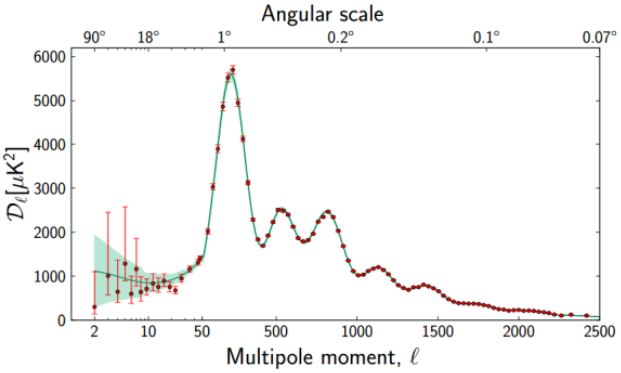

Figure 1: Compilation of power spectrum measurements of the CMB temperature made by various experiments [9–21].

7 Conclusions

In the analysis shown in this article, it can be seen that the study of anisotropies has a very important role in modern cosmology [22–28] because they provide information about the universe from the moment baryonic matter and radiation decouple until our days. In addition, with the help of the observations and the numerical codes, it is possible to infer about the characteristics that different cosmological models would have at different times when interpreting the generated spectra of the CMB.

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