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Relativistic theory of gravitation for Modified Newtonian Dynamics

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Abstract
Modified Newtonian dynamics (MOND) is a hypothesis that proposes a modification of Newton's law of universal gravitation to account for observed properties of galaxies. This theory is an alternative to the hypothesis of dark matter in terms of explaining why galaxies do not appear to obey the currently understood laws of physics. In this work, we present a relativistic theory of gravitation for MOND.

Keywords: Field equations, MOND.

Teoría relativista de gravitación para la dinámica Newtoniana modificada

Resumen
La dinámica newtoniana modificada (MOND) es una hipótesis que propone una modificación de la ley de gravitación universal de Newton para dar cuenta de las propiedades observadas de las galaxias. Esta teoría es una alternativa a la hipótesis de la materia oscura en términos de explicar por qué las galaxias no parecen obedecer las leyes de la física actualmente entendidas. En este trabajo presentamos una teoría relativista de la gravitación para MOND.

Palabras clave: Ecuaciones de campo, MOND.

1 Introduction

A viable explanation for the questions that have been raised about dark matter is to assume that gravity is not well understood. If we adopt the modified Newtonian dynamics (MOND) theory [1–6], this theory corrects Newton's laws for small accelerations [7] [8].

However, the construction of a Relativistic MOND theory has been generating difficulties and it is not clear how it can be reconciled with gravitational lensing measurements of the bending of light around galaxies. The main relativistic MOND theory was proposed by Jacob Bekenstein in 2004 [9]. Bekenstein's theory solves many of the problems of earlier attempts. A modified theory of gravity proposed by John Moffat [10] [11] and others nonsymmetric theories of gravitation [12–24].

2 Modified Newtonian dynamics theory (MOND)

MOND is a phenomenological modification to Newtonian dynamics, i. e., from Newton's second law. In the Newtonian theory of gravity, the gravitational acceleration in the static, spherically symmetric field of a point mass M at a distance R from the source can be written in the

$$\overrightarrow{\mathbf{a}} = -\nabla\Phi,$$

where G is Newton's constant of gravitation and $\Phi = -\frac{GM}{R}$ is the gravitational potential. The corresponding force acting on a test mass m is

$$\overrightarrow{\mathbf{F}} = m\frac{d^2\overrightarrow{\mathbf{r}}}{dt^2} = -m\nabla\Phi, \qquad (1)$$

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where the acceleration has been defined as the second derivative of the trajectory of the test particle with respect to time.

MOND is characterized by a scalar acceleration a_0, not a scalar distance, and its derivation from the Newtonian prediction is time dependent. To explain the anomalies of the rotation curves of spiral galaxies, Milgrom proposed a modification to equation (1) of the form

$$\mu\left(\frac{\|\overrightarrow{\mathbf{a}}\|}{a_0}\right)\overrightarrow{\mathbf{a}} = -\nabla\Phi. \tag{2}$$

Here $\mu(x) \approx x$, if $x \ll 1$ and $\mu(x) \to 1$ for $x \gg 1$. Note that MOND is not a complete theory, that is, it violates the principle of conservation of linear momentum.

3 AQUAL: Reformulation of the non-relativistic field of MOND

Empirically, when describing the motions of test particles, eg stars in the collective field of a galaxy, equation (2) is not quite correct. It can be verified that a pair of accelerated particles, one in the field of the other, according to equation (2) does not preserve the conservation of momentum. Therefore, the MOND formula itself is not a physical theory. Instead of this equation, we preserve the Galilean and rotational invariance of a Lagrangian density, $\mathscr{L}$, which gives us Poisson's equation, but we keep the linearity requirement of the differential equation in place. By virtue of the previous considerations, for the conservation of such a physical law (principle of conservation of momentum), we propose a first non-relativistic generalization of MOND. This theory, called AQUAL (for its acronym in English, A QUAdratic Lagrangian) is based on the Lagrangian density

$$\mathscr{L} = -\frac{a_0^2}{8\pi G} f\left(\frac{\|\nabla\Phi\|^2}{a_0^2}\right) - \rho\Phi, \tag{3}$$

where ρ is the mass density, $a_0 = 1\times10^{-8}\,\frac{\text{cm}}{s^2}$ is an acceleration scale introduced for dimensional consistency, and f is an arbitrary function dimensionless. In the Newtonian theory of gravitation (Poisson's equation) it corresponds to the change $f(y) = y$. From equation (3), we can find the equation of gravitational field

$$\nabla\cdot\left[\mu\left(\frac{\|\nabla\Phi\|}{a_0}\right)\nabla\Phi\right] = 4\pi G\rho, \tag{4}$$

where $\mu\left(\sqrt{y}\right) = \frac{df(y)}{dy}$. By virtue of the Lagrangian density, this theory has been called AQUAL.

For systems with spherical, cylindrical or planar symmetry, equation (4) can be integrated. With the usual definition of Newtonian gravitational acceleration, $\mathbf{a} = -\nabla\Phi$, the solution corresponds precisely to equation (2), MOND formula. This is no longer true for low symmetries. However, numerical integration reveals that equation (2) has an approximate degree of truth. The mentioned inaccuracy of the MOND equation for systems of a discrete collection of particles is in violation of the conservation laws.

Summarizing, when the parts of a system lacking high symmetry move with small accelerations on the scale a_0, the field $\nabla\Phi$ is calculated from the solution of the AQUAL equation, that is, the equation (4). AQUAL becomes the non-relativistic theory of the gravitational field, from which the relativistic formulation of the MOND paradigm is modeled, that is, the TeVeS theory.

4 Relativistic theory of gravitation

A relativistic MOND theory is essential if gravitational lensing for extragalactic and cosmological systems is to be understood without relying on dark matter. The fundamental principles on which the TeVeS relativistic MOND theory is based (the acronym for TeVeS means, for the acronym in English, Tensor-Vector-Scalar gravity) are: least action, relativistic invariance, equivalence, causality, and derivations of Newtonian gravity [4].

Fields and Actions

TeVeS is based on three dynamic gravitational fields: an Einsteinian metric $g_{\mu\nu}$ with a well-defined inverse $g^{\mu\nu}$, a time quadrivector U_μ, such that $g^{\alpha\beta}U_\alpha U_\beta = -1$, and a scalar field ϕ and a non-dynamic scalar field σ. The physical metric in TeVeS is obtained by extending the Einstein metric in space-time directions orthogonal to $U^\alpha = g^{\alpha\beta}U_\beta$ by a factor $\exp(-2\phi)$, while going back by the same factor in the direction parallel to U^α:

$$\begin{aligned}\widetilde{g}_{\mu\nu} = \exp(-2\phi)\left(g_{\mu\nu} + U_\mu U_\nu\right) - \exp(2\phi)\,U_\mu U_\nu = \\ \exp(-2\phi)\,g_{\mu\nu} - 2U_\mu U_\nu \sinh(2\phi)\,.\end{aligned} \tag{5}$$

So the inverse metric is

$$\widetilde{g}^{\mu\nu} = \exp(2\phi)\,g^{\mu\nu} + 2U^\mu U^\nu \sinh(2\phi)\,. \tag{6}$$

The TeVeS action can be written as

$$S_{TeVeS} = \int\left(\mathscr{L}_G + \mathscr{L}_s + \mathscr{L}_v + \mathscr{L}_m\right)d^4x. \tag{7}$$

The geometric part of the action is formed by the Ricci tensor $R_{\mu\nu}$ in terms of the metric tensor $g_{\mu\nu}$, that is

$$S_G = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,g^{\mu\nu}R_{\mu\nu}, \tag{8}$$

where $\sqrt{-g} = \sqrt{-\left|g_{\mu\nu}\right|}$.

In terms of two positive constant parameters, k and ℓ, the action for the pair of scalar fields is expressed as

$$S_s = -\frac{1}{2}\int d^4x\sqrt{-g} \left[\sigma^2 h^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu} + \frac{1}{2}G\frac{\sigma^4}{\ell^2}F\left(kG\sigma^2\right)\right], \tag{9}$$

where $h^{\mu\nu} = g^{\mu\nu} - U^\mu U^\nu$, F is a free function, ϕ is dimensionless, the terms of σ^2 have dimension G^{-1}. Consequently, k is a dimensionless constant, while ℓ is a constant length.

The action of the vector U^μ is given by the equation

$$S_v = -\frac{K}{32\pi G}\int d^4x\sqrt{-g}[g^{\alpha\beta}g^{\mu\nu}\left(\frac{\partial U_\alpha}{\partial x^\mu} - \frac{\partial U_\mu}{\partial x^\alpha}\right)\times \left(\frac{\partial U_\beta}{\partial x^\nu} - \frac{\partial U_\nu}{\partial x^\beta}\right) - 2\left(\frac{\lambda}{K}\right)\left(g^{\mu\nu}U_\mu U_\nu + 1\right)], \tag{10}$$

here λ is the space-time dependent Lagrange multiplier to fulfill the normalization of the equation $g^{\alpha\beta}U_\alpha U_\beta = -1$ while K is a dimensionless constant, since U^μ is dimensionless. So TeVeS has two dimensionless parameters, k and K, in addition to the dimensional constants G and ℓ.

According to the equivalence principle, the action for matter in TeVeS is obtained from transcribing the Lagrangian density for flat spacetime $\mathscr{L}\left(\eta_{\mu\nu}, f^\alpha, \partial_\mu f^\alpha, ...\right)$ times a Lagrangian density, for the material fields, using the following formula

$$S_m = \int d^4x\sqrt{-\tilde{g}}\mathscr{L}\left(\tilde{g}_{\mu\nu}, f^\alpha, \nabla_\mu f^\alpha, ...\right), \tag{11}$$

where $\sqrt{-\tilde{g}} = \exp\left(-2\phi\right)\sqrt{-g}$.

Field Equations

To find the field equations of the TeVeS gravitational theory, we vary the total action $S = S_G + S_s + S_v + S_m$ with respect to the fields $g^{\mu\nu}$, ϕ, σ and U_μ. For this, we take the variation of equation (6)

$$\begin{aligned} \delta\tilde{g}^{\mu\nu} = \exp\left(2\phi\right)\delta g^{\mu\nu} + \\ 2\sinh\left(2\phi\right)U_\rho\left(\delta g^{\rho\mu}U^\nu + \delta g^{\rho\nu}U^\mu\right) + \\ 2\sinh\left(2\phi\right)\left(U^\mu g^{\nu\rho} + U^\nu g^{\mu\rho}\right)\delta U_\rho + \\ 2\left[\exp\left(2\phi\right)g^{\mu\nu} + 2U^\mu U^\nu\cosh\left(2\phi\right)\right]\delta\phi. \end{aligned} \tag{12}$$

When we vary S_G with respect to $g^{\mu\nu}$, that is, we carry out a variation with respect to $\mathfrak{g}^{\mu\nu}$, we obtain [25]

$$\delta S = -\int\sqrt{-g}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\delta g_{\mu\nu}d^4x,$$

where we have used the equation

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}.$$

By virtue of the principle of gravitational action, we obtain the field equations

$$-\sqrt{-g}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) = -\sqrt{-g}G^{\mu\nu} = 0. \tag{13}$$

On the other hand, if we consider the total action S, and take its variation, equation (13) takes the form:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\times \left\{\tilde{T}_{\mu\nu} + \left[1 - \exp\left(-4\phi\right)\right]U^\rho\left(\tilde{T}_{\rho\mu}U_\nu + \tilde{T}_{\rho\nu}U_\beta\right) + \Delta_{\mu\nu}\right\} + \Theta_{\mu\nu} \tag{14}$$

where:

$$\Delta_{\mu\nu} \equiv \sigma^2\left[\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta} - \frac{1}{2}g^{\rho\eta}\frac{\partial\phi}{\partial x^\rho}\frac{\partial\phi}{\partial x^\eta}g_{\mu\nu} - U^\rho\frac{\partial\phi}{\partial x^\rho}\left(U_\mu\frac{\partial\phi}{\partial x^\nu} + U_\nu\frac{\partial\phi}{\partial x^\mu} - \frac{1}{2}U^\eta\frac{\partial\phi}{\partial x^\eta}g_{\mu\nu}\right)\right] -\frac{1}{4}G\frac{\sigma^4}{\ell^2}F\left(kG\sigma^2\right)g_{\mu\nu},$$

$$\Theta_{\mu\nu} \equiv K\left[g^{\rho\eta}\left(\frac{\partial U_\rho}{\partial x^\mu} - \frac{\partial U_\mu}{\partial x^\rho}\right)\left(\frac{\partial U_\eta}{\partial x^\nu} - \frac{\partial U_\nu}{\partial x^\eta}\right) - \frac{1}{4}g^{\alpha\beta}g^{\rho\eta}\left(\frac{\partial U_\alpha}{\partial x^\rho} - \frac{\partial U_\rho}{\partial x^\alpha}\right)\left(\frac{\partial U_\beta}{\partial x^\eta} - \frac{\partial U_\eta}{\partial x^\beta}\right)g_{\mu\nu}\right] - \lambda U_\mu U_\nu.$$

If we vary the action S_s, with respect to σ, we obtain a relationship between σ and $\frac{\partial\phi}{\partial x^\mu}$

$$-kG\sigma^2F - \frac{1}{2}\left(kG\sigma^2\right)^2\frac{dF\left(\mu\right)}{d\mu} = k\ell^2h^{\alpha\beta}\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta}, \tag{15}$$

where $\mu = kG\sigma^2$.

Next, to carry out the variation with respect to ϕ, it must be remembered that this quantity enters into action for material fields exclusively through $\widetilde{g}^{\mu\nu}$, so using equations (12) and take the variation of the action S_m, we found that

$$\nabla_\nu\left(\sigma^2 h^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\right)=\left\{g^{\mu\nu}+\left[1+\exp\left(-4\phi\right)\right]U^\mu U^\nu\right\}\widetilde{T}_{\mu\nu}. \tag{16}$$

Since equation (16) is an equation for ϕ only, with $\widetilde{T}_{\mu\nu}$ as the source. We define a function $\mu\left(y\right)$ of the form

$$-\mu F\left(\mu\right)-\frac{1}{2}\mu^2\frac{dF\left(\mu\right)}{d\mu}=y,$$

so, $kG\sigma^2=\mu\left(k\ell^2h^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\beta}\right)$. Consequently, equation (16) takes the form

$$\nabla_\nu\left[\mu\left(k\ell^2h^{\rho\eta}\frac{\partial\phi}{\partial x^\rho}\frac{\partial\phi}{\partial x^\eta}\right)h^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\right]=\left\{g^{\mu\nu}+\left[1+\exp\left(-4\phi\right)\right]U^\mu U^\nu\right\}\widetilde{T}_{\mu\nu}. \tag{17}$$

In this equation, in quasi-static situations, we can replace $h^{\mu\nu}$ by $g^{\mu\nu}$, so equation (17) has the same structure as equation (4).

The variation of S with respect to U_μ, and using equation (12), gives the vector equation

$$K\nabla_\beta\left(\nabla^\alpha U^\beta-\nabla^\beta U^\alpha\right)+\lambda U^\alpha+8\pi G\sigma^2U^\beta\frac{\partial\phi}{\partial x^\beta}g^{\alpha\rho}\frac{\partial\phi}{\partial x^\rho}=8\pi G\left[1-\exp\left(-4\phi\right)\right]g^{\alpha\mu}U^\beta\widetilde{T}_{\mu\beta}. \tag{18}$$

As already mentioned above, λ is a Lagrange multiplier. Therefore, by contraction of equation (18) with U_μ and substituting the result, we obtain

$$\begin{aligned}K\nabla_\beta\left(\nabla^\alpha U^\beta-\nabla^\beta U^\alpha\right)+KU^\alpha U_\rho\nabla_\beta\left(\nabla^\rho U^\beta-\nabla^\beta U^\rho\right)+8\pi G\sigma^2\left[U^\beta\frac{\partial\phi}{\partial x^\beta}g^{\alpha\lambda}\frac{\partial\phi}{\partial x^\lambda}+U^\alpha\left(U^\beta\frac{\partial\phi}{\partial x^\beta}\right)^2\right]\\=8\pi G\left[1-\exp\left(-4\phi\right)\right]\left(g^{\alpha\mu}U^\beta\widetilde{T}_{\mu\beta}+U^\alpha U^\beta U^\lambda\widetilde{T}_{\lambda\beta}\right).\end{aligned} \tag{19}$$

This equation has only three independent components, that is, taking into account that both sides of the equation are orthogonal to U_μ.

5 Concluding remarks

In the extragalactic region, where the Newtonian theory of gravitation has had an excellent description, the accelerations of the stars and gases are much larger than those generated by the Newtonian gravitational field of the visible matter in the system. This is the "mass loss" or "acceleration discrepancy" problem. It is elegant to infer the existence of dark matter in systems the size of dwarf spheroidal galaxies with mass $\sim 10^6 M_\odot$ for galaxy clusters in the $10^{13}M_\odot$ regime [26,27]. And again, galaxies and galaxy clusters are found by gravitational lensing. When interpreted by general relativity, gravitational lensing is large and anomalous, unless the presence of dark matter, in quantities, with distribution similar to dwarf galaxies, is assumed to explain the accelerations of stars and gases. But dark matter has not been detected at all, despite much experimental and observational effort.

References

[1] B. Berendzen, R. Hart, and D. Seeley. Man Descoveries the Galaxies, Columbia University Press, New York (1987).

[2] F. Zwicky. Morphological Astronomy. Springer, Berlin (1957).

[3] C. Will. Theory and experiment in Gravitational Physics. Cambridge University, Cambridge (1986).

[4] M. Milgrom. A modification of the Newtonian dynamics as a posible alternative to the hidden mass hypothesis, The Astrophysical Journal, 270 (1983) 365.

[5] M. Milgrom. A modification of the Newtonian dy-

namics: Implications for galaxies, The Astrophysical Journal, 270 (1983) 371.

[6] M. Milgrom. A modification of the Newtonian dynamics: Implications for galaxy systems, The Astrophysical Journal, 270 (1983) 384.

[7] M. Milgrom. MOND theoretical aspects, The Astrophysical Journal 571 (2002) 81.

[8] A. Vázquez-González y T. Matos. La materia oscura del Universo: retos y perspectivas, Revista Mexicana de Física E, 54 (2008) 193.

[9] J. D. Bekenstein. Relativistic gravitation theory for the MOND paradigm, Physical Review D, 70 (2004) 083509; Erratum-ibid.D71 (2005) 069901.

[10] J. W. Moffat. Nonsymmetric Gravitational Theory, Physical Letters B, 355 (1995) 447.

[11] J. W. Moffat. Scalar-Tensor-Vector Gravity Theory, Journal of Cosmology and Astrophysics, 0603 (2006) 004.

[12] R. G. Wishwakarma. A scale-invariant, Machian theory of gravitation and electrodynamics unified, International Journal of Geometric Methods in Modern Physics, 15(10) (2018) 1850178.

[13] S. Ragusa. Nonsymmetric theory of gravitation, Physical Review D, 56 (1997) 864.

[14] S. Ragusa. Conservation Laws in a Metric Nonsymmetric Theory of Gravitation, Brazilian Journal of Physics, 35 (2005) 1020.

[15] S. Ragusa. Nonexistence of regular stationary solution in a metric nonsymmetric theory of gravitation, Brazilian Journal of Physics, 38(3a) (2008).

[16] S. Ragusa y D. Bosquetti. Metric nonsymmetric theory of gravitation: the analogue of the theorem of Birkhoff, Brazilian Journal of Physics, 36 (2006) 576.

[17] S. Ragusa y Lucas C. Céleri. Interior Problem in a Nonsymmetric Theory of Gravitation, Brazilian Journal of Physics, 33 (2003) 821.

[18] M. Valenzuela. A relativistic theory of the field, Revista de Investigación de Física, 24(2) (2021) 72-79. Doi: https://doi.org/10.15381/rif.v24i2.14245

[19] M. Valenzuela. A relativistic theory of the field. II: Hamilton's principle and Bianchi's identities, Revista de Investigación de Física, 24(3) (2021) 12-24. Doi: https://doi.org/10.15381/rif.v24i3.14375

[20] M. Valenzuela. General relativistic theory of gravitation and electrodynamics, Revista de Investigación de Física, 25(1) (2022) 1-9. Doi: https://doi.org/10.15381/rif.v25i1.14972

[21] M. Valenzuela. Corrigendum: A relativistic theory of the field, Revista de Investigación de Física, 25(2) (2022) 35-36. Doi: https://doi.org/10.15381/rif.v25i2.22486

[22] M. Valenzuela. Corrigendum: A relativistic theory of the field II, Revista de Investigación de Física, 25(2) (2022) 37-38. Doi: https://doi.org/10.15381/rif.v25i2.22487

[23] M. Valenzuela. On the relativistic theory of the asymmetric field, Revista de Investigación de Física, 25(2) (2022) 21-34. Doi: https://doi.org/10.15381/rif.v25i2.21449

[24] M. Valenzuela. Corrigendum: On the relativistic theory of the asymmetric field, Revista de Investigación de Física, 25(3) 45-47 (2022). Doi: https://doi.org/10.15381/rif.v25i3.23719

[25] M. Valenzuela. Gravitation, cosmology and dark matter, Revista de Investigación de Física, 26(1) (2023) 35-68. Doi:https://doi.org/10.15381/rif.v26i1.22806

[26] V. Trimble. Existence and nature of Dark Matter in the Universe, Annual Review of Astronomy and Astrophysics, 25 (1987) 425.

[27] Jan S. Nilsson, Bo-Sture Skagerstam y B. Gustafsson. The Birth and Early Evolution of our Universe, Proceedings of Nobel Simposium #79, Gräftavallen, Östersund, Sweden, Physica Scripta, T36 (1991).