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Global symmetry breaking and topological defects

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Abstract
In this article, we present the fundamentals of Goldstone's theorem and the breaking of global symmetries. As well as one of the important aspects to take into account in the process of spontaneous symmetry breaking are the possible topological defects generated by the breaking. For this reason, we will expose the concept of topological defect and the calculation of the relic density of the axion, considered as a candidate to dark matter.

Keywords: Global symmetry breaking, topological defect, axion.

Ruptura de la simetría global y defectos topológicos

Resumen
En este artículo, presentamos los fundamentos del teorema de Goldstone y el rompimiento de las simetrías globales. Así como uno de los aspectos importantes a tener en cuenta en el proceso del rompimiento espontáneo de la simetría son los posibles defectos topológicos generados debido a esta ruptura. Por esta razón, expondremos el concepto de defecto topológico y el cálculo de la densidad reliquia del axión, considerado como un candidato para la materia oscura.

Palabras clave: Rompimiento global de la simetría, defecto topológico, axión.

1 Introduction

The dynamics of a system [1–4] involving a quantum field, $\Phi(x)$, is determined in terms of the action defined as

$$S=\int_{\Gamma} dtL=\int_{\Gamma} d^4x\mathscr{L}\left(\Phi\left(x\right),\partial_\mu\Phi\left(x\right)\right), \tag{1}$$

where $\mathscr{L}$ is the Lagrangian density and Γ a region of spacetime. Thus, any transformation that satisfies the condition

$$S\rightarrow S'=S, \tag{2}$$

is a symmetry transformation. The relations between symmetries and conservation laws is described by the Noether's theorem. For each symmetry transformation there is a current density, called the Noether current, which is conserved if the symmetry is exact. Each corresponding charge, denoted by the operator Q, satisfies the equation

$$\frac{dQ}{dt}=0 \tag{3}$$

for exact symmetries, this leads to conservation laws for that charge. In electrodynamics and quantum chromodynamics, local gauge symmetry is responsible for the conservation of color and electric charges.

Symmetry transformations are described in terms of groups. To each symmetry corresponds an abstract element of a symmetry group $\mathscr{G}$. This group is represented

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by a set of operators $\mathcal{G}$. These are like basis change operators on the Hilbert space of quantum fields

The Hilbert space is a complete, separable, infinite-dimensional unit vector space. It is also a space constituted by square-integrable (wave) functions $\psi(q)$, that is, whose integral over its domain of definition $\mathcal{D}$, $\int_{\mathcal{D}} \psi * (q)\, \psi(q)\, dq \equiv \|\psi(q)\|^2$, exists. The vector space thus constituted is usually denoted as $\mathcal{L}^2(\mathcal{D})$ and the above integral gives the square of the modulus of $\psi(q)$.

Quantum fields have bundles of energy and momentum, that is, particles. The state of a free particle with quadrimomentum p, mass m, satisfying $p^2 = m^2 > 0$, spin s, projection σ into the z axis with respect to a reference frame at rest, is created by applying the operator $a_\sigma^\dagger(p)$ to the empty state σ, of the system $|0\rangle$

$$|q, \tau\rangle = a_\sigma^\dagger(q)\,|0\rangle\,,$$

where p^0 is the relativistic energy of the particle $p^0 = +\sqrt{|\mathbf{p}|^2 + m^2}$. Thus, the states of a system of particles are built from the vacuum state, creating a Hilbert space, called a Fock space[1]. The creation operator is used to compute the inner product

$$\langle q, \tau\, |p, \sigma\rangle = \langle 0|\, a_\tau(q)\, a_\sigma^\dagger(p)\,|0\rangle\,, \tag{4}$$

where $a_\tau(q)$ is the annihilation operator that satisfies $a_\tau(q)\,|0\rangle = 0$. Commutation relations are generated by involving the creation and annihilation operators. In the case of a real scalar field ($s = 0$), the commutation relations are defined as

$$\begin{gathered} [a(q)\,, a(p)] = 0, \\ \left[a^\dagger(q)\,, a^\dagger(p)\right] = 0 \\ \left[a(q)\,, a^\dagger(p)\right] = 2p^0\delta^{(3)}(\mathbf{q} - \mathbf{p})\,. \end{gathered} \tag{5}$$

Then the inner product has the form

$$\langle q\, |p\rangle = 2p^0\delta^{(3)}(\mathbf{q} - \mathbf{p})\,. \tag{6}$$

When considering a system of particles represented by plane waves, that is, free particles, the wave function of each of them has the form $\exp(\pm ipx)$. Thus the real scalar field for the free particle is written in terms of a Fourier transform

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\mathbf{p}}{2p^0} \left[\exp(-ipx)\, a(p) + \exp(-ipx)\, a^\dagger(p)\right] \tag{7}$$

Therefore, the wavefunctions are determined by

$$\begin{aligned} \langle p|\,\phi(x)\,|0\rangle &= \tfrac{1}{(2\pi)^{3/2}} \exp(ipx)\,, \\ \langle 0|\,\phi(x)\,|p\rangle &= \tfrac{1}{(2\pi)^{3/2}} \exp(-ipx)\,. \end{aligned} \tag{8}$$

Symmetries can be classified into space-time and internal symmetries. Space-time symmetries transform the coordinates of a point in space-time and into the field in a particular way, depending on its class (scalar, vector or spin). Internal symmetries only transform the field. Both classifications can be divided into global and local symmetries. Thus, global space-time symmetries are associated with special relativity and local ones with general relativity; the global internal symmetries are associated with the approximate symmetries of flavor, isospin, among others, while the local ones with the color norm and electroweak symmetries.

The representation of each element g that belongs to the group of internal symmetries $\mathscr{G}$ is a linear operator $\mathscr{U}$. If the internal symmetry depends on the parameter θ, then the global and local transformations are given by

$$\begin{gathered} \phi(x) \longmapsto \phi'(x) = \mathscr{U}(g(\theta))\,\phi(x)\,, \\ \phi(x) \longmapsto \phi'(x) = \mathscr{U}(g(\theta(x)))\,\phi(x)\,, \end{gathered} \tag{9}$$

respectively.

2 Lagrangian Formalism

The variation of the action before the infinitesimal transformations

$$x'^\mu = x^\mu + \delta x^\mu, \tag{10}$$

$$\Phi'\left(x'\right) = \Phi(x) + \delta\Phi(x) \tag{11}$$

result

$$\delta S = \int \left[\delta\left(d^4x\right)\mathscr{L} + d^4x\delta\mathscr{L}\right], \tag{12}$$

where the variation of the hypervolume differential is given by

$$\delta\left(d^4x\right) = d^4x' - d^4x = d^4x\partial_\mu\left(\delta x^\mu\right) \tag{13}$$

and the variation of the Lagrangian density

$$\delta\mathscr{L} = \frac{\partial\mathscr{L}}{\partial\Phi}\delta\Phi + \frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\Phi\right)}\delta\left(\partial_\mu\Phi\right) \tag{14}$$

with $\delta(\partial_\mu\Phi) \neq \partial_\mu(\delta\Phi)$ given (10).

The total variation of the field is composed of

$$\delta\Phi(x) = \Phi'(x) - \Phi(x) + (\partial_\mu\Phi)\,\delta x^\mu, \tag{15}$$

where the first term represents the direct variation of the field $\delta_0\Phi = \Phi'(x) - \Phi(x)$ and the second term arises as a consequence of the variation in x^ρ. The variation of the derivative of the field is given by

$$\delta\left(\partial_\mu\Phi\right) = \partial'_\mu\Phi' - \partial_\mu\Phi.$$

Under the coordinate transformation, the right-hand side of this equation can be written as

$$(\partial'_\mu x^\nu)\partial_\nu\Phi' - \partial_\mu\Phi,$$

[1] The Fock space is the Hilbert space prepared as a direct sum of the tensor products of the Hilbert spaces for a particle.

or

$$\delta\left(\partial_\mu \Phi\right)=\left(\partial'_\mu x^\nu\right)\partial_\nu\left(\Phi+\delta\Phi\right)-\partial_\mu\Phi, \tag{16}$$

where $\partial'_\mu x^\nu$ is an element of the Jacobian matrix of the transformation $x'^\mu \longmapsto x^\mu$, which is given by

$$\partial'_\mu x^\nu=\delta_\mu^\nu-\partial_\mu\left(\delta x^\nu\right). \tag{17}$$

Thus, we get the variation

$$\delta\left(\partial_\mu\Phi\right)=\partial_\mu\left(\delta\Phi\right)-\partial_\mu\left(\delta x^\nu\right)\partial_\nu\Phi. \tag{18}$$

Therefore, the variation of the action is

$$\begin{aligned}\delta S=&-\int d^4x\left[\frac{\partial\mathscr{L}}{\partial\Phi}-\partial_\mu\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\Phi\right)}\right]\partial_\nu\Phi\delta x^\nu+\\&\int d^4x\left[\frac{\partial\mathscr{L}}{\partial\Phi}-\partial_\mu\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\Phi\right)}\right]\delta\Phi-\\&\int d^4x\partial_\mu\left[\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\Phi\right)}\left(\left(\partial_\nu\Phi\right)\delta x^\nu-\delta\Phi\right)-\delta x^\mu\mathscr{L}\right]\end{aligned} \tag{19}$$

Infinitesimal transformations represent symmetries, if and only if $\delta S=0$. Consequently, the equation of motion of the quantum field is obtained, called the Euler-Lagrange equation [5–15]

$$\frac{\partial\mathscr{L}}{\partial\Phi}-\partial_\mu\left[\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\Phi\right)}\right]=0. \tag{20}$$

Thus, the Noether current [5] is defined as

$$J^\mu\left(x\right)=\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\Phi\right)}\left(\left(\partial_\nu\Phi\right)\delta x^\nu-\delta\Phi\right)-\delta x^\mu\mathscr{L}, \tag{21}$$

and satisfies the continuity equation

$$\partial_\mu J^\mu\left(x\right)=0. \tag{22}$$

Integrating this equation in a finite volume V

$$\int\limits_V d^3x\partial_\mu J^\mu\left(x\right)=0,$$

separate the temporal from the spatial derivatives and by virtue of Einstein's notation [16]

$$\int\limits_V d^3x\partial_0 J^0\left(x\right)+\int\limits_V d^3x\partial_k J^k\left(x\right)=0$$

as well as by applying Gauss's theorem [17], we find

$$\frac{d}{dt}\int\limits_V d^3xJ^0\left(x\right)+\oint\limits_{\partial V}d^3xn_kJ^k\left(x\right). \tag{23}$$

The volume integral is defined as the charge

$$Q_V=\int\limits_V d^3xJ^0\left(x\right). \tag{24}$$

If there is no input or output flow at the boundary (isolated system), then we get

$$\frac{dQ_V}{dt}=0, \tag{25}$$

that is, charge is conserved.

Every field behaves like a scalar field ($\delta\Phi=0$) in space-time translations, therefore, the current density is

$$J^\mu\left(x\right)=\left[\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\Phi\right)}\left(\partial_\nu\Phi\right)-\delta_\nu^\mu\mathscr{L}\right]\epsilon^\nu, \tag{26}$$

from which the energy-momentum tensor is defined

$$T_\nu^\mu=\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\Phi\right)}\left(\partial_\nu\Phi\right)-\delta_\nu^\mu\mathscr{L}. \tag{27}$$

Using the Minkowsky metric tensor $\eta^{\nu\rho}$ to raise the subscript ν, $T^{\mu\nu}=\eta^{\nu\rho}T_\rho^\mu$ and integrating the continuity equation in a spatial volume V, we find

$$\frac{d}{dt}\int\limits_V d^3xT^{0\nu}+\oint\limits_{\partial V}d^3xn_kT^{k\nu}=0. \tag{28}$$

Consequently, the associated conserved charge is the quadromoment, which is the generator of infinitesimal space-time translations,

$$P^\nu=\int\limits_V d^3xT^{0\nu}. \tag{29}$$

3 Global Symmetries

Let the quantum field $\Phi_k\left(x\right)$ be members of a multiplet, the transformation before the element $g\left(\theta_1,\theta_2,...,\theta_N\right)$ from the symmetry group $SU\left(n\right)$ results

$$\Phi'_k\left(x\right)=\sum_{l=1}^n\left(\mathbf{U}\right)_{kl}\Phi_l\left(x\right), \tag{30}$$

where $\left(\mathbf{U}\right)_{kl}$ is an element of the matrix that represents the transformation. Since the finite transformation can always be constructed by the composition of infinitesimal transformations, the variation under the global infinitesimal transformation becomes

$$\begin{aligned}\delta\Phi_k\left(x\right)=\sum_{l=1}^n\left[\delta_{kl}+i\theta^a\left(\mathbf{T}_a\right)_{kl}\right]\Phi_l\left(x\right)-\Phi_k\left(x\right)=\\ i\theta^a\sum_{l=1}^n\left(\mathbf{T}_a\right)_{kl}\Phi_l\left(x\right),\end{aligned} \tag{31}$$

with θ^a as infinitesimals and $a=1,2,3,...,n^2-1$. Thus, we arrive at the currents of Noether

$$J_a^\mu\left(x\right)=\sum_l\left[\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\Phi_k\right)}\left(-i\left(\mathbf{T}_a\right)_{kl}\Phi_l\left(x\right)\right)\right], \tag{32}$$

we associate the charges

$$Q_{V,a}=-i\sum_l\int\limits_V d^3x\frac{\partial\mathscr{L}}{\partial\left(\partial_0\Phi_k\right)}\left(\mathbf{T}_a\right)_{kl}\Phi_l\left(x\right). \tag{33}$$

It is enough to show that the n^2-1 charges $Q_{V,a}$ are precisely the generators of the symmetries. For this, a symmetry operation represents a change of base. The components of the field are transformed by

$$\mathscr{U}^{\dagger}(g)\,\Phi_k(x)\,\mathscr{U}(g)=\Phi'_k(x)=\sum_l(\mathbf{U})_{kl}\,\Phi_l(x)\,. \quad (34)$$

Therefore, for an infinitesimal transformation

$$(1-i\theta^a\mathscr{T}_a)\,\Phi_k(x)\,(1+i\theta^a\mathscr{T}_a)=\sum_l(\mathbf{U})_{kl}\,\Phi_l(x) \quad (35)$$

we obtain the expression that indicates how the components are transformed in the group

$$[\mathscr{T}_a,\Phi_k]=-\sum_l(\mathbf{T}_a)_{kl}\,\Phi_l(x)\,. \quad (36)$$

For the proof, we first define the conjugate moment

$$\Pi^k(x)=\frac{\partial\mathscr{L}}{\partial(\partial_0\Phi_k)}, \quad (37)$$

and then the commutator is evaluated in equal times, $x_0=y_0$

$$[Q_{V,a},\Phi_m(y)]=\left[-i\sum_l\int_V d^3x\Pi^k(x)\,(\mathbf{T}_a)_{kl}\,\Phi_l(x)\,,\Phi_m(y)\right],$$

rearranging the right side

$$-i\sum_l\int_V d^3x\left[\Pi^k(x)\,(\mathbf{T}_a)_{kl}\,\Phi_l(x)\,,\Phi_m(y)\right],$$

and by virtue of the properties of the Lie brackets

$$\begin{aligned}&[Q_{V,a},\Phi_m(y)]=\\ -i\sum_l\int_V d^3x&\left\{\Pi^k(x)\,(\mathbf{T}_a)_{kl}\,[\Phi_l(x)\,,\Phi_m(y)]\right.\\ &\left.+\left[\Pi^k(x)\,,\Phi_m(x)\right](\mathbf{T}_a)_{kl}\,\Phi_l(x).\right.\end{aligned} \quad (38)$$

Consequently, the commutation relations of the fields Φ_k and Π^k are needed to proceed. In bosonic fields, these are defined as

$$\begin{aligned}\left[\Pi^k(x)\,,\Phi_l(y)\right]\big|_{x^0=y^0}&=\tfrac{1}{i}\delta^k_l\delta^3(\mathbf{x}-\mathbf{y})\,,\\ [\Phi_l(x)\,,\Phi_m(y)]\big|_{x^0=y^0}&=0,\\ \left[\Pi^k(x)\,,\Pi^m(y)\right]\big|_{x^0=y^0}&=0.\end{aligned} \quad (39)$$

The commutator results

$$[Q_{V,a},\Phi_m(y)]=-\sum_l(\mathbf{T}_a)_{ml}\,\Phi_l(y)\,. \quad (40)$$

Finally, the commutator is evaluated by

$$[Q_{V,a},Q_{V,b}]=\left[-i\sum_l\int_V d^3x\Pi^k(x)\,(\mathbf{T}_a)_{kl}\,\Phi_l(x)\,,-i\sum_r\int_V d^3x\Pi^s(y)\,(\mathbf{T}_b)_{sr}\,\Phi_r(y)\right]\Bigg|_{x^0=y^0},$$

rearranging terms, on the right side of the equation we have

$$\sum_l\sum_r\int_V d^3x\int_V d^3y\,\left[\Pi^k(x)\,(\mathbf{T}_a)_{kl}\,\Phi_l(x)\,,\Pi^s(y)\,(\mathbf{T}_b)_{sr}\,\Phi_r(y)\right]\Big|_{x^0=y^0}.$$

After applying the properties of the Lie brackets, we have

$$\begin{aligned}&\sum_l\sum_r\int_V d^3x\int_V d^3y\Pi^k(x)\,(\mathbf{T}_a)_{kl}\,[\Pi^s(y)\,,\Phi_l(x)]|_{x^0=y^0}\,(\mathbf{T}_b)_{sr}\,\Phi_r(y)\\ &-\sum_l\sum_r\int_V d^3x\int_V d^3y\Pi^k(x)\,\Pi^s(y)\,(\mathbf{T}_a)_{kl}\,(\mathbf{T}_b)_{sr}\,[\Phi_l(x)\,,\Phi_r(y)]|_{x^0=y^0}\\ &-\sum_l\sum_r\int_V d^3x\int_V d^3y\,\left[\Pi^k(x)\,,\Pi^s(y)\right]\Big|_{x^0=y^0}(\mathbf{T}_a)_{kl}\,\Phi_l(x)\,(\mathbf{T}_b)_{sr}\,\Phi_r(y)\\ &-\sum_l\sum_r\int_V d^3x\int_V d^3y\Pi^k(y)\,(\mathbf{T}_b)_{sr}\,\left[\Pi^k(x)\,,\Phi_r(y)\right]\Big|_{x^0=y^0}(\mathbf{T}_a)_{kl}\,\Phi_l(x)\,.\end{aligned}$$

By virtue of equations (39) one finds

$$-i\sum_l\sum_r\int_V d^3x\int_V d^3y\Pi^k(x)\,(\mathbf{T}_a)_{kl}\,\delta^s_l\delta^3(\mathbf{y}-\mathbf{x})\,(\mathbf{T}_b)_{sr}\,\Phi_r(y)$$

$$+i\sum_l\sum_r\int\limits_V d^3x\int\limits_V d^3y\Pi^s(y)(\mathbf{T}_b)_{sr}\,\delta^k_r\delta^3(\mathbf{x}-\mathbf{y})(\mathbf{T}_a)_{kl}\,\Phi_l(x).$$

Furthermore, the properties of the Dirac delta function [18] will be used to obtain

$$-i\sum_l\sum_r\int\limits_V d^3x\Pi^k(x)(\mathbf{T}_a)_{kl}\,\delta^s_l(\mathbf{T}_b)_{sr}\,\Phi_r(x)$$

$$+i\sum_l\sum_r\int\limits_V d^3x\Pi^s(x)(\mathbf{T}_b)_{sr}\,\delta^k_r(\mathbf{T}_a)_{kl}\,\Phi_l(x),$$

or

$$-i\sum_r\int\limits_V d^3x\Pi^k(x)(\mathbf{T}_a)_{kl}(\mathbf{T}_b)_{lr}\,\Phi_r(x)+i\sum_l\int\limits_V d^3x\Pi^s(x)(\mathbf{T}_b)_{sk}(\mathbf{T}_a)_{kl}\,\Phi_l(x).$$

With the help of the definition of the Lie commutator

$$-i\sum_r\int\limits_V d^3x\Pi^k(x)([\mathbf{T}_a,\mathbf{T}_b])_{kr}\,\Phi_r(x)$$

or in its equivalent form

$$-i^2f^c_{ab}\sum_r\int\limits_V d^3x\Pi^k(x)(\mathbf{T}_c)_{kr}\,\Phi_r(x).$$

Finally, we have

$$[Q_{V,a},Q_{V,b}]=if^c_{ab}Q_{V,c}. \tag{41}$$

With equations (36), (40) and (41) we conclude that the charges are the generators of the symmetry. If $\mathcal{H}$ is the Hamiltonian of the theory, the Heisenberg equation [19] [20] implies

$$\frac{dQ_{V,a}}{dt}=[\mathcal{H},Q_{V,a}]=[\mathcal{H},\mathscr{T}_a]=0, \tag{42}$$

which expresses the invariance of the theory under transformations of the group $\mathscr{G}$.

4 Gauge symmetries

If the parameters of the global transformation are considered to be space-time dependent, we are dealing with a local transformation

$$\mathbf{U}=\mathbf{U}(g(\theta^a(x))). \tag{43}$$

In a global transformation we have

$$\partial_\mu\Phi_k\to\partial_\mu\Phi'_k=$$
$$\sum_{l=!}^{n}\left[(\mathbf{U})_{kl}\,\partial_\mu\Phi_l(x)+\partial_\mu(\mathbf{U})_{kl}\,\Phi_l(x)\right]=$$
$$(\partial_\mu\Phi_k)'+\sum_{l=1}^{n}\partial_\mu(\mathbf{U})_{kl}\,\Phi_l(x). \tag{44}$$

If the theory is claimed to be invariant under a local transformation, the covariant derivative is introduced

$$\mathscr{D}_\mu\Phi_k=\sum_{r=1}^{n}\left[\delta_{kr}\partial_\mu-ig(\mathbf{T}_a)_{kr}\,W^a_\mu\right]\Phi_r, \tag{45}$$

where the n^2-1 quantum fields W^a_μ are called gauge fields and g is the coupling constant. We demanding

$$\mathscr{D}'_\mu\Phi'_k=(\mathscr{D}_\mu\Phi_k)', \tag{46}$$

then, we get the infinitesimal transformation of the fields

$$W^a_\mu\to W'^a_\mu=W^a_\mu+f^a_{bc}W^b_\mu\theta^c+\frac{1}{g}\partial_\mu\theta^a. \tag{47}$$

This transformation is known as the gauge transformation of W^a_μ, which only depends on the group $\mathcal{G}$ and not on how the fields Φ_k, called material fields, are transformed. Considering an infinitesinal global transformation, the gauge fields are transformed by

$$W^a_\mu\to W'^a_\mu=W^a_\mu+f^a_{bc}W^b_\mu\theta^c, \tag{48}$$

and since the structure constants generate the adjoint representation, we have

$$W^a_\mu=W^a_\mu+i\theta^c(\mathbf{A}_b)^a_c\,W^b_\mu. \tag{49}$$

By generalizing the electromagnetic tensor, the Yang-Mills field [21] is defined as

$$\mathscr{F}^a_\mu\to\mathscr{F}'^a_\mu=\mathscr{F}^a_\mu+f^a_{bc}\mathscr{F}^b_\mu\theta^c, \tag{50}$$

is directly transformed into the adjoint representation. The conclusion is that for a Lagrangian density $\mathscr{L}(\Phi_k(x),\partial_\mu\Phi_k(x))$ invariant under a global transformation, it can be invariant under a local gauge transformation by introducing the gauge fields W^a_μ, which enter the covariant derivatives $\mathscr{D}_\mu\Phi_k$ and the Yang-Mills field $\mathscr{F}_{\mu\nu}$. The Lagrangian density is

$$\mathscr{L}=\mathscr{L}(\Phi_k(x),\mathscr{D}_\mu\Phi_k(x))-\frac{1}{4}\mathscr{F}^{\mu\nu}_a\mathscr{F}^a_{\mu\nu}, \tag{51}$$

where the last contribution contains the kinetic energy terms and the self-interactions of the non-abelian gauge fields W^a_μ. In general, the Lagrangian density $\mathscr{L}(\Phi_k(x), \partial_\mu \Phi_k(x))$ contains the kinetic energy terms $\mathscr{L}_0(\Phi_k(x), \partial_\mu \Phi_k(x))$ and the interactions between the particles $\mathscr{L}_{int}(\Phi_k(x), \partial_\mu \Phi_k(x))$. Therefore, the material fields become the fundamental representation and the norm fields the adjoining representation. Also, the symmetries determine the interactions between the fields by means of the gauge principle [22]. By promoting a global symmetry to a local one, the introduced gauge fields induce the terms of the interactions, therefore, these terms are the mediators of the interactions.

5 Spontaneous breaking of symmetry

Since for a global symmetry the generators satisfy

$$[\mathscr{T}, \mathcal{H}] = 0, \tag{52}$$

these do not necessarily leave the vacuum state invariant. The Wigner-Weyl symmetry leaves the vacuum state invariant and, consequently, the mass spectrum of the particles in the theory is degenerate. The Nambu-Goldstone symmetry does not leave the vacuum state invariant, and therefore the particle spectrum must contain a particle with zero mass, known as the Goldstone boson. The Nambu-Goldstone type symmetry is a spontaneously broken symmetry.

When considering the expectation value of the vacuum state to (36), we obtain

$$\langle 0| [\mathscr{T}_a, \Phi_k] |0\rangle = -\sum_l (\mathbf{T}_a)_{kl} \langle 0| \Phi_l(x) |0\rangle . \tag{53}$$

In a Wigner-Weyl symmetry we have that the generators

$$\mathscr{T}_a |0\rangle = 0, \tag{54}$$

consequently

$$\langle \Phi_k \rangle_0 = \langle 0| \Phi_k(x) |0\rangle = 0. \tag{55}$$

In a spontaneously broken symmetry

$$\langle \Phi_k \rangle_0 = \langle 0| \Phi_k(x) |0\rangle \neq 0. \tag{56}$$

Goldstone bosons are a consequence of Goldstone's theorem [23], which is based on the conservation of currents

$$\partial_\mu J^\mu_a(x) = 0. \tag{57}$$

Expression (53) is rewritten in terms of the current

$$\int_V d^3y \langle 0| \left[J^0_a(y), \Phi_k(x)\right] |0\rangle = -\sum_l (\mathbf{T}_a)_{kl} \langle 0| \Phi_l(x) |0\rangle , \tag{58}$$

or

$$\int_V d^3y \left[\langle 0| J^0_a(y) \Phi_k(x) |0\rangle - \langle 0| \Phi_k(x) J^0_a(y) |0\rangle\right] = -\sum_l (\mathbf{T}_a)_{kl} \langle 0| \Phi_l(x) |0\rangle \neq 0. \tag{59}$$

Since the current can be written in terms of the translation operators of the Poincare group

$$J^0_a(y) = \exp(-iy^\mu P_\mu) J^0_a(0) \exp(iy^\mu P_\mu) \tag{60}$$

thus, these operators leave invariant the state of the vacuum, then

$$\int_V d^3y \left[\langle 0| J^0_a(0) \exp(iy^\mu P_\mu) \Phi_k(x) |0\rangle - \langle 0| \Phi_k(x) \exp(-iy^\mu P_\mu) J^0_a(0) |0\rangle \neq 0.\right. \tag{61}$$

If we insert $\sum_{n^G} \left|n^G\right\rangle \left\langle n^G\right| = 1$, the left-hand side of equation (61) becomes

$$\sum_{n^G} \langle 0| J^0_a(0) \left|n^G\right\rangle \left\langle n^G\right| \Phi_k(x) |0\rangle e^{iy^0 P^G_0} \int_V d^3y e^{-i\vec{\mathbf{y}}\cdot\vec{\mathbf{P}}^G} - \sum_{n^G} \langle 0| \Phi_k(x) \left|n^G\right\rangle \left\langle n^G\right| J^0_a(0) |0\rangle e^{-iy^0 P^G_0} \int_V d^3y e^{i\vec{\mathbf{y}}\cdot\vec{\mathbf{P}}^G} \neq 0. \tag{62}$$

By defining the coefficient

$$c_{n^G} = \langle 0| J^0_a(0) \left|n^G\right\rangle \left\langle n^G\right| \Phi_k(x) |0\rangle , \tag{63}$$

and since

$$\int_V d^3y e^{-i\vec{\mathbf{y}}\cdot(\pm\vec{\mathbf{P}}^G)} = (2\pi)^3 \Delta_V\left(\pm\vec{\mathbf{P}}^G\right), \tag{64}$$

with $\Delta^3_V\left(\vec{\mathbf{P}}\right)$ as an approximation of the Dirac delta, the expression (61) becomes

$$\lim_{V\to\infty} (2\pi)^3 \times \sum_{n^G} \left\{c_{n^G} \exp\left(iy^0 P^G_0\right) - c^*_{n^G} \exp\left(-iy^0 P^G_0\right)\right\}_{\vec{\mathbf{P}}^G = 0} \neq 0. \tag{65}$$

Since the left-hand side is independent of y^0, it follows that

$$P^G_0\Big|_{\vec{\mathbf{P}}^G} = 0. \tag{66}$$

Thus, the massless states $\left|n^G\right\rangle$ correspond to the Goldstone bosons. The number of Goldstone bosons is equal to the number of generators that do not annihilate the vacuum state.

To satisfy the norm principle, the global transformation must be promoted to a local one. In the Nambu-Goldstone case, the Goldstone bosons associated with the spontaneous symmetry-breaking generators disappear via a local gauge transformation and the corresponding gauge fields gain mass.

6 Goldstone's theorem

If we consider the global invariant Lagrangian density $U(1)$ composed of a complex scalar field $\Phi(x)$ and its complex conjugate $\Phi^*(x)$ then

$$\begin{aligned}\mathscr{L} &= \partial_\mu \Phi^* \partial^\mu \Phi - V(\Phi^* \Phi), \\ V(\Phi^*\Phi) &= m^2 \Phi^* \Phi + \lambda (\Phi^* \Phi)^2 .\end{aligned} \quad (67)$$

The described system of the Lagrangian density (67) transforms with the Lagrangian density composed of the two real fields φ_1 and φ_2, which are related to Φ and Φ^* through from $\Phi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$ and $\Phi^* = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2)$, as follows

$$\mathscr{L} = \frac{1}{2}\partial_\mu \varphi_1 \partial^\mu \varphi_1 + \frac{1}{2}\partial_\mu \varphi_2 \partial^\mu \varphi_2 - V\left(\varphi_1^2 + \varphi_2^2\right). \quad (68)$$

Equation (68) has the symmetry $\mathcal{O}(2)$, that is, the Lagrangian density (68) is invariant when considering the following transformation

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \to \begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad (69)$$

so that the symmetry $U(1)$ is equivalent to the symmetry $\mathcal{O}(2)$.

In quantum field theory, excitations of particles in a field are defined as quantum fluctuations of the field around the lowest energy state, i. e., the state related to vacuum. The constant value of the field corresponding to the lower energy state is called the vacuum expectation value (VEV), i. e. $\langle 0|\Phi|0\rangle \equiv \Phi_0$. To find the spectrum of the particle, the potential is expanded around the minimum corresponding to the lowest energy state

$$V(\varphi_1,\varphi_2) = V(\varphi_{01},\varphi_{02}) + \sum_{a=1,2} \left.\frac{\partial V}{\partial \varphi_a}\right|_0 \Delta\varphi_a + \frac{1}{2}\sum_{a,b=1,2} \left.\frac{\partial^2 V}{\partial \varphi_a \partial \varphi_b}\right|_0 \Delta\varphi_a \Delta\varphi_b + ..., \quad (70)$$

where $\Delta\varphi_a = \varphi_a - \varphi_{0a}$, $\Delta\varphi = \varphi_b - \varphi_{0b}$ and $\Phi_0 = (\varphi_{01}, \varphi_{02})$ is the VEV of $\Phi = (\varphi_1, \varphi_2)$, i. e. $\varphi_{0a} = \langle 0|\varphi_0|0\rangle$ $(a = 1, 2)$. Since the potential V has a minimum in $\Phi = \Phi_0$, the second term of the right side of equation (70) is zero. The $\left.\frac{\partial^2 V}{\partial\varphi_a \partial\varphi_b}\right|_0 \equiv m_{ab}^2$ in the third term of the right side of equation (70) is called the mass matrix and is diagonalized to generate the spectrum of the particle mass.

Now, two possible cases of V are possible. (1) The first case corresponds to the single vacuum state and is called "Wigner phase". Considering the parameters m^2 and λ for the potential in expression (67) as positive $m^2 > 0$ and $\lambda > 0$,

$$V\left(\varphi_1^2 + \varphi_2^2\right) = \frac{m^2}{2}\left(\varphi_1^2 + \varphi_2^2\right) + \frac{\lambda}{4}\left(\varphi_1^2 + \varphi_2^2\right)^2, \quad (71)$$

and requires the following condition for the vacuum

$$\left(\frac{\partial V}{\partial \varphi_1}\right)_0 = m^2 \varphi_{01} + \lambda\varphi_{01}\left(\varphi_{01}^2 + \varphi_{02}^2\right) = 0, \quad (72)$$

$$\left(\frac{\partial V}{\partial \varphi_2}\right)_0 = m^2 \varphi_{02} + \lambda\varphi_{02}\left(\varphi_{01}^2 + \varphi_{02}^2\right) = 0. \quad (73)$$

Thus, through equations (72) and (73), it is obtained to the single vacuum with $\varphi_{01} = \varphi_{02} = 0$. The mass matrix becomes diagonal in this case

$$m_{ab}^2 = \begin{pmatrix} m^2 & 0 \\ 0 & m^2 \end{pmatrix}, \quad (74)$$

which means that φ_1 and φ_2 have equal masses m as can be seen from equation (71). (2) On the other hand, in which the vacuum state is non-unique is called "Nambu-Goldstone phase". In this case, for example, in continuously or infinitely degenerate vacuum states with $\varphi_{01} \neq 0$ and/or $\varphi_{02} \neq 0$, for the potential with $m^2 = -\mu^2$ $(\mu^2 > 0)$ and $\lambda > 0$

$$V\left(\varphi_1^2 + \varphi_2^2\right) = -\frac{\mu^2}{2}\left(\varphi_1^2 + \varphi_2^2\right) + \frac{\lambda}{4}\left(\varphi_1^2 + \varphi_2^2\right)^2. \quad (75)$$

The minimum of V is in the following partial derivatives

$$\left(\frac{\partial V}{\partial \varphi_1}\right)_0 = -\mu^2 \varphi_{01} + \lambda\varphi_{01}\left(\varphi_{01}^2 + \varphi_{02}^2\right) = 0, \quad (76)$$

$$\left(\frac{\partial V}{\partial \varphi_2}\right)_0 = -\mu^2 \varphi_{02} + \lambda\varphi_{02}\left(\varphi_{01}^2 + \varphi_{02}^2\right) = 0, \quad (77)$$

leading to the condition

$$\varphi_{01}^2 + \varphi_{02}^2 = v^2 = \frac{\mu^2}{\lambda}, \text{or} \qquad (\Phi^*\Phi)_0 = |\Phi_0|^2 = \frac{v^2}{2} = \frac{\mu^2}{2\lambda}. \quad (78)$$

In other words, all points on a circle with radius $v = \sqrt{\mu^2/\lambda}$ in the plane (φ_1, φ_2) correspond to the minimum of V, that is, the vacuum state is already non-unique but has $\mathcal{O}(2)$ symmetry. From expression (75) we obtain

$$\frac{\partial^2 V}{\partial \varphi_1^2} = -\mu^2 + \lambda\left(\varphi_1^2 + \varphi_2^2\right) + 2\lambda\varphi_1^2, \quad (79)$$

$$\frac{\partial^2 V}{\partial \varphi_2^2} = -\mu^2 + \lambda\left(\varphi_1^2 + \varphi_2^2\right) + 2\lambda\varphi_2^2, \quad (80)$$

$$\frac{\partial^2 V}{\partial\varphi_1\partial\varphi_2} = 2\lambda\varphi_1\varphi_2. \quad (81)$$

Therefore, if we choose a point $(\varphi_{01} = v, \quad \varphi_{02} = 0)$ as a physical vacuum, the mass matrix can be expressed as

$$m_{ab}^2 = \begin{pmatrix} 2\lambda v^2 & 0 \\ 0 & 0 \end{pmatrix}. \quad (82)$$

Therefore, $\varphi_1' = \varphi_1 - v$ corresponds to a particle with mass $m^2 = 2\lambda v^2$, while $\varphi_2' = \varphi_2$ is massless. φ_2' is called the "Goldstone boson". If these new fields are used, the Lagrangian density (68) is expressed

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu\varphi_1'\right)^2 + \frac{1}{2}\left(\partial_\mu\varphi_2'\right)^2 - \frac{1}{2}\left(2\lambda v^2\right)\left(\varphi_1'\right)^2 + \lambda v\varphi_1'\left[\left(\varphi_1'\right)^2 + \left(\varphi_2'\right)^2\right] - \frac{\lambda}{4}\left[\left(\varphi_1'\right)^2 + \left(\varphi_2'\right)^2\right]^2. \quad (83)$$

The Lagrangian density no longer has $\mathcal{O}(2)$ symmetry, although the Lagrangian density (68) does. That is, the symmetry of the Lagrangian density has been broken by the symmetry breaking of the vacuum state. This symmetry is called a hidden symmetry or spontaneous symmetry breaking (SSB).

In short, starting from the Lagrangian density with a global symmetry, negative parameter $m^2 = -\mu^2$ and breaking the symmetry of the vacuum states by choosing a particular point among the states of symmetrically degenerate vacuum, it was found that a massless particle (Goldstone boson) appeared. This mechanism is called Goldstone's theorem [23] [24] [25]. In general, massless Goldstone bosons [26] appear depending on symmetry properties when a global symmetry is spontaneously broken.

7 Finite temperature effects

The usual methods used in quantum field theories are adequate to describe situations in a vacuum, such as occur in accelerators. However, in the primitive Universe the conditions are quite different, being characterized by a high-temperature plasma, with an energy density that cannot be neglected. In these conditions, it is necessary to find other methods, halfway between thermodynamics and quantum field theories, that allow realistic calculations to be made under these conditions where the environment is characterized by a thermal bath. These methods are developed by finite temperature field theory [27].

Effective potential

The function that contains all the finite temperature effects is the effective potential, ${V_{eff}}^\beta$. This is composed of the classical potential in vacuum, V_0, plus the term that describes the quantum and temperature effects V^β

$$V_{eff}^\beta(\phi_c) = V_0(\phi_c) + V^\beta(\phi_c), \quad (84)$$

where $\phi_c \equiv \overline{\phi}(x)$ is the constant value in space that the field takes, for a translational invariant theory.

There are two formalisms to calculate the second term of equation (84), that is, the potential V^β, both giving the same results, at least to first order. One formalism is the imaginary time method to describe equilibrium situations, and the other is the real time method, with which certain non-equilibrium systems can be investigated. Taking into account the contributions of the diagrams to 1 loop, it follows that the effective potential is [27]

$$V_1^\beta(\phi_c) = \int \frac{d^3p}{(2\pi)^3}\left[\frac{\omega}{2} + \frac{1}{\beta}\log\left(1 - e^{-\beta\omega}\right)\right], \quad (85)$$

where $\beta \equiv 1/T$ and ω is

$$\omega = \sqrt{|\overrightarrow{p}|^2 + m^2(\phi_c)}, \quad (86)$$

where $m^2(\phi_c)$ is the curvature of the potential, also called displaced mass

$$m^2(\phi_c) \equiv \frac{\partial^2 V_0(\phi_c)}{\partial\phi_c^2}. \quad (87)$$

The first part of the integral (85) accounts for the quantum corrections in vacuum, giving the effective potential at zero temperature, $V_1|_{T=0}$. The part of (85) that depends on temperature can be written as

$$\frac{1}{\beta}\int \frac{d^3p}{(2\pi)^3}\left[\frac{\omega}{2} + \frac{1}{\beta}\log\left(1 - e^{-\beta\omega}\right)\right] = \frac{1}{2\pi^2\beta^4}J_B\left[m^2(\phi_c)\beta^2\right], \quad (88)$$

where the bosonic thermal function, J_B, has been defined as

$$J_B\left[m^2\beta^2\right] = \int_0^\infty dx\left(x^2\right)\log\left[1 - \exp\left(-\sqrt{x^2 + \beta^2 m^2}\right)\right]. \quad (89)$$

In this way, the effective potential at 1 loop consists of the following parts

$$V_{eff}^\beta(\phi_c) = V_0(\phi_c) + V_1(\phi_c)|_{T=0} + \frac{1}{2\pi^2\beta^4}J_B\left[m^2(\phi_c)\beta^2\right]. \quad (90)$$

The function J_B [27] admits an expansion for high temperatures

$$J_B\left[m^2/T^2\right] = -\frac{1}{45}\pi^4 + \frac{1}{12}\pi^2\frac{m^2}{T^2} - \frac{\pi}{6}\left(\frac{m^2}{T^2}\right)^{3/2} - \frac{1}{32}\frac{m^4}{T^4}\log\left(\frac{m^2}{a_b T^2}\right) - 2\sqrt{\pi^7}\sum_{l=1}^{\infty}(-1)^l\frac{\zeta(2l+1)}{(l+1)!}\Gamma\left(l+\frac{1}{2}\right)\left(\frac{m^2}{4\pi^2T^2}\right)^{l+2} \quad (91)$$

where ζ is the Riemann zeta function and $a_b = 16\pi^2\exp\left(\frac{3}{2}-2\gamma_E\right)$, with $\gamma_E \simeq 0.577$ the Euler-Mascheroni constant.

With these approximations we have that, at high temperatures, the part of the effective potential to a loop that depends on temperature approaches

$$V_1^\beta \simeq \frac{1}{24}m^2T^2 - \frac{1}{12\pi}m^3T - \frac{1}{64\pi^2}m^4\log\left(\frac{m^2}{a_bT^2}\right) + \mathcal{O}\left(\frac{m^6}{T^2}\right), \quad (92)$$

where the terms that are constant with the field have been neglected.

8 Phase transitions

One of the most relevant consequences of finite temperature effects is the influence they exert on phase transitions. The key point is that at finite temperature the fundamental state of the system does not correspond to the minimum of $V_0(\phi)$, but is the value that minimizes the potential $V_{eff}^\beta(\phi)$, given in equation (84). Thus, the state that defines the vacuum is a function of temperature, $\langle\phi(T)\rangle$.

In the context of cosmology, the Kirzhnits effect [28] is very important. In the Big Bang theory, initially the Universe is at very high temperatures, so it is to be expected that, due to temperature effects, the symmetries are not broken. In this way, for a discrete symmetry, the minimum of the corresponding potential will be found at $\langle\phi(T)\rangle = 0$. For a certain critical temperature value, T_c, the minimum at $\phi = 0$ will become metastable and a phase transition will occur.

There are two types of phase transitions: first order and second order. In the case of first order transitions, for $T \gg T_c$ the potential only has a minimum at $\phi = 0$. As the temperature decreases, but still with $T > T_c$, a local minimum develops at $\phi \neq 0$. When the temperature reaches its critical value, the minima are degenerate and for $T < T_c$ the minimum at $\phi \neq 0$ becomes the absolute minimum. In this case, between the local minimum at $\phi = 0$ and the global minimum at $\phi \neq 0$ there is a potential barrier, so a classical transition cannot occur between the two minima. This has to be developed by quantum tunneling, which implies the formation of real vacuum bubbles that rapidly expand up to speeds close to light. On the other hand, in the second order transitions, said potential barrier does not appear between the minima and the transition takes place gradually. Therefore, the origin $\phi = 0$ goes from being a global minimum for $T > T_c$ to a local minimum for $T < T_c$.

An important point, especially at the calculation level, in the study of phase transitions produced by temperature effects is that the first order (a loop) of the usual perturbation theory, in powers of coupling constants, loses its validity for temperatures above the critical temperature [29]. Then it is necessary to replace this theory of perturbations by an improved expansion, where diagrams of more than one loop are taken into account. In this improved theory an infinite number of diagrams are summarized for the order of the expansion, which is known as Daisy summarization [27]. The effect of this summarization is to introduce a shift in the mass that depends on the temperature

$$m^2(\phi_c) \longrightarrow m_{eff}^2 = m^2(\phi_c) + \Pi \quad (93)$$

where $\Pi \propto T^2$ is the self-energy corresponding to the propagator when they have taken into account all the diagrams with loops added to the propagator to the first order in temperature. Obviously, this Π function depends on the potential being considered.

Since we believe that the Universe evolved through such a symmetry breaking sequence. There are many questions we can ask ourselves about the cosmological implications of the regime.

9 Topological defects

Another important aspect to take into account in the process of spontaneous symmetry breaking are the possible topological defects generated by the breaking. Topological defects [30] [31] [32] are concentrations of energy that are generated after the spontaneous breaking of a symmetry, in case the resulting vacuum has a non-trivial topology. To understand non-trivial topology, let's look at a couple of examples: domain walls [33] [34] and cosmic strings [35] [36]. The existence and stability of these objects will be dictated by topological considerations, with numerical simulations being the only treatment that allows studying their evolution.

As a first example, we consider the breaking of symmetry in a potential with a symmetry under reflection Z_2, that is, an invariance before the change $\phi \longrightarrow -\phi$. Given the breaking of this symmetry, the expected value of the field in a vacuum can take the value $\langle\phi\rangle = +v$, or the value $\langle\phi\rangle = -v$. Considering that the theory was translational invariant, it has been assumed that all space is in the same ground state, an assumption that is not true. There will be regions where $\langle\phi\rangle = +v$ and others where $\langle\phi\rangle = -v$, there being no reason for the field to take one or the other value. Then, since between the regions where $\langle\phi\rangle$ is different, the field has to pass from $\phi = +v$ to $\phi = -v$ continuously, there must be regions in space in which $\phi = 0$, that is, zones of false vacuum. These zones are two-dimensional regions, with a certain thickness, that separate the different boundaries where the expected value of the field has different values. These zones are called domain walls and they appear whenever a discrete symmetry is broken. In this case, the vacuum is said to have a non-trivial topology because it consists of two disconnected states. If we look at the shape of the potential after the break, we can see that in areas where $\phi = 0$, the energy is greater than in the rest of the space. As a conclusion, these domain walls are two-dimensional regions with a high concentration of energy.

Another example of a topological defect is that of cosmic strings [35] [36]. In this case, the symmetry to consider would be, for example, the $N = 2$ case of the $O(N)$ symmetry, respected by the potential (103) (see appendix A). A particular case would be a complex field, $\phi = |\phi| \exp(i\theta)$. In that case, once the symmetry is broken, the expectation value of the field in vacuum can be found at any point on a circle of radius v. That is, the expected value of the module of the field is fixed at $\langle|\phi|\rangle = v$, but the value of its phase can be any. Thus, there is a phase that depends on the position, $\theta(\overrightarrow{r})$, which can be different at each point in space. However, the ϕ field can only take on a single value at each point in space, so the change in phase, $\Delta\theta$, around any closed path must be a multiple of 2π. We take, for example, a path in which $\Delta\theta = 2\pi$. As we compress the path into a point, Theta cannot continuously vary from $\Delta\theta = 2\pi$ to $\Delta\theta = 0$. For this reason, there will be a point in said path where the time phase is undefined, the only possibility of this undefined being that $\langle|\phi|\rangle = 0$. This false vacuum point on the considered path is part of a one-dimensional false vacuum tube. These tubes are called cosmic strings and would be the one-dimensional analogue to domain walls. In this case, the topology of the vacuum is non-trivial because there is no unique way to map the manifold that defines it to a circle. As in the case of domain walls, in the region where $\langle|\phi|\rangle = v$, the energy is greater than the rest of space, so these strings are one-dimensional concentrations of energy.

Other examples of topological defects are magnetic monopoles [37] [38] and textures [39]. The former are zero-dimensional analogues to strings and domain walls and appear in breaking spherical symmetries. Textures are delocalized objects of higher dimensions, which appear when breaking the most complicated symmetry groups. In addition, there can also be topological defects that are a combination of those described, such as [40] domain walls linked by cosmic strings, strings that end in monopoles, etc.

The specific effects vary depending on the topological defect considered. For example, domain walls and monopoles are cosmologically catastrophic. Any model in which there is a production of these types of defects has to face the problem that defects evolve in such a way that they contradict different irrefutable observational facts of the Universe, such as producing too large anisotropies in the background of radiation or predicting densities of $\Omega \sim 10^{11}\rho_c$. Therefore, the models must be discarded.

However, cosmic strings and possibly textures are much more benign. Among other characteristics, they can play the role of seeds that give rise to the formation of the large-scale structure that is currently observed, as well as the anisotropies of the radiation background. Another important contribution would be to the dark matter of the Universe.

The spontaneous breakage of many theories predicts the existence of one or more types of topological defects. These objects are inherently nonperturbative and probably cannot be produced in the high-energy collisions that take place in terrestrial accelerators. Instead, they can be produced in phase transitions that occur in the early Universe. For this reason, these defects are considered to be traces of the first moments of the Universe. Although they are not minimum energy configurations, monopoles, strings, and domain walls (boundary domain) are topologically stable and, as predicted by the Kibble mechanism [41], their production is inevitable at phase transitions. One of the most important consequences of these objects is that they crucially affect the evolution of the Universe. The concrete effects vary depending on the topological defect considered. For example, domain walls and monopoles are cosmologically catastrophic. Any model in which there is production of these types of defects has to face the problem that the defects evolve in such a way that they contradict different irrefutable observational facts of the Universe, such as producing excessively large anisotropies in the Cosmic Microwave Background.

10 The axion

The axion is one of the best-known pseudo Goldstone bosons, although experiments looking for it have so far failed to detect it. Its motivation is theoretical and arises as a consequence of the spontaneous rupture of Peccei-Quinn (PQ). This symmetry is the most elegant solution to the CP problem of strong interactions, so if the axion finally does not exist, an alternative solution to this problem will have to be found [42]. Axion physics has very precise properties, all of which depend on a single free parameter, f_a, the symmetry breaking scale of PQ. The importance of the axion is mainly due to the possibility that it is the main component of the Dark Matter of the Universe [43] [44] [45] [46].

Cosmological production

There are different mechanisms that cause the production of axions, which we will group into thermal and non-thermal mechanisms, the latter being the ones that predominate. Among the non-thermal mechanisms we can basically find two classes, those originated in the oscillations of the axion field and those that are produced from the decay of the topological defects generated in the spontaneous breaking of the symmetry PQ [47] [48].

The cosmological history [15] [49] of the axion begins at temperatures $T \sim f_a$, when the $U(1)_{PQ}$ symmetry breaks spontaneously. In principle, all possible expected values for the axion field, $\langle a \rangle$, are equally likely, but naturally $\langle a \rangle$ is expected to be of the order of the scale of PQ.

The vacuum expectation value (VEV) for the axion field is subjected to the evolution equation

$$\frac{d^2 \langle a \rangle}{dt^2} + 3H(t) \frac{d \langle a \rangle}{dt} + m_a^2(t) \langle a \rangle = 0, \qquad (94)$$

where H is the Hubble parameter and $m_a(t)$ is the mass of the axion, which evolves over time (as a function of temperature and temperature as a function of time). In the interval $f_a > T \gg \Lambda_{QCD}$, the Hubble parameter is much larger than m_a and $\langle a \rangle$ remains constant, since H acts as a friction. The mass of the axion is suppressed at high energies and it is on the QCD scale that it begins to be relevant, quickly becoming $m_a \simeq H$. From this moment on, the friction induced by H is inefficient and the field of the axion begins to evolve. Thus, for a temperature $T \sim \Lambda_{QCD}$ we have

$$m_a \simeq H \simeq \frac{\Lambda_{QCD}^2}{M_P}, \qquad (95)$$

moment from which the axion field begins to oscillate around $\langle a \rangle = 0$.

The mass of the axion increases adiabatically as T decreases, so the oscillation is approximately sinusoidal with an amplitude that decreases with time [50] [51]

$$\langle a \rangle \simeq A(t) \cos\left(m_a(t)\, t\right), \qquad (96)$$

to which corresponds a density of axions

$$n_a \sim m_a A^2 \sim T^3, \qquad (97)$$

and an energy density

$$\rho_a = m_a n_a. \qquad (98)$$

From these reasonings, we can estimate the current energy density of axions produced from these oscillations

$$\rho_0 \sim \left(m_a^2 A^2\right)_0 \sim m_{a0} \left(m_a A^2\right)_i \left(\frac{T_0}{\Lambda_{QCD}}\right)^3. \qquad (99)$$

m_{a0} it must be interpreted as the mass of the axion at zero temperature, given by $m_a = 0.6eV\left(10^7 GeV/f_a\right)$. Approximately it can be accepted that $m_{a0} \simeq \Lambda_{QCD}^2/f_a$. Taking m_{ai} from formula (99) and considering that $A_i \sim \langle a \rangle_i \sim f_a$, we find

$$\rho_0 \sim \frac{\Lambda_{QCD}^2}{f_a} \frac{\Lambda_{QCD}^2}{M_P} f_a^2 \left(\frac{T_0}{\Lambda_{QCD}}\right)^3 \sim f_a \frac{T_0^3 \Lambda_{QCD}}{M_P}, \qquad (100)$$

so we see that the relic axion density increases with f_a. A detailed calculation of these oscillations [52] yields an axion density

$$\Omega_a h^2 = C_a \left(\frac{f_a}{10^{12} GeV}\right)^{1.175} \theta_i^2, \qquad (101)$$

where C_a is a constant between 0.5 and 10.

Another source of non-thermal axion production is the decay of topological defects. As seen above, the spontaneous breaking of a global symmetry gives rise to a series of topological defects.

The last production mechanism to be discussed is thermal processes [53]. Taking into account the interactions of the axions with quarks and photons, in [54] it is found that the axions were in thermal contact with the photon bath if $f_a < 10^{18} GeV$ holds. As a conclusion, it is found that in the early Universe there is thermal production of axions if it is true that $f_a < 10^{12} GeV$.

11 Conclusions

We can conclude, from the Lagrangian density with a global symmetry, negative parameter $m^2 = -\mu^2$ and breaking the symmetry of the vacuum states by choosing a particular point among the states of symmetrically degenerate vacuum, it was found that a massless particle (Goldstone boson) appeared. This mechanism is called Goldstone's theorem. In general, massless Goldstone bosons appear depending on symmetry properties when a global symmetry is spontaneously broken.

Many examples of spontaneously broken global symmetries giving rise to Goldstone bosons can be found in nature. One of them is family symmetry. Said symmetry related to the number and properties of the families of the standard model.

The spontaneous breaking of symmetry of many theories predicts the existence of one or more types of topological defects. These objects are inherently nonperturbative and probably cannot be produced in the high-energy collisions that take place in terrestrial accelerators. Instead, they can be produced in phase transitions that occur in the early Universe. For this reason, these defects are considered to be traces of the first moments of the Universe. Although they are not minimum energy configurations, monopoles, strings, and domain walls (boundary domain) are topologically stable and their production is inevitable at transitions of cosmological phases. One of the most important consequences of these objects is that they crucially affect the evolution of the Universe. The concrete effects vary depending on the topological defect considered.

Finally, there are different mechanisms that cause the production of axions, which we will group into thermal and non-thermal mechanisms, the latter being the ones that predominate. Among the non-thermal mechanisms we can basically find two classes, those originated in the oscillations of the axion field and those that are produced from the decay of the topological defects generated in the spontaneous breaking of the symmetry PQ.

Appendix A: Sigma model

The Lagrangian density of the linear sigma model contains a set of N real scalar fields

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \phi^i\right)^2 + \frac{1}{2}\mu^2\left(\phi^i\right)^2 - \frac{\lambda}{4}\left[\left(\phi^i\right)^2\right], \quad (102)$$

where there is a sum at i in each factor of $\left(\phi^i\right)^2$. The Lagrangian density (102) is invariant under symmetry $\phi^i \rightarrow R_N^{ij}\phi^j$, for any matrix R_N, orthogonal and $N \times N$. The group of transformations $\phi^i \rightarrow R_N^{ij}\phi^j$, is the group of rotations in N dimensions, also called $O(N)$, N-dimensional orthogonal group.

The classical configuration of the field that represents the ground state, ϕ_0^i, is the one that minimizes the potential

$$V\left(\phi^i\right) = -\frac{1}{2}\mu^2\left(\phi^i\right)^2 + \frac{\lambda}{4}\left[\left(\phi^i\right)^2\right], \quad (103)$$

for any value ϕ_0^i that satisfies ϕ_0^i

$$\left(\phi_0^i\right)^2 = \frac{\mu^2}{\lambda}. \quad (104)$$

This condition only fixes the length of the vector ϕ_0^i, its direction being arbitrary. Without losing generality, we can choose that the vector ϕ_0^i points in the N-th direction

$$\phi_0^i = (0, 0, ..., 0, v) = \left(0, 0, ..., 0, \frac{\mu}{\sqrt{\lambda}}\right). \quad (105)$$

For $N = 2$, the potential $V\left(\phi^i\right)$ has the typical shape of the Mexican hat, where the minimum can be found at any point on a circle of radius v. It is convenient to redefine the field $\phi^i(x)$, introducing the fields $\pi^i(x)$ and $\eta(x)$

$$\phi^i(x) = \left(\pi^i(x), v + \eta(x)\right), \quad (106)$$

where $i = 1, 2, 3, ..., N-1$.

Introducing the change (106) in (102), the Lagrangian density takes after spontaneous symmetry breaking

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \pi^i\right)^2 + \frac{1}{2}\left(\partial_\mu \eta\right)^2 - \frac{1}{2}\left(2\mu^2\right)\eta^2 - \sqrt{\lambda}\mu\eta^3 - \sqrt{\lambda}\mu\left(\pi^i\right)^2 - \frac{1}{4}\lambda\eta^4 - \frac{1}{2}\lambda\left(\pi^i\right)^2\eta^2 - \frac{1}{4}\lambda\left[\left(\pi^i\right)^2\right]^2. \quad (107)$$

Analyzing the new Lagrangian density (107), we observe that the spectrum consists of a massive η field and a set of $N-1$ massless fields π^i. Symmetry $O(N)$ original is hidden, leaving the symmetry subgroup explicit $O(N-1)$, that transforms the fields π^i, that is, $\pi^i \rightarrow R_N^{ij}\pi^j$. The massive field η describes oscillations of the field ϕ_i in the radial direction, where the potential has a non-zero second derivative. The massless fields π^i describe oscillations of ϕ^i in the tangential directions, along the valley of the potential. This valley is a $(N-1)$-dimensional surface, where all $N-1$ directions have no slope and are equivalent, which shows the symmetry $O(N-1)$.

In the example used, initially we had $N(N-1)/2$ continuous symmetries, corresponding to the different orthogonal axes about which a rotation can be performed $O(N)$, in N dimensions. After the spontaneous break, the subgroup remained $O(N-1)$, containing $(N-1)(N-2)/2$ continuous symmetries. Thus, the difference $N-1$ will be the number of symmetries that have been broken. This is precisely the number of massless particles that have appeared in theory, as predicted by Goldstone's theorem.

In nature, one can find many examples of spontaneously broken global symmetries that give rise to Goldstone bosons. One of them is family symmetry. This symmetry related to the number and properties of the families of the standard model.

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