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Hamiltonian formalism of Bianchi models with cosmological constant

Mississippi Valenzuela *[1,2,3,4]

[1] *Universidad Nacional Autónoma de México, Instituto de Física, Departamento de Física Teórica, México*
[2] *Universidad Autónoma de Zacatecas, Unidad Académica de Ciencia y Tecnología de la Luz y la Materia*
[3] *Instituto de México Zacatecas por el Bien Común, Departamento de Preparatoria, Zacatecas, México*
[4] *Universidad Sierra Madre, Zacatecas, México*

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Abstract
Cosmic Background Radiation (CMB) has resulted in anomalies or deviations from the standard model of cosmology. Consequently, we propose to study Bianchi spacetimes with cosmological constant by applying the Arnowitt-Desser-Misnes (ADM) formalism of general relativity in the Hamiltonian version. From the Lagrangian density $\mathscr{L}$ and with the use of the Legendre transformation we can calculate the Hamiltonian density $\mathcal{H}$ and the Poisson parentheses. Subsequently, we present the equations of motion for each of the Bianchi spacetimes. In addition, we discuss some theoretical consequences in these equations when we take the limit $\Omega \to -\infty$ and the parameters β_+ and β_- fixed, consequently, we find that the dependent part of the gravitational potential of the Hamiltonian density tends to zero and from the equations of motion we find the constant of motion, $p_\Omega = p_{\beta_+} = p_{\beta_-} =$ constant. On the other hand, Friedmann-Lemaitre-Robertson-Walker (FLRW) models can be generalized only to some Bianchi models. The Bianchi type I and VII models are a generalization of the Euclidean FLRW model ($k = 0$), the Bianchi type IX for the spherical FLRW model ($k = 1$) and the Bianchi types V and VII are for the hyperbolic FLRW model ($k = -1$). The rest of the Bianchi models do not contain the FLRW models as a particular case.

Keywords: Cosmology, Bianchi models, ADM formalism.

Formalismo Hamiltoniano de los modelos de Bianchi con constante cosmológica

Resumen
La Radiación Cósmica de Fondo (CMB) ha dado como resultado anomalías o desviaciones con respecto al modelo estándar de la cosmología. En consecuencia, proponemos estudiar los espacio-tiempo de Bianchi con constante cosmológica aplicando el formalismo de Arnowitt-Deser-Misner (ADM) de relatividad general en su versión Hamiltoniana. A partir de la densidad Lagrangiana $\mathscr{L}$ y con el uso de la transformación de Legendre podemos calcular densidad Hamiltoniana $\mathcal{H}$ y los paréntesis de Poisson. Posteriormente, presentamos las ecuaciones de movimiento para cada uno de los espacio-tiempo de Bianchi. Además, discutimos algunas consecuencias de carácter teórico en dichas ecuaciones cuando tomamos el límite $\Omega \to -\infty$ y los parámetros β_+ y β_- fijos, en consecuencia, encontramos que la parte dependiente del potencial gravitacional de la densidad Hamiltoniana tiende a cero y de las ecuaciones de movimiento encontramos la constante de movimiento, $p_\Omega = p_{\beta_+} = p_{\beta_-} =$ constante. Por otro lado, los modelos cosmológicos Friedmann-Lemaitre-Robertson-Walker (FLRW) se pueden generalizar sólo a algunos modelos cosmológicos de Bianchi. Los modelos Bianchi tipo I y VII_0 son una generalización del modelo FLRW Euclídiano ($k = 0$), el Bianchi tipo IX para el modelo cosmológico FLRW esférico ($k = 1$) y los Bianchis tipo V y VII_h lo son para el modelo FLRW hiperbólico ($k = -1$). El resto de los modelos de Bianchi no contiene a los modelos FLRW como un caso particular.

Palabras clave: Cosmología, modelos de Bianchi, formalismo ADM.

*mvalenzuelalumat@uaz.edu.mx

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1 Introduction

The homogeneity and isotropy of cosmological models are related to the intrinsic symmetries of a variety. A very viable way to classify the different cosmological models is through their symmetries. Symmetries or isometries leave invariant to the metric tensor. The fields that generate these symmetries are called Killing vector fields [1–6]. These vector fields are defined as [7]:

$$\mathfrak{L}_X g_{\mu\nu} = 0. \tag{1}$$

Bianchi's cosmological models are homogeneous, therefore, these models have Killing vectors associated with this symmetry. However, Killing vectors satisfy the property:

$$[X_\mu, X_\nu] = C^\lambda_{\mu\nu} X_\lambda,$$

where $C^\lambda_{\mu\nu}$ are the structure constants. Bianchi models are classified according to the structure constants [8–11].

In this article the ADM formalism [12–22] of general relativity is applied to the different cases of the Bianchi models type A and B. These cosmological models are analyzed with cosmological constant. First, a general model for Bianchi's cosmological models is described; where we deduced the Lagrangian density $\mathscr{L}$. Subsequently, we develop the Hamiltonian density $\mathcal{H}$. Finally, the Hamiltonian density $\mathcal{H}$ is applied to deduce the dynamics of the Bianchi models.

2 General model

In Misner's notation, the metric of the Bianchi models can be written as [23]

$$ds^2 = -N^2 dt^2 + e^{2\Omega(t)} e^{2\beta_{ij}(t)} \omega^i \omega^j, \tag{2}$$

where $N(t)$ is the lapse function, ω^i called the differential 1-form, $e^{2\Omega(t)}$ is the scale factor of the Universe and β_{ij} determines the anisotropic parameters $\beta_+(t)$ and $\beta_-(t)$ as follows

$$\beta_{ij} = \begin{bmatrix} \beta_+ + \sqrt{3}\beta_- & 0 & 0 \\ 0 & \beta_+ - \sqrt{3}\beta_- & 0 \\ 0 & 0 & -2\beta_+ \end{bmatrix}. \tag{3}$$

In this general model of the Bianchi models, the shift function is not stipulated in the metric of equation (2), therefore in the subsequent developments for the Bianchi models that will be treated it will not appear as a dynamic variable. The term $h_{ij} = e^{2\Omega(t)} e^{\beta_{ij}(t)}$ of the second term of equation (2) is compared with the term g_{ab} of the ADM formalism of general relativity and we intuit that the dynamical variables for the Bianchis are Ω, β_+, β_-, since the lapse function with $N = \exp(3\Omega)$ is the physical norm.

The non-zero components of extrinsic curvature; using equations (2) and (3) and equation

$$K_{\mu\nu} = \frac{h^\rho_\mu h^\sigma_\nu}{2N} \left(\frac{\partial h_{\rho\sigma}}{\partial t} - \nabla_\rho N_\sigma - \nabla_\sigma N_\rho \right),$$

are given by:

$$K_{11} = \frac{1}{N} \left(\frac{d\Omega}{dt} + \frac{d\beta_+}{dt} + \sqrt{3}\frac{d\beta_-}{dt} \right) \exp\left[2\left(\Omega + \beta_+ + \sqrt{3}\beta_- \right) \right], \tag{4}$$

$$K_{22} = \frac{1}{N} \left(\frac{d\Omega}{dt} + \frac{d\beta_+}{dt} - \sqrt{3}\frac{d\beta_-}{dt} \right) \exp\left[2\left(\Omega + \beta_+ - \sqrt{3}\beta_- \right) \right], \tag{5}$$

$$K_{33} = \frac{1}{N} \left(\frac{d\Omega}{dt} - 2\frac{d\beta_+}{dt} \right) \exp\left[2\left(\Omega - 2\beta_+ \right) \right]. \tag{6}$$

The trace of the extrinsic curvature is given by

$$K = h^{ij} K_{ij} = -\frac{3}{N} \frac{d\Omega}{dt}. \tag{7}$$

Given

$$\sqrt{\det(h_{ij})} = \exp[3\Omega(t)],$$

and inserting equations (4), (5), (6) and (7) into the Einstein-Hilbert action of the ADM variables (see appendix A)

$$S[g_{ab}, N, N^a] = \int dt \int d^3x N \sqrt{\det(h_{ij})} \left({}^{(3)}R - K^2 + K_{\mu\nu}K^{\mu\nu} - \Lambda \right), \tag{8}$$

the Lagrangian density is expressed by

$$\mathscr{L} = \frac{6\exp(3\Omega)}{N} \left[-\left(\frac{d\Omega}{dt} \right)^2 + \left(\frac{d\beta_+}{dt} \right)^2 + \left(\frac{d\beta_-}{dt} \right)^2 \right] + N\exp(3\Omega)\left({}^{(3)}R - \Lambda \right) \tag{9}$$

The conjugate moments for the dynamic variables Ω, β_+, β_- are given by

$$p_\Omega = \frac{\partial \mathscr{L}}{\partial \dot{\Omega}} = -\frac{12}{N} \frac{d\Omega}{dt} \exp(3\Omega), \tag{10}$$

$$p_{\beta_+} = \frac{\partial \mathscr{L}}{\partial \dot{\beta}_+} = \frac{12}{N} \frac{d\beta_+}{dt} \exp(3\Omega), \tag{11}$$

$$p_{\beta_-} = \frac{\partial \mathscr{L}}{\partial \dot{\beta}_-} = \frac{12}{N} \frac{d\beta_-}{dt} \exp(3\Omega). \tag{12}$$

Using the Legendre transformation [24, 25], equation (9) and equations (10), (11) and (12); we calculate the Hamiltonian density from the equation

$$\mathcal{H} = p_\Omega \frac{d\Omega}{dt} + p_{\beta_+} \frac{d\beta_+}{dt} + p_{\beta_-} \frac{d\beta_-}{dt} - \mathscr{L},$$

resulting

$$\mathcal{H} = \frac{N}{24}\exp\left(-3\Omega\right)\left(-p_{\Omega}^{2} + p_{\beta_+}^{2} + p_{\beta_-}^{2}\right) - N\exp\left(3\Omega\right)\left({}^{(3)}R - \Lambda\right), \qquad (13)$$

where the three-dimensional curvature scalar is given by [26–29]

$${}^{(3)}R = C^{i}_{jk}C^{l}_{mn}h_{il}h^{km}h^{jn} + 2C^{i}_{jk}C^{k}_{li}h^{jl} + 4C^{i}_{ik}C^{j}_{jm}h^{km}, \qquad (14)$$

where C^{i}_{jk} are the structure constants and $h_{ij} = e^{2\Omega(t)}e^{\beta_{ij}(t)}$.

Equation (13) constitutes a Hamiltonian constraint in the ADM formalism of general relativity (see appendix A). Therefore, $\mathcal{H} \approx 0$ reproduces Einstein's field equations. The shift function N^a does not appear in equation (2), therefore, the equation $-2h_{ac}N^{c}D_{b}\pi^{ab} = N^{c}\mathcal{H}_{c}$ will not be considered, then there are not diffeomorphism constraints for the Bianchi models.

The classical Poisson brackets for the dynamic variables considered are

$$\begin{aligned} \{x_i, x_j\} &= 0,\\ \{p_i, p_j\} &= 0,\\ \{x_i, p_j\} &= \delta_{ij}, \end{aligned} \qquad (15)$$

where $x_i = \Omega, \beta_+, \beta_-$ and $p_i = p_\Omega, p_{\beta_+}, p_{\beta_-}$ with $i = 1, 2, 3$.

Next, we develop the formalism of the Bianchi models A and B.

3 Class A

Bianchi I

The constants of the Bianchi I are null, that is, $C^{i}_{jk} = 0$ [9]. Therefore; from equation (13) and using equation (14), the Hamiltonian density is expressed by the equation

$$\mathcal{H}_I = \frac{1}{24}\left(-p_{\Omega}^{2} + p_{\beta_+}^{2} + p_{\beta_-}^{2}\right) + \Lambda\exp\left(6\Omega\right), \qquad (16)$$

where $N = \exp(3\Omega)$. From equation (16) we find the equations of motion

$$\frac{d\Omega}{dt} = \{\Omega, \mathcal{H}_I\} = \frac{\partial\Omega}{\partial\Omega}\frac{\partial\mathcal{H}_I}{\partial p_\Omega} - \frac{\partial\mathcal{H}_I}{\partial\Omega}\frac{\partial\Omega}{\partial p_\Omega} = -\frac{1}{12}p_\Omega, \quad (17)$$

$$\frac{d\beta_+}{dt} = \{\beta_+, \mathcal{H}_I\} = \frac{\partial\beta_+}{\partial\beta_+}\frac{\partial\mathcal{H}_I}{\partial p_{\beta_+}} - \frac{\partial\mathcal{H}_I}{\partial\beta_+}\frac{\partial\beta_+}{\partial p_{\beta_+}} = \frac{1}{12}p_{\beta_+}, \qquad (18)$$

$$\frac{d\beta_-}{dt} = \{\beta_-, \mathcal{H}_I\} = \frac{\partial\beta_-}{\partial\beta_-}\frac{\partial\mathcal{H}_I}{\partial p_{\beta_-}} - \frac{\partial\mathcal{H}_I}{\partial\beta_-}\frac{\partial\beta_-}{\partial p_{\beta_-}} = \frac{1}{12}p_{\beta_-}, \qquad (19)$$

$$\frac{dp_\Omega}{dt} = \{p_\Omega, \mathcal{H}_I\} = \frac{\partial p_\Omega}{\partial\Omega}\frac{\partial\mathcal{H}_I}{\partial p_\Omega} - \frac{\partial\mathcal{H}_I}{\partial\Omega}\frac{\partial p_\Omega}{\partial p_\Omega} = -6\Lambda\exp\left(3\Omega\right), \qquad (20)$$

$$\frac{dp_{\beta_+}}{dt} = \{p_{\beta_+}, \mathcal{H}_I\} = \frac{\partial p_{\beta_+}}{\partial\beta_+}\frac{\partial\mathcal{H}_I}{\partial p_{\beta_+}} - \frac{\partial\mathcal{H}_I}{\partial\beta_+}\frac{\partial p_{\beta_+}}{\partial p_{\beta_+}} = 0, \qquad (21)$$

$$\frac{dp_{\beta_-}}{dt} = \{p_{\beta_-}, \mathcal{H}_I\} = \frac{\partial p_{\beta_-}}{\partial\beta_-}\frac{\partial\mathcal{H}_I}{\partial p_{\beta_-}} - \frac{\partial\mathcal{H}_I}{\partial\beta_-}\frac{\partial p_{\beta_-}}{\partial p_{\beta_-}} = 0. \qquad (22)$$

Integrating the ordinary differential equations (21) and (22), we obtain the results

$$p_{\beta_+} = p_{0\beta_+} = constante, \qquad (23)$$

$$p_{\beta_-} = p_{0\beta_-} = constante, \qquad (24)$$

and from equations (18) and (19)

$$\beta_+ = \frac{1}{12}p_{0\beta_+}t + \beta_{0+}, \qquad (25)$$

$$\beta_- = \frac{1}{12}p_{0\beta_-}t + \beta_{0-}. \qquad (26)$$

To integrate equations (17) and (20), we note that

$$\frac{dp_\Omega}{dt} = 4A^2 - \frac{1}{4}p_\Omega^2, \qquad (27)$$

where A is a constant. The integration of equation (27) depends on the sign of the cosmological constant. If $\Lambda > 0$, then from equation (20) we have $\frac{dp_\Omega}{dt} < 0$. The integration of the differential equation (27) gives

$$\ln\left(\frac{4A + p_\Omega}{4A - p_\Omega}\right) = 2A\left(t + t_0\right),$$

then p_Ω is

$$p_\Omega = 4A\tanh\left[A\left(t + t_0\right)\right]. \qquad (28)$$

The integration of equations (17) and (20) generates

$$\Omega\left(t\right) = \frac{1}{6}\ln\left\{\frac{2A^2}{-3\Lambda}\left[1 - \tanh^2 A\left(t + t_0\right)\right]\right\}. \qquad (29)$$

If $\Lambda < 0$, then from equation (20) we have $\frac{dp_\Omega}{dt} > 0$. The integration of the differential equation (27) generates

$$p_\Omega = 4A\coth\left[A\left(t + t_0\right)\right]. \qquad (30)$$

The integration of equations (17) and (20) leads to

$$\Omega(t) = \frac{1}{6}\ln\left\{\frac{2A^2}{3\Lambda}\left[\coth^2 A(t+t_0) - 1\right]\right\}. \qquad (31)$$

Finally, for the Bianchi I with null cosmological constant, $\Lambda = 0$, we have that $\frac{dp_\Omega}{dt} = 0$ and the equation of motion for Ω is similar to β_+ and β_-

$$p_\Omega = p_{0\Omega}, \quad \Omega(t) = -\tfrac{1}{12}p_{0\Omega}t + \Omega_0. \qquad (32)$$

For a non-zero cosmological constant, $\Lambda \neq 0$, these solutions contain a singularity in $\Omega \to -\infty \left(e^{2\Omega} \to 0\right)$ as $t \to 0$.

Bianchi II

The structure constants of the Bianchi II are [9]

$$C^1_{23} = -C^1_{32} = 1.$$

Using the structure constants and equation (14), the curvature scalar is determined by

$$^{(3)}R_{II} = -2\exp\left(-2\Omega + 4\beta_+ + 4\sqrt{3}\beta_-\right). \qquad (33)$$

Introducing equation (14) into equation (13), the Hamiltonian density for the Bianchi II model is

$$\mathcal{H}_{II} = \frac{1}{24}\left(-p^2_\Omega + p^2_{\beta_+} + p^2_{\beta_-}\right) + 2\exp\left(4\Omega + 4\beta_+ + 4\sqrt{3}\beta_-\right) + \Lambda\exp(6\Omega), \qquad (34)$$

where $N = \exp(3\Omega)$.

From equation (34), we can obtain the equations of motion

$$\frac{d\Omega}{dt} = \{\Omega, \mathcal{H}_{II}\} = \frac{\partial\mathcal{H}_{II}}{\partial p_\Omega} = -\frac{1}{12}p_\Omega, \qquad (35)$$

$$\frac{d\beta_+}{dt} = \{\beta_+, \mathcal{H}_{II}\} = \frac{\partial\mathcal{H}_{II}}{\partial p_{\beta_+}} = \frac{1}{12}p_{\beta_+}, \qquad (36)$$

$$\frac{d\beta_-}{dt} = \{\beta_-, \mathcal{H}_{II}\} = \frac{\partial\mathcal{H}_{II}}{\partial p_{\beta_-}} = \frac{1}{12}p_{\beta_-}, \qquad (37)$$

$$\frac{dp_\Omega}{dt} = \{p_\Omega, \mathcal{H}_{II}\} = -8\exp\left(4\Omega + 4\beta_+ + 4\sqrt{3}\beta_-\right) - 6\Lambda\exp(6\Omega), \qquad (38)$$

$$\frac{dp_{\beta_+}}{dt} = \left\{p_{\beta_+}, \mathcal{H}_{II}\right\} = -\frac{\partial\mathcal{H}_{II}}{\partial\beta_+} = -8\exp\left(4\Omega + 4\beta_+ + 4\sqrt{3}\beta_-\right), \qquad (39)$$

$$\frac{dp_{\beta_-}}{dt} = \left\{p_{\beta_-}, \mathcal{H}_{II}\right\} = -\frac{\partial\mathcal{H}_{II}}{\partial\beta_-} = -8\sqrt{3}\exp\left(4\Omega + 4\beta_+ + 4\sqrt{3}\beta_-\right). \qquad (40)$$

The dynamics of the Bianchi II are considered below. Assuming fixed anisotropic parameters β_+ and β_-, Consequently, the term $\exp\left(\Omega + 4\beta_+ + 4\sqrt{3}\beta_-\right) \to 0$ conforming $\Omega \to -\infty$. From the preceding consideration and from equations (38), (39), and (40) taking into account that $\Omega \to -\infty$, it results that each conjugate moment is constant. Additionally, let us observe the term that contains the cosmological constant tends to zero if $\Omega \to -\infty$.

Bianchi VI$_0$

The Bianchi VI$_0$ has the structure constants [26]

$$C^1_{23} = -C^1_{32} = 1, \quad C^2_{31} = -C^2_{13} = -1.$$

With the previous structure constants and using equation (14) the curvature scalar is

$$^{(3)}R_{VI_0} = -4\exp(-2\Omega + 4\beta_+)\left[\cosh\left(4\sqrt{3}\beta_-\right) + 1\right]. \qquad (41)$$

If we use equation (41), equation (13) becomes

$$\mathcal{H}_{VI_0} = \frac{1}{24}\left(-p^2_\Omega + p^2_{\beta_+} + p^2_{\beta_-}\right) + 4\exp(4\Omega + 4\beta_+) \times \left[\cosh\left(4\sqrt{3}\beta_-\right) + 1\right] + \Lambda\exp(6\Omega). \qquad (42)$$

With this Hamiltonian density we can write the equations of motion

$$\frac{d\Omega}{dt} = \{\Omega, \mathcal{H}_{VI_0}\} = \frac{\partial\mathcal{H}_{VI_0}}{\partial p_\Omega} = -\frac{1}{12}p_\Omega, \qquad (43)$$

$$\frac{d\beta_+}{dt} = \{\beta_+, \mathcal{H}_{VI_0}\} = \frac{\partial\mathcal{H}_{VI_0}}{\partial p_{\beta_+}} = \frac{1}{12}p_{\beta_+}, \qquad (44)$$

$$\frac{d\beta_-}{dt} = \{\beta_-, \mathcal{H}_{VI_0}\} = \frac{\partial\mathcal{H}_{VI_0}}{\partial p_{\beta_-}} = \frac{1}{12}p_{\beta_-}, \qquad (45)$$

$$\frac{dp_\Omega}{dt} = \{p_\Omega, \mathcal{H}_{VI_0}\} = -16\exp(4\Omega + 4\beta_+) \times \left[\cosh\left(4\sqrt{3}\beta_-\right) + 1\right] - 6\Lambda\exp(6\Omega), \qquad (46)$$

$$\frac{dp_{\beta_+}}{dt} = \left\{p_{\beta_+}, \mathcal{H}_{VI_0}\right\} = -\frac{\partial\mathcal{H}_{VI_0}}{\partial\beta_+} = -16\exp(4\Omega + 4\beta_+)\left[\cosh\left(4\sqrt{3}\beta_-\right) + 1\right], \qquad (47)$$

$$\frac{dp_{\beta_-}}{dt} = \left\{p_{\beta_-}, \mathcal{H}_{VI_0}\right\} = -\frac{\partial \mathcal{H}_{VI_0}}{\partial \beta_-} = -16\sqrt{3}\exp\left(4\Omega + 4\beta_+\right)\sinh\left(4\sqrt{3}\beta_-\right). \quad (48)$$

Assuming the anisotropic parameters β_+ and β_- are fixed in the second term of equation (42), consequently, the last term of equation (42) tends to 0, according to $\Omega \to -\infty$, where each conjugate moment is constant, therefore $p_\Omega = p_{\beta_+} = p_{\beta_-} = constant$. Just as the term containing the cosmological constant tends to zero if $\Omega \to -\infty$.

Bianchi VII$_0$

The Bianchi VII$_0$ has structure constants given by [9,26]:

$$C^1_{23} = -C^1_{32} = -1,$$
$$C^2_{31} = -C^2_{13} = -1.$$

With the previous structure constants and equation (14), we found that the curvature scalar is expressed by:

$$^{(3)}R_{VII_0} = -4\exp\left(-2\Omega + 4\beta_+\right)\left[\cosh\left(4\sqrt{3}\beta_-\right) - 1\right]. \quad (49)$$

If we use equation (49), equation (13) becomes

$$\mathcal{H}_{VII_0} = \frac{1}{24}\left(-p^2_\Omega + p^2_{\beta_+} + p^2_{\beta_-}\right) + 4\exp\left(4\Omega + 4\beta_+\right) \times \left[\cosh\left(4\sqrt{3}\beta_-\right) - 1\right] + \Lambda\exp\left(6\Omega\right). \quad (50)$$

With this Hamiltonian density; that is, equation (43), we can write the equations of motion

$$\frac{d\Omega}{dt} = \left\{\Omega, \mathcal{H}_{VII_0}\right\} = \frac{\partial \mathcal{H}_{VII_0}}{\partial p_\Omega} = -\frac{1}{12}p_\Omega, \quad (51)$$

$$\frac{d\beta_+}{dt} = \left\{\beta_+, \mathcal{H}_{VII_0}\right\} = \frac{\partial \mathcal{H}_{VII_0}}{\partial p_{\beta_+}} = \frac{1}{12}p_{\beta_+}, \quad (52)$$

$$\frac{d\beta_-}{dt} = \left\{\beta_-, \mathcal{H}_{VII_0}\right\} = \frac{\partial \mathcal{H}_{VII_0}}{\partial p_{\beta_-}} = \frac{1}{12}p_{\beta_-}, \quad (53)$$

$$\frac{dp_\Omega}{dt} = \left\{p_\Omega, \mathcal{H}_{VII_0}\right\} = -16\exp\left(\Omega + 4\beta_+\right) \times \left[\cosh\left(4\sqrt{3}\beta_-\right) - 1\right] - \Lambda\exp\left(6\Omega\right), \quad (54)$$

$$\frac{dp_{\beta_+}}{dt} = \left\{p_{\beta_+}, \mathcal{H}_{VII_0}\right\} = -\frac{\partial \mathcal{H}_{VII_0}}{\partial \beta_+} = -16\exp\left(\Omega + 4\beta_+\right)\left[\cosh\left(4\sqrt{3}\beta_-\right) - 1\right], \quad (55)$$

$$\frac{dp_{\beta_-}}{dt} = \left\{p_{\beta_-}, \mathcal{H}_{VII_0}\right\} = -\frac{\partial \mathcal{H}_{VII_0}}{\partial \beta_-} = -16\sqrt{3}\exp\left(\Omega + 4\beta_+\right)\sinh\left(4\sqrt{3}\beta_-\right). \quad (56)$$

In this model we assume the anisotropic parameters β_+ and β_- are fixed, consequently, equation (50) tends to 0, according to $\Omega \to -\infty$ and each conjugate moment is constant $p_\Omega = p_{\beta_+} = p_{\beta_-} = constant$.

Bianchi VIII

For this model the structure constants are [9,26]

$$C^1_{23} = -C^1_{32} = -1,$$
$$C^2_{31} = -C^2_{13} = -1,$$
$$C^3_{12} = -C^3_{21} = 1.$$

Using these structure constants and inserting them into equation (14), the scalar of curvature is given by the scalar equation

$$^{(3)}R_{VIII} = -4\exp\left(-2\Omega + 4\beta_+\right)\cosh\left(4\sqrt{3}\beta_+\right) - 2\exp\left(-2\Omega - 8\beta_+\right) - 4\exp\left(-2\Omega + 4\beta_+\right) + 8\exp\left(-2\Omega - 2\beta_+\right)\cosh\left(2\sqrt{3}\beta_-\right), \quad (57)$$

and the Hamiltonian density is

$$\mathcal{H}_{VIII} = \frac{1}{24}\left(-p^2_\Omega + p^2_{\beta_+} + p^2_{\beta_-}\right) + \exp\left(4\Omega\right)\left[W\left(\beta_+, \beta_-\right) - 1\right] + \Lambda\exp\left(6\Omega\right), \quad (58)$$

where

$$W\left(\beta_+, \beta_-\right) = 1 + 4e^{4\beta_+}\cosh\left(4\sqrt{3}\beta_+\right) + 2e^{-8\beta_+} - 8e^{-2\beta_+}\cosh\left(2\sqrt{3}\beta_-\right) + 4e^{4\beta_+}.$$

From equation (58) we find that the equations of motion are:

$$\frac{d\Omega}{dt} = \left\{\Omega, \mathcal{H}_{VIII}\right\} = \frac{\partial \mathcal{H}_{VIII}}{\partial p_\Omega} = -\frac{1}{12}p_\Omega, \quad (59)$$

$$\frac{d\beta_+}{dt} = \left\{\beta_+, \mathcal{H}_{VIII}\right\} = \frac{\partial \mathcal{H}_{VIII}}{\partial p_{\beta_+}} = \frac{1}{12}p_{\beta_+}, \quad (60)$$

$$\frac{d\beta_-}{dt} = \left\{\beta_-, \mathcal{H}_{VIII}\right\} = \frac{\partial \mathcal{H}_{VIII}}{\partial p_{\beta_-}} = \frac{1}{12}p_{\beta_-}, \quad (61)$$

$$\frac{dp_\Omega}{dt} = \left\{p_\Omega, \mathcal{H}_{VIII}\right\} = -4\exp\left(4\Omega\right)\left[W\left(\beta_+, \beta_-\right) - 1\right] - 6\Lambda\exp\left(6\Omega\right), \quad (62)$$

$$\frac{dp_{\beta_+}}{dt} = \left\{p_{\beta_+}, \mathcal{H}_{VIII}\right\} = -\frac{\partial \mathcal{H}_{VIII}}{\partial \beta_+} = -4\exp\left(4\Omega\right)\frac{\partial W}{\partial \beta_+} \quad (63)$$

$$\frac{dp_{\beta_-}}{dt} = \left\{p_{\beta_-}, \mathcal{H}_{VIII}\right\} = -\frac{\partial \mathcal{H}_{VIII}}{\partial \beta_-} = -4\exp\left(4\Omega\right)\frac{\partial W}{\partial \beta_-}. \tag{64}$$

The dynamics of the Bianchi VIII cosmological model can be seen as the dynamics of a particle in a time-dependent potential. The simplest movements are obtained by assuming the anisotropic parameters β_+ and β_- fixed, thus the last term of equation (58) that contains $W\left(\beta_+, \beta_-\right)$ tends to zero in the limit $\Omega \to -\infty$, as well as the term containing the cosmological constant. From the preceding consideration and equations (62), (63) and (64) we follows that each conjugate moment is constant, i.e. $p_\Omega = p_{\beta_+} = p_{\beta_-} = constant$.

For large values of β of $W\left(\beta_+, \beta_-\right)$, we can find that in the limit $\beta_+ \to -\infty$ the value of $W\left(\beta_+, \beta_-\right)$ of equation (58) behaves as

$$W\left(\beta_+ \to -\infty, \beta_-\right) \sim 2\exp\left(-8\beta_+\right)$$
$$-8\exp\left(-2\beta_+\right)\cosh\left(2\sqrt{3}\beta_-\right),$$

and for the limit $\beta \to +\infty$ taking into account $\beta_- \ll 1$, the anisotropic potential behaves in the way

$$W\left(\beta_+ \to +\infty, \beta_-\right) \sim 1 + 4\left(2 + 24\beta_-^2\right)\exp\left(4\beta_+\right).$$

Bianchi IX

This model has the structure constants [9, 26, 30]

$$C^1_{23} = -C^1_{32} = 1,$$
$$C^2_{31} = -C^2_{13} = 1,$$
$$C^3_{12} = -C^3_{21} = 1.$$

If we substitute these structure constants into equation (14) we obtain the three-dimensional curvature scalar

$$\begin{gathered} {}^{(3)}R_{IX} = -2\exp\left(-2\Omega - 8\beta_+\right) + \\ 8\exp\left(-2\Omega - 2\beta_+\right)\cosh\left(2\sqrt{3}\beta_-\right) \\ -4\exp\left(-2\Omega + 4\beta_+\right)\left[\cosh\left(4\sqrt{3}\beta_+\right) + 1\right] \end{gathered} \tag{65}$$

and then we substitute equation (65) into (13) to obtain

$$\begin{gathered} \mathcal{H}_{IX} = \frac{1}{24}\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) + \\ \exp\left(4\Omega\right)\left[V\left(\beta_+, \beta_-\right) - 1\right] + \Lambda\exp\left(6\Omega\right), \end{gathered} \tag{66}$$

where

$$V\left(\beta_+, \beta_-\right) = 1 + 2e^{-8\beta_+} - 8e^{-2\beta_+}\cosh\left(2\sqrt{3}\beta_-\right) +$$
$$4e^{4\beta_+}\left[\cosh\left(4\sqrt{3}\beta_-\right) + 1\right].$$

With equation (66) we can write the equations of motion as:

$$\frac{d\Omega}{dt} = \left\{\Omega, \mathcal{H}_{IX}\right\} = \frac{\partial \mathcal{H}_{IX}}{\partial p_\Omega} = -\frac{1}{12}p_\Omega, \tag{67}$$

$$\frac{d\beta_+}{dt} = \left\{\beta_+, \mathcal{H}_{IX}\right\} = \frac{\partial \mathcal{H}_{IX}}{\partial p_{\beta_+}} = \frac{1}{12}p_{\beta_+}, \tag{68}$$

$$\frac{d\beta_-}{dt} = \left\{\beta_-, \mathcal{H}_{IX}\right\} = \frac{\partial \mathcal{H}_{IX}}{\partial p_{\beta_-}} = \frac{1}{12}p_{\beta_-}, \tag{69}$$

$$\begin{gathered} \frac{dp_\Omega}{dt} = \left\{p_\Omega, \mathcal{H}_{IX}\right\} = -\frac{\partial \mathcal{H}_{IX}}{\partial \Omega} = \\ -4\exp\left(4\Omega\right)\left[V\left(\beta_+, \beta_-\right) - 1\right] - \Lambda\exp\left(6\Omega\right), \end{gathered} \tag{70}$$

$$\frac{dp_{\beta_+}}{dt} = \left\{p_{\beta_+}, \mathcal{H}_{IX}\right\} = -\frac{\partial \mathcal{H}}{\partial \beta_+} = -4\exp\left(4\Omega\right)\frac{\partial V}{\partial \beta_+} \tag{71}$$

$$\frac{dp_{\beta_-}}{dt} = \left\{p_{\beta_-}, \mathcal{H}_{IX}\right\} = -\frac{\partial \mathcal{H}}{\partial \beta_-} = -4\exp\left(\Omega\right)\frac{\partial V}{\partial \beta_-} \tag{72}$$

The dynamics of the Bianchi IX model can be seen as that of a particle in a time-dependent potential. The simple movements are obtained by assuming the anisotropic parameters β_+ and β_- fixed and the last term of equation (66) containing the anisotropic potential $V\left(\beta_+, \beta_-\right)$ is negligible, according to $\Omega \to -\infty$, where it turns out that each conjugate moment is constant.

From the preceding limit in the Hamiltonian constriction (66) we find that the conjugate moments are constant. Another viable way to verify such a statement, we take the limit when $\Omega \to -\infty$ in equations (70), (71) and (72), and consequently $p_\Omega = p_{\beta_+} = p_{\beta_-} = constant$.

For the asymptotic description; That is, for large β, we can be found that in the limit $\beta_+ \to -\infty$ the value of the anisotropic potential of equation (66) behaves as

$$V\left(\beta_+ \to -\infty, \beta_-\right) \sim 2\exp\left(-8\beta_+\right) -$$

$$8\exp\left(-2\beta_+\right)\cosh\left(2\sqrt{3}\beta_-\right),$$

and finally for the opposite case, in addition to $\beta_- \ll 1$, the anisotropic potential behaves as

$$V\left(\beta_+ \to +\infty, \beta_-\right) \sim 1 + 96\beta_-^2\exp\left(4\beta_+\right).$$

4 Class B

Bianchi III

In the Bianchi III model we have [9,31]

$$C^1_{13} = -C^1_{31} = 1.$$

Using these structure constants and equation (14), the curvature scalar is determined by

$${}^{(3)}R_{III} = 2C^1_{13}C^1_{31}h_{11}h^{33}h^{11} + 2C^1_{31}C^1_{31}h^{33} + 4C^i_{ik}C^j_{jm}h^{km}$$

and we find

$${}^{(3)}R_{III} = 4\exp\left(-2\Omega + 4\beta_+\right). \tag{73}$$

If we take equation (73) and substitute it into equation (13), we find the Hamiltonian density

$$\mathcal{H}_{III} = \frac{1}{24}\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) - 4\exp\left(4\Omega + 4\beta_+\right) + \Lambda\exp\left(6\Omega\right). \tag{74}$$

With this Hamiltonian density we can write the equations of motion:

$$\frac{d\Omega}{dt} = \{\Omega, \mathcal{H}_{III}\} = \frac{\partial\mathcal{H}_{III}}{\partial p_\Omega} = -\frac{1}{12}p_\Omega, \tag{75}$$

$$\frac{d\beta_+}{dt} = \{\beta_+, \mathcal{H}_{III}\} = \frac{\partial\mathcal{H}_{III}}{\partial p_{\beta_+}} = \frac{1}{12}p_{\beta_+}, \tag{76}$$

$$\frac{d\beta_-}{dt} = \{\beta_-, \mathcal{H}_{III}\} = \frac{\partial\mathcal{H}_{III}}{\partial p_{\beta_-}} = \frac{1}{12}p_{\beta_-}, \tag{77}$$

$$\frac{dp_\Omega}{dt} = -\frac{\partial\mathcal{H}_{III}}{\partial\Omega} = 16\exp\left(4\Omega + 4\beta_+\right) - 6\Lambda\exp\left(6\Omega\right), \tag{78}$$

$$\frac{dp_{\beta_+}}{dt} = \left\{p_{\beta_+}, \mathcal{H}_{III}\right\} = -\frac{\partial\mathcal{H}_{III}}{\partial\beta_+} = 16\exp\left(4\Omega + 4\beta_+\right), \tag{79}$$

$$\frac{dp_{\beta_-}}{dt} = \left\{p_{\beta_-}, \mathcal{H}_{III}\right\} = -\frac{\partial\mathcal{H}_{III}}{\partial\beta_-} = 0. \tag{80}$$

Assuming the anisotropic parameters β_+ and β_- fixed, consequently, the last term of equation (74) tends to zero, according to $\Omega \to -\infty$. From the above it follows that each conjugate moment is constant. Since equations (78), (79) and (80) tend to zero as $\Omega \to -\infty$, therefore $p_\Omega = p_{\beta_+} = p_{\beta_-} = constant$ [32].

Bianchi IV

This cosmological model has the structure constants [9,31]

$$C^1_{13} = -C^1_{31} = 1,$$
$$C^1_{23} = -C^1_{32} = 1,$$
$$C^2_{23} = -C^2_{32} = 1.$$

Using equation (14) we obtain

$${}^{(3)}R_{IV} = 2C^1_{23}C^1_{32}h_{11}h^{33}h^{22} + 4C^i_{ik}C^j_{jm}h^{km},$$

where the scalar of intrinsic curvature is given by

$${}^{(3)}R_{IV} = -2\exp\left(-2\Omega + 4\beta_+ + 4\sqrt{3}\beta_-\right) + 8\exp\left(-2\Omega + 4\beta_+\right). \tag{81}$$

Using equations (81) and (13), we find the Hamiltonian density

$$\mathcal{H}_{IV} = \frac{1}{24}\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2\right) + 2\exp\left(4\Omega + 4\beta_+ + 4\sqrt{3}\beta_-\right) - 8\exp\left(4\Omega + 4\beta_+\right) + \Lambda\exp\left(6\Omega\right) \tag{82}$$

From equation (82), we can write the equations of motion

$$\frac{d\Omega}{dt} = \{\Omega, \mathcal{H}_{IV}\} = \frac{\partial\mathcal{H}_{IV}}{\partial p_\Omega} = -\frac{1}{12}p_\Omega, \tag{83}$$

$$\frac{d\beta_+}{dt} = \{\beta_+, \mathcal{H}_{IV}\} = \frac{\partial\mathcal{H}_{IV}}{\partial p_{\beta_+}} = \frac{1}{12}p_{\beta_+}, \tag{84}$$

$$\frac{d\beta_-}{dt} = \{\beta_-, \mathcal{H}_{IV}\} = \frac{\partial\mathcal{H}_{IV}}{\partial p_{\beta_-}} = \frac{1}{12}p_{\beta_-}, \tag{85}$$

$$\frac{dp_\Omega}{dt} = \{p_\Omega, \mathcal{H}_{IV}\} = -\frac{\partial\mathcal{H}_{IV}}{\partial\Omega} = -8\left[\exp\left(4\sqrt{3}\beta_-\right) - 4\right] \times \exp\left(\Omega + 4\beta_+\right) - \Lambda\exp\left(6\Omega\right) \tag{86}$$

$$\frac{dp_{\beta_+}}{dt} = \left\{p_{\beta_+}, \mathcal{H}_{IV}\right\} = -\frac{\partial\mathcal{H}_{IV}}{\partial\beta_+} = -8\left[\exp\left(4\sqrt{3}\beta_-\right) - 4\right]\exp\left(\Omega + 4\beta_+\right), \tag{87}$$

$$\frac{dp_{\beta_-}}{dt} = \left\{p_{\beta_-}, \mathcal{H}_{IV}\right\} = -\frac{\partial\mathcal{H}_{IV}}{\partial\beta_-} = -8\sqrt{3}\exp\left(\Omega + 4\beta_+ + 4\sqrt{3}\beta_-\right). \tag{88}$$

Assuming the anisotropic parameters β_+ and β_- fixed, consequently, the last two terms of equation (82) tend to zero as $\Omega \to -\infty$; in other words, the last two terms of equation (82) become very small if Ω becomes very large. Taking into consideration the previous analysis, from equations (86), (87) and (88) we find that $\frac{dp_\Omega}{dt} = \frac{dp_{\beta_+}}{dt} = \frac{dp_{\beta_-}}{dt} = 0$ as $\Omega \to -\infty$; so $p_\Omega = p_{\beta_+} = p_{\beta_-} = constant$.

Bianchi V

This cosmological model is characterized by the structure constants [9, 31]

$$C^1_{13} = -C^1_{31} = 1,\quad C^2_{23} = -C^2_{32} = 1.$$

Using equation (14), we find

$$^{(3)}R_V = 8\exp(-2\Omega + 4\beta_+)\,. \tag{89}$$

If we substitute equation (89) into equation (13) we find the Hamiltonian density

$$\mathcal{H}_V = \frac{1}{24}\left(-p^2_\Omega + p^2_{\beta_+} + p^2_{\beta_-}\right) - 8\exp(4\Omega + 4\beta_+) + \Lambda\exp(6\Omega)\,. \tag{90}$$

Using equation (90), we find

$$\frac{d\Omega}{dt} = \{\Omega, \mathcal{H}_V\} = \frac{\partial\mathcal{H}_V}{\partial p_\Omega} = -\frac{1}{12}p_\Omega, \tag{91}$$

$$\frac{d\beta_+}{dt} = \{\beta_+, \mathcal{H}_V\} = \frac{\partial\mathcal{H}_V}{\partial p_{\beta_+}} = \frac{1}{12}p_{\beta_+}, \tag{92}$$

$$\frac{d\beta_-}{dt} = \{\beta_-, \mathcal{H}_V\} = \frac{\partial\mathcal{H}_V}{\partial p_{\beta_-}} = \frac{1}{12}p_{\beta_-}, \tag{93}$$

$$\frac{dp_\Omega}{dt} = \{p_\Omega, \mathcal{H}_V\} = -32\exp(4\Omega + 4\beta_+) - 6\Lambda\exp(6\Omega)\,, \tag{94}$$

$$\frac{dp_{\beta_+}}{dt} = \left\{p_{\beta_+}, \mathcal{H}_V\right\} = -\frac{\partial\mathcal{H}_V}{\partial\beta_+} = -32\exp(4\Omega + 4\beta_+)\,, \tag{95}$$

$$\frac{dp_{\beta_-}}{dt} = \left\{p_{\beta_-}, \mathcal{H}_V\right\} = -\frac{\partial\mathcal{H}_V}{\partial\beta_-} = 0. \tag{96}$$

Assuming the anisotropic parameter β_+ fixed, consequently, the second term of equation (90) tends to zero, according to $\Omega \to -\infty$. Since equations (94), (95) and (96) tend to zero as $\Omega \to -\infty$, then we have the result $p_\Omega = p_{\beta_+} = p_{\beta_-} = constant$.

Bianchi VI$_h$

In the Bianchi VI$_h$ [9, 31]

$$C^1_{23} = -C^1_{32} = 1,\quad C^2_{31} = -C^2_{13} = -1$$
$$C^1_{13} = -C^1_{31} = 1,\quad C^2_{23} = -C^2_{32} = h.$$

With the previous structure constants substituting them into equation (14), we find

$$^{(3)}R_{VI_h} = 2C^1_{23}C^1_{32}h_{11}h^{33}h^{22} + 2C^2_{13}C^2_{31}h_{22}h^{11}h^{33} + 4C^1_{32}C^2_{31}h^{33} + 4\left[\left(C^1_{13}\right)^2 + \left(C^2_{23}\right)^2 + 2C^1_{13}C^2_{23}\right]h^{33},$$

or

$$^{(3)}R_{VI_h} = -4\exp(-2\Omega + 4\beta_+)\left[\cosh\left(4\sqrt{3}\beta_-\right) - 1\right] + 4(1+h)^2\exp(-2\Omega + 4\beta_+) \tag{97}$$

From equation (96), we find the Hamiltonian density

$$\mathcal{H}_{VI_h} = \tfrac{1}{24}\left(-p^2_\Omega + p^2_{\beta_+} + p^2_{\beta_-}\right) + 4\exp(4\Omega + 4\beta_+)\left[\cosh\left(4\sqrt{3}\beta_-\right) - 1\right] - 4(1+h)^2\exp(4\Omega + 4\beta_+) + \Lambda\exp(6\Omega)\,. \tag{98}$$

With this Hamiltonian density, we can write the equations of motion

$$\frac{d\Omega}{dt} = \{\Omega, \mathcal{H}_{VI_h}\} = \frac{\partial\mathcal{H}_{VI_h}}{\partial p_\Omega} = -\frac{1}{12}p_\Omega, \tag{99}$$

$$\frac{d\beta_+}{dt} = \{\beta_+, \mathcal{H}_{VI_h}\} = \frac{\partial\mathcal{H}_{VI_h}}{\partial p_{\beta_+}} = \frac{1}{12}p_{\beta_+}, \tag{100}$$

$$\frac{d\beta_-}{dt} = \{\beta_-, \mathcal{H}_{VI_h}\} = \frac{\partial\mathcal{H}_{VI_h}}{\partial p_{\beta_-}} = \frac{1}{12}p_{\beta_-}, \tag{101}$$

$$\tfrac{dp_\Omega}{dt} = -\tfrac{\partial\mathcal{H}_{VI_h}}{\partial\Omega} = -6\Lambda\exp(6\Omega) - 16\exp(4\Omega + 4\beta_+)\cosh\left(4\sqrt{3}\beta_-\right) + 16\exp(4\Omega + 4\beta_+) + 16(1+h)^2\exp(4\Omega + 4\beta_+)\,, \tag{102}$$

$$\frac{dp_{\beta_+}}{dt} = -\frac{\partial\mathcal{H}_{VI_h}}{\partial\beta_+} = -16\exp(\Omega + 4\beta_+)\times\left[\cosh\left(4\sqrt{3}\beta_-\right) - 1\right] + 16(1+h)^2\exp(4\Omega + 4\beta_+) \tag{103}$$

$$\frac{dp_{\beta_-}}{dt} = \left\{p_{\beta_-}, \mathcal{H}_{VI_h}\right\} = -\frac{\partial\mathcal{H}_{VI_h}}{\partial\beta_-} = -16\sqrt{3}\exp(4\Omega + 4\beta_+)\sinh\left(4\sqrt{3}\beta_-\right)\,. \tag{104}$$

If we assume the anisotropic parameters β_+ and β_- fixed, consequently, the last two terms of equation (98) tend to zero as $\Omega \to -\infty$. Furthermore, from equations (102), (103) and (104) we find that $\frac{dp_\Omega}{dt} = \frac{dp_{\beta_+}}{dt} = \frac{dp_{\beta_-}}{dt} = 0$ according to $\Omega \to -\infty$; therefore, we conclude that $p_\Omega = p_{\beta_+} = p_{\beta_-} = constant$.

Bianchi VII$_h$

In the Bianchi VII$_h$ [33]

$$C^1_{23} = -C^1_{32} = -1, \quad C^2_{31} = -C^2_{13} = -1$$
$$C^1_{13} = -C^1_{31} = h, \quad C^2_{23} = -C^2_{32} = h.$$

Applying the structure constants to equation (14), we find the intrinsic curvature scalar

$$^{(3)}R_{VII_h} = 2C^1_{23}C^1_{32}h_{11}h^{33}h^{22} + 2C^2_{13}C^2_{31}h_{22}h^{11}h^{33} + 4C^1_{32}C^2_{31}h^{33} + 4\left[\left(C^1_{13}\right)^2 + \left(C^2_{23}\right)^2 + 2C^1_{13}C^2_{23}\right]h^{33},$$

or

$$^{(3)}R_{VII_h} = -4\exp\left(-2\Omega + 4\beta_+\right) \times \left[\cosh\left(4\sqrt{3}\beta_-\right) + 1\right] + 4h^2\exp\left(-2\Omega + 4\beta_+\right) \quad (105)$$

From equations (14) and equation (105), we find the Hamiltonian density expressed by

$$\mathcal{H}_{VII_h} = \tfrac{1}{24}\left(-p^2_\Omega + p^2_{\beta_+} + p^2_{\beta_-}\right) + 4\exp\left(4\Omega + 4\beta_+\right) \times \left[\cosh\left(4\sqrt{3}\beta_-\right) + 1\right] - 4h^2\exp\left(4\Omega + 4\beta_+\right). \quad (106)$$

With equation (106), we can calculate the equations of motion

$$\frac{d\Omega}{dt} = \{\Omega, \mathcal{H}_{VII_h}\} = \frac{\partial \mathcal{H}_{VII_h}}{\partial p_\Omega} = -\frac{1}{12}p_\Omega, \quad (107)$$

$$\frac{d\beta_+}{dt} = \{\beta_+, \mathcal{H}_{VII_h}\} = \frac{\partial \mathcal{H}_{VII_h}}{\partial p_{\beta_+}} = \frac{1}{12}p_{\beta_+}, \quad (108)$$

$$\frac{d\beta_-}{dt} = \{\beta_-, \mathcal{H}_{VII_h}\} = \frac{\partial \mathcal{H}_{VII_h}}{\partial p_{\beta_-}} = \frac{1}{12}p_{\beta_-}, \quad (109)$$

$$\frac{dp_\Omega}{dt} = \{p_\Omega, \mathcal{H}_{VII}\} = -\frac{\partial \mathcal{H}_{VII_h}}{\partial \Omega} = -6\Lambda\exp\left(6\Omega\right) -16\exp\left(4\Omega + 4\beta_+\right)\cosh\left(4\sqrt{3}\beta_-\right) -16\exp\left(4\Omega + 4\beta_+\right) + 4h^2\exp\left(4\Omega + 4\beta_+\right), \quad (110)$$

$$\frac{dp_{\beta_+}}{dt} = \{p_{\beta_+}, \mathcal{H}_{VII_H}\} = -\frac{\partial \mathcal{H}_{VII_h}}{\partial \beta_+} = -16\exp\left(4\Omega + 4\beta_+\right)\left[\cosh\left(4\sqrt{3}\beta_-\right) + 1\right] + 16h^2\exp\left(4\Omega + 4\beta_+\right), \quad (111)$$

$$\frac{dp_{\beta_-}}{dt} = \{p_{\beta_-}, \mathcal{H}_{VII_h}\} = -\frac{\partial \mathcal{H}_{VII_h}}{\partial \beta_-} = -16\sqrt{3}\exp\left(4\Omega + 4\beta_+\right)\sinh\left(4\sqrt{3}\beta_-\right). \quad (112)$$

We set the anisotropic parameters β_+ and β_-, consequently, the last two terms of equation (106) tend to zero as $\Omega \to -\infty$. From equations (110), (111) and (112) we find that $\frac{dp_\Omega}{dt} = \frac{dp_{\beta_+}}{dt} = \frac{dp_{\beta_-}}{dt} = 0$ according to $\Omega \to -\infty$, therefore $p_\Omega = p_{\beta_+} = p_{\beta_-} = constant$.

5 Concluding remarks

We study Bianchi spacetimes with cosmological constant by applying the ADM formalism of general relativity in the Hamiltonian version. From the Lagrangian density $\mathscr{L}$ and with the use of the Legendre transformation we calculate the Hamiltonian density $\mathcal{H}$ and the Poisson parentheses. Then, we develop the dynamics of the Bianchi models from the Hamiltonian constriction $\mathcal{H}$ and not from the moment constriction $\mathcal{H}_a$. Starting from $\mathcal{H}$ we study the dynamics of each of the Bianchi models with the calculation of each of the Poisson brackets of each canonical variable, and through which we conclude that in the limit when $\Omega \to -\infty$ we interpret each Hamiltonian constriction as a time-dependent gravitational potential. We present the way to construct the Lagrangian density and the Hamiltonian density for each Bianchi with a cosmological constant without a scalar field. However, it has not been mentioned that the curvature scalar $^{(3)}R$ depends on the structure constants $C^\lambda_{\mu\nu}$. The structure constants are important in this article.

We further conclude that the FLRW cosmological models [34–37] can be generalized only to some Bianchi cosmological models. The Bianchi type I and VII$_0$ cosmological models are a generalization of the Euclidean FLRW model ($k = 0$), the Bianchi type IX for the spherical FLRW cosmological model ($k = 1$) and the Bianchi type V and VII$_0$ are for the hyperbolic FLRW model ($k = -1$). The rest of the Bianchi models do not contain the FLRW models as a particular case.

6 Appendix A: ADM formalism of general relativity

6.1 Decomposition of space-time

Suppose we have been given a hypersurface in a four-dimensional Riemann space that can be imagined as an element of a family of surfaces; the normal vectors n^a for this family of surfaces would be:

$$n_\mu n^\mu = -1.$$

This equation, geometrically, we can interpret as the non-zero product of the normal vector n^μ to the hypersurface.

Now let us take these surfaces as the coordinate surfaces $x_0 = constant$ (space-time is being cut into slices, that is, into foliations) of a coordinate system that is not necessarily orthogonal and we denote the components of the normal vectors by

$$n_\mu = (-N, 0, 0, 0), \quad (113)$$

$$n^\mu = \left(\frac{1}{N}, -\frac{N^a}{N}\right) \quad (114)$$

where $\mu, \nu = 0,1,2,3$ y $a, b = 1,2,3$.

Geometrically, equation (113) represents only the ratio of the proper time flow τ with respect to the function t in the opposite direction, as the movement normal to the hypersurface is carried out, we have as a consequence of the fact that the other components of the vector are null. In equation (114), the contravariant vector n^μ to the hypersurface, not necessarily orthogonal to n_μ, measures in it is last three components the amount of tangential displacement.

Let us start an analysis to describe some quantities on the hypersurface. Let us consider a vector flow t^μ, which we decompose into its normal and tangential part to the hypersurface as

$$t^\mu = Nn^\mu + N^\mu, \tag{115}$$

where n^μ is a unit vector to the hypersurface and N^μ a tangent vector. The scalar N is called the "lapse" function, and the function N^μ is called the "shift" function. These quantities, together with the metric g_{ab} constitute the so-called ADM variables. The lapse function represents how far one hypersurface is separated from another, in other words, it measures the ratio of the flow of proper time with respect to the function t, as the movement normal to the hypersurface is carried out, and therefore we have $d\tau = Ndt$. On the other hand, the spatial part of the shift function measures the amount of tangential displacement for the hypersurface contained in the vector field t^μ.

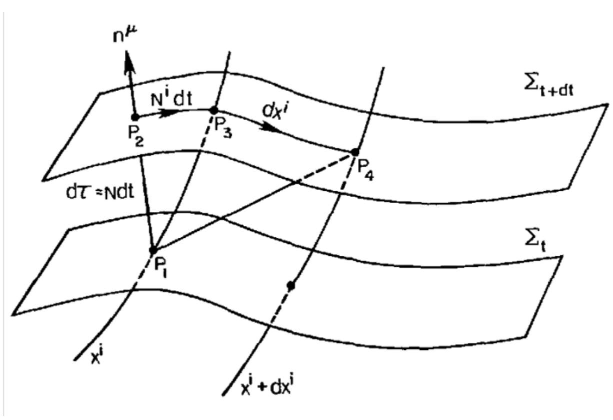

Figure 1: $3+1$ decomposition of the manifold, with lapse function N, and shift vector N^i.

Geometrically, the vector flow t^μ, using equation (115), can be interpreted in the following way: let us consider two infinitesimally close hypersurfaces, the term Nn^μ tells us how much we move perpendicular to the hypersurface, on the other hand, with the vector N^μ we can affirm that it tells us how far we move tangentially to the hypersurface. In general, we know that the vector field t^μ is not perpendicular or tangential at a point on the hypersurface, therefore, we conclude from equation (115) that the vector field t^μ can be in a arbitrary direction, in addition we can decompose this vector field into two vectors; one perpendicularNn^μ and the other parallel to the hypersurface N^μ.

The metric tensor g_{ab} of the hypersurface

$$^{(3)}ds^2 = g_{ab}dx^a dx^b,$$

and the metric tensor space-time is related by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(Ndx^0\right)^2 + g_{ab}\left(dx^a + N^a dx^0\right)\left(dx^b + N^b dx^0\right), \tag{116}$$

where $(dx^a + N^a dx^0)$ is the displacement on the base hypersurface and Ndt is the proper time between them, or, rearranging terms

$$ds^2 = \left(N^a N_a - N^2\right)\left(dx^0\right)^2 + 2N_a dx^a dx^0 + g_{ab}dx^a dx^b,$$

where we have taken space-time with the signature $(-,+,+,+)$. From the last equation we can see that the components of the metric tensor are given by

$$g_{\mu\nu} = \begin{pmatrix} N_a N^a - N^2 & N_b \\ N_a & g_{ab} \end{pmatrix}, \tag{117}$$

where g_{ab} denotes the spatial metric tensor. The contravariant components of the metric tensor are found by inverting the matrix $g_{\mu\nu}$, so we have

$$g^{\mu\nu} = \begin{pmatrix} -1/N^2 & N^b/N^2 \\ N^a/N^2 & g_{ab} - N^a N^b/N^2 \end{pmatrix}. \tag{118}$$

With the help of the tensor

$$h^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu, \tag{119}$$

which has the properties

$$h_{\mu\nu}h^\nu_\sigma = h_{\mu\sigma}, \quad h_{\mu\nu}n^\mu = 0, \quad h_{ab} = g_{ab}, \quad h^{ab} = g^{ab}, \quad h^0_\nu = 0,$$

We can decompose each tensor into its parallel or perpendicular components for the surface normal vector.

6.2 Extrinsic curvature

We construct for an arbitrary vector u_μ at a point p belonging to the hypersurface a covariant derivative D associated with the metric tensor $h^{\mu\nu}$ by

$$D_\mu u_\nu = h^\rho_\mu h^\sigma_\nu \nabla_\rho u_\sigma = h^\rho_\mu h^\sigma_\nu \left(\frac{\partial u_\sigma}{\partial x^\rho} - \Gamma^\lambda_{\rho\sigma}u_\lambda\right).$$

$$D_\mu u_\nu = h^\rho_\mu h^\sigma_\nu \nabla_\rho u_\sigma. \tag{120}$$

and we extend it for arbitrary tensors by linearity and Leibniz's rule. Here ∇_ρ denotes the covariant derivative associated with $g_{\mu\nu}$, that is, the equation $\nabla_\mu u_\nu = \partial_\mu u_\nu - \Gamma^\rho_{\mu\nu}u_\rho$. Taking into account equation (120), we can see that the covariant derivative D_μ obeys the following conditions

$$\begin{aligned} D_\rho h_{\mu\nu} &= 0, \\ [D_\mu, D_\nu] f &= 0. \end{aligned} \tag{121}$$

A consequence of the tensor $h_{\mu\nu}$ is define an intrinsic curvature ${}^{(3)}R^\sigma_{\mu\nu\rho}$, but however, this tensor of intrinsic curvature defined in the hypersurface is not sufficient in the sense that it is necessary to describe an extrinsic curvature that gives us information about how this hypersurface curves, this extrinsic curvature is given by

$$K_{\mu\nu} = h^\rho_\mu h^\sigma_\nu \nabla_\rho n_\sigma. \tag{122}$$

Geometrically, the $K_{\mu\nu}$ tensor describes like the vectors normal to the hypersurface converge or diverge, determining the geometry of an infinitesimal parallel surface.

Taking into account the result $\nabla_\mu (n_\nu n^\nu) = 2n^\nu \nabla_\mu n_\nu = 0$ we have the following important quantities in terms of extrinsic curvature

$$\begin{aligned} K = h^{\mu\nu} K_{\mu\nu} &= \nabla_\mu n^\mu, \\ K^{\mu\nu} K_{\mu\nu} &= (\nabla_\mu n^\nu)(\nabla_\nu n^\mu). \end{aligned} \tag{123}$$

Another identity that we will consider here is obtained by considering the generalization of the Lie derivative $\mathfrak{L}$ [38] to a tensor of the type $T^{\mu_1 \cdots \mu_m}_{\nu_1 \ldots \nu_n}$ is given by

$$\mathfrak{L}_u T^{\mu_1 \cdots \mu_n}_{\nu_1 \ldots \nu_n} = u^\alpha \nabla_\alpha T^{\mu_1 \cdots \mu_m}_{\nu_1 \ldots \nu_n} + \sum_{j=1}^{n} T^{\mu_1 \cdots \mu_m}_{\nu_1 \ldots \alpha \ldots \nu_n} \nabla_{\nu_j} u^\alpha - \sum_{i=1}^{m} T^{\mu_1 \ldots \alpha \ldots \mu_m}_{\nu_1 \ldots \nu_n} \nabla_\alpha u^{\mu_i} \tag{124}$$

where $T^{\mu_1 \cdots \mu_n}_{\nu_1 \ldots \nu_n}$ is an arbitrary tensor, and the Lie derivative is taken with respect to u^α. If we use equation (124) we take the Lie derivative of the tensor $g_{\mu\nu}$ and the tensor $h_{\mu\nu}$ with respect to the vector n^α we obtain the results

$$\begin{aligned} \mathfrak{L}_n g_{\mu\nu} &= n^\rho \nabla_\rho g_{\mu\nu} + g_{\rho\nu} \nabla_\mu n^\rho + g_{\rho\mu} \nabla_\nu n^\rho, \\ \mathfrak{L}_n h_{\mu\nu} &= \nabla_\mu n_\nu + \nabla_\nu n_\mu + n_\mu n^\rho \nabla_\rho n_\nu + n_\nu n^\rho \nabla_\rho n_\mu. \end{aligned} \tag{125}$$

Multiplying $\mathfrak{L}_n h_{\mu\nu}$ by the mixed tensors $h^\mu_\mu, h^\nu_\nu,$ we have

$$K_{\mu\nu} = \frac{1}{2} \mathfrak{L}_n h_{\mu\nu}. \tag{126}$$

Taking the second of equations (125) and substituting equation (115) in this last equation we arrive at the result

$$\mathfrak{L}_n h_{\mu\nu} = \frac{1}{N} \left(\mathfrak{L}_t h_{\mu\nu} - N^\rho \nabla_\rho h_{\mu\nu} - h_{\rho\nu} \nabla_\mu N^\rho - h_{\mu\rho} \nabla_\nu N^\rho \right), \tag{127}$$

consequently, the extrinsic curvature can be represented by the equation

$$K_{\mu\nu} = \frac{1}{2N} \left(\mathfrak{L}_t h_{\mu\nu} - N^\rho \nabla_\rho h_{\mu\nu} - h_{\rho\nu} \nabla_\mu N^\rho - h_{\mu\rho} \nabla_\nu N^\rho \right). \tag{128}$$

Finally, if we consider the definition of the covariant derivative on the hypersurface D_μ given in equation (120) and the covariant derivative of a first-rank covariant vector

$$K_{\mu\nu} = h^\rho_\mu h^\sigma_\nu \nabla_\rho n_\sigma = h^\rho_\mu h^\sigma_\nu \left(\frac{\partial n_\sigma}{\partial x^\rho} - \Gamma^\tau_{\rho\sigma} n_\tau \right),$$

we have, using equation (113)

$$K_{\mu\nu} = h^\rho_\mu h^\sigma_\nu N \Gamma^0_{\rho\sigma},$$

with the help of the symmetric tensor

$$K_{\mu\nu} = h^\rho_\mu h^\sigma_\nu \left(N g^{00} \Gamma_{0\rho\sigma} + N g^{0c} \Gamma_{c\rho\sigma} \right)$$

where $c = 1, 2, 3,$ and with the help of equation (118) in addition to considering the Christoffel symbols of the first kind

$$K_{\mu\nu} = \frac{1}{2N} h^\rho_\mu h^\sigma_\nu \left[2N^a \Gamma_{a\rho\sigma} - \left(\frac{\partial g_{0\sigma}}{\partial x^\rho} + \frac{\partial g_{\rho 0}}{\partial x^\sigma} - \frac{\partial g_{\rho\sigma}}{\partial x^0} \right) \right].$$

Rearranging the previous equation

$$K_{\mu\nu} = \frac{1}{2N} h^\rho_\mu h^\sigma_\nu \times \left[\frac{\partial g_{\rho\sigma}}{\partial x^0} - \left(\frac{\partial g_{0\sigma}}{\partial x^\rho} - N^a \Gamma_{a\rho\sigma} \right) - \left(\frac{\partial g_{\rho 0}}{\partial x^\sigma} - N^a \Gamma_{a\rho\sigma} \right) \right]$$

and considering equation (117), we can alternatively write the extrinsic curvature as

$$K_{\mu\nu} = \frac{1}{2N} h^\rho_\mu h^\sigma_\nu \left(\frac{\partial h_{\rho\sigma}}{\partial t} - \nabla_\rho N_\sigma - \nabla_\sigma N_\rho \right),$$

or

$$K_{\mu\nu} = \frac{1}{2N} \left(\frac{\partial h_{\mu\nu}}{\partial t} - D_\mu N_\nu - D_\nu N_\mu \right). \tag{129}$$

Note that $K_{\mu\nu}$ does not depend on the derivatives with respect to t de N^μ.

6.3 Curvature scalar

The Riemann tensor is defined by

$$[\nabla_\mu, \nabla_\nu] u_\rho = R^\sigma_{\mu\nu\rho} u_\sigma, \tag{130}$$

and the scalar of curvature is given by

$$R = R_{\mu\nu\rho\sigma} g^{\mu\rho} g^{\nu\sigma}.$$

Using equation (119) in the curvature scalar and calculating the corresponding products we arrive at

$$R = h^{\mu\rho} h^{\nu\sigma} R_{\mu\nu\rho\sigma} - R_{\mu\nu\rho\sigma} h^{\nu\sigma} n^\mu n^\rho - R_{\mu\nu\rho\sigma} h^{\mu\rho} n^\nu n^\sigma + R_{\mu\nu\rho\sigma} n^\mu n^\rho n^\nu n^\sigma,$$

furthermore using the symmetry of the Riemann tensor, the last term is zero, so the above equation reduces to

$$R = R_{\mu\nu\rho\sigma}h^{\mu\rho}h^{\nu\sigma} - 2h^{\mu\rho}n^{\nu}R_{\mu\nu\rho\sigma}n^{\sigma},$$

if we now take into account equation (130), the scalar of curvature takes the form

$$R = R_{\mu\nu\rho\sigma}h^{\mu\rho}h^{\nu\sigma} - 2h^{\mu\rho}n^{\nu}h_{\tau\rho}\left[\nabla_{\mu},\nabla_{\nu}\right]n^{\tau},$$

or

$$R = R_{\mu\nu\rho\sigma}h^{\mu\rho}h^{\nu\sigma} - 2n^{\nu}\left[\nabla_{\mu},\nabla_{\nu}\right]n^{\mu}. \qquad (131)$$

Now, our objective is transform the second term of equation (131), that is, we will transform the equation

$$n^{\nu}\left[\nabla_{\mu},\nabla_{\nu}\right]n^{\mu} = n^{\nu}\nabla_{\mu}\left(\nabla_{\nu}n^{\mu}\right) - n^{\nu}\nabla_{\nu}\left(\nabla_{\mu}n^{\mu}\right),$$

and for this we take into account these small tricks

$$n^{\nu}\left[\nabla_{\mu},\nabla_{\nu}\right]n^{\mu} = n^{\nu}\nabla_{\mu}\left(\nabla_{\nu}n^{\mu}\right) - n^{\nu}\nabla_{\nu}\left(\nabla_{\mu}n^{\mu}\right) - \left(\nabla_{\mu}n^{\nu}\right)\left(\nabla_{\nu}n^{\mu}\right) + \left(\nabla_{\mu}n^{\nu}\right)\left(\nabla_{\nu}n^{\mu}\right) + \left(\nabla_{\nu}n^{\nu}\right)\left(\nabla_{\mu}n^{\mu}\right) - \left(\nabla_{\mu}n^{\mu}\right)\left(\nabla_{\nu}n^{\nu}\right),$$

to arrive at the equation

$$n^{\nu}\left[\nabla_{\mu},\nabla_{\nu}\right]n^{\mu} = \nabla_{\mu}\left(n^{\nu}\nabla_{\nu}n^{\mu}\right) - \nabla_{\nu}\left(n^{\nu}\nabla_{\mu}n^{\mu}\right) - \left(\nabla_{\mu}n^{\nu}\right)\left(\nabla_{\nu}n^{\mu}\right) + \left(\nabla_{\nu}n^{\nu}\right)\left(\nabla_{\mu}n^{\mu}\right),$$

which is equivalent to

$$n^{\nu}\left[\nabla_{\mu},\nabla_{\nu}\right]n^{\mu} = \nabla_{\mu}\left(n^{\nu}\nabla_{\nu}n^{\mu} - n^{\mu}\nabla_{\nu}n^{\nu}\right) - \left(\nabla_{\mu}n^{\nu}\right)\left(\nabla_{\nu}n^{\mu}\right) + \left(\nabla_{\nu}n^{\nu}\right)\left(\nabla_{\mu}n^{\mu}\right).$$

From this last equation when considering equations (123) it is transformed into

$$n^{\nu}\left[\nabla_{\mu},\nabla_{\nu}\right]n^{\mu} = \nabla_{\mu}\left(n^{\nu}\nabla_{\nu}n^{\mu} - n^{\mu}\nabla_{\nu}n^{\nu}\right) - K^{\mu\nu}K_{\mu\nu} + K^{2},$$

and consequently the scalar of curvature is given by

$$R = R_{\mu\nu\rho\sigma}h^{\mu\rho}h^{\nu\sigma} - 2\times\left[K^{2} - K^{\mu\nu}K_{\mu\nu} + \nabla_{\mu}\left(n^{\nu}\nabla_{\nu}n^{\mu} - n^{\mu}\nabla_{\nu}n^{\nu}\right)\right] \qquad (132)$$

Next, we want to find the Riemann curvature tensor on the hypersurface related to the covariant derivative D_{μ} which is expressed in equation (120) by the equations

$$\left[D_{\mu},D_{\nu}\right]u_{\rho} = {}^{(3)}R^{\sigma}_{\mu\nu\rho}u_{\sigma}. \qquad (133)$$

Using equation (120) we can write

$$D_{\mu}D_{\nu}u_{\rho} = D_{\mu}\left(h^{\nu'}_{\nu}h^{\rho'}_{\rho}\nabla_{\nu'}u_{\rho'}\right),$$

and equation (120) again,

$$D_{\mu}D_{\nu}u_{\rho} = h^{\mu'}_{\mu}h^{\nu''}_{\nu}h^{\rho''}_{\rho}\nabla_{\mu'}\left(h^{\nu'}_{\nu''}h^{\rho'}_{\rho''}\nabla_{\nu'}u_{\rho'}\right),$$

then after applying Leibniz's rule twice

$$D_{\mu}D_{\nu}u_{\rho} = h^{\mu'}_{\mu}h^{\nu''}_{\nu}h^{\rho''}_{\rho}\left(h^{\nu'}_{\nu''}h^{\rho'}_{\rho''}\nabla_{\mu'}\nabla_{\nu'}u_{\rho'}\right) + h^{\mu'}_{\mu}h^{\nu''}_{\nu}h^{\rho''}_{\rho}h^{\nu'}_{\nu''}\left(\nabla_{\mu'}h^{\rho'}_{\rho''}\right)\nabla_{\nu'}u_{\rho'} + h^{\mu'}_{\mu}h^{\nu''}_{\nu}h^{\rho''}_{\rho}\left(\nabla_{\mu'}h^{\nu'}_{\nu''}\right)h^{\rho'}_{\rho''}\nabla_{\nu'}u_{\rho'}.$$

and with the help of the result

$$h^{\mu'}_{\mu}h^{\nu'}_{\nu}\nabla_{\mu'}h^{\sigma}_{\nu'} = h^{\mu'}_{\mu}h^{\nu'}_{\nu}\nabla_{\mu'}\left(g^{\sigma}_{\nu'} + n^{\sigma}n_{\nu'}\right) = K_{\mu\nu}n^{\sigma},$$

we get the equation

$$D_{\mu}D_{\nu}u_{\rho} = h^{\mu'}_{\mu}h^{\nu'}_{\nu}h^{\rho'}_{\rho}\nabla_{\mu'}\nabla_{\nu'}u_{\rho'} + h^{\nu'}_{\nu}K_{\mu\rho}n^{\rho'}\nabla_{\nu'}u_{\rho'} + h^{\rho'}_{\rho}K_{\mu\nu}n^{\nu'}\nabla_{\nu'}u_{\rho'}. \qquad (134)$$

Transforming the second term of equation (134)

$$D_{\mu}D_{\nu}u_{\rho} = h^{\mu'}_{\mu}h^{\nu'}_{\nu}h^{\rho'}_{\rho}\nabla_{\mu'}\nabla_{\nu'}u_{\rho'} - h^{\nu'}_{\nu}K_{\mu\rho}u_{\rho'}\nabla_{\nu'}n^{\rho'} + h^{\rho'}_{\rho}K_{\mu\nu}n^{\nu'}\nabla_{\nu'}u_{\rho'}$$

or

$$D_{\mu}D_{\nu}u_{\rho} = h^{\mu'}_{\mu}h^{\nu'}_{\nu}h^{\rho'}_{\rho}\nabla_{\mu'}\nabla_{\nu'}u_{\rho'} - K_{\mu\rho}K^{\rho'}_{\nu}u_{\rho'} + h^{\rho'}_{\rho}K_{\mu\nu}n^{\nu'}\nabla_{\nu'}u_{\rho'}. \qquad (135)$$

We have, by exchanging indices in equation (135) and subtracting the resulting equation from this

$$\left[D_{\mu},D_{\nu}\right]u_{\rho} = h^{\mu'}_{\mu}h^{\nu'}_{\nu}h^{\rho'}_{\rho}\left[\nabla_{\mu},\nabla_{\nu}\right]u_{\rho'} - K_{\mu\rho}K^{\rho'}_{\nu}u_{\rho'} + K_{\nu\rho}K^{\rho'}_{\mu}u_{\rho},$$

and with the help of equations (130) and (133)

$${}^{(3)}R^{\sigma}_{\mu\nu\rho}u_{\sigma} = h^{\mu'}_{\mu}h^{\nu'}_{\nu}h^{\rho'}_{\rho}R^{\sigma'}_{\mu'\nu'\rho'}u_{\sigma'} + \left(-K_{\mu\rho}K^{\rho'}_{\nu} + K_{\nu\rho}K^{\rho'}_{\mu}\right)u_{\rho'}$$

or

$${}^{(3)}R^{\sigma}_{\mu\nu\rho}u_{\sigma} = \left(h^{\mu'}_{\mu}h^{\nu'}_{\nu}h^{\rho'}_{\rho}h^{\sigma}_{\sigma'}R^{\sigma'}_{\mu'\nu'\rho'} - K_{\mu\rho}K^{\sigma}_{\nu} + K_{\nu\rho}K^{\sigma}_{\mu}\right)u_{\sigma} \qquad (136)$$

equation that expresses the Riemann tensor on the hypersurface in terms of the Riemann tensor in spacetime and the extrinsic curvature. This equation is called the Gauss-Codazzi equation [39].

Finally we will obtain the spatial curvature scalar from the equation

$${}^{(3)}R = {}^{(3)}R_{\mu\nu\rho\sigma}h^{\mu\rho}h^{\nu\sigma}, \qquad (137)$$

which when using equation (136) is written as

$${}^{(3)}R = h^{\mu\rho}h^{\nu\sigma}R_{\mu\nu\rho\sigma} - K^{2} + K_{\mu\nu}K^{\mu\nu},$$

and when taking into account equation (132)

$${}^{(3)}R = R + K^{2} - K_{\mu\nu}K^{\mu\nu} + 2\nabla_{\mu}\left(n^{\nu}\nabla_{\nu}n^{\mu} - n^{\mu}\nabla_{\nu}n^{\nu}\right), \qquad (138)$$

or we also write it in the form

$$^{(3)}R = R + K^2 - K_{\mu\nu}K^{\mu\nu} + 2\nabla_\mu\left(\Delta^\mu\right),$$

where $\Delta^\mu = n^\nu\nabla_\nu n^\mu - n^\mu\nabla_\nu n^\nu$. The last term of this equation is the covariant derivative of the term Δ^μ and when introduced into the action, through the divergence theorem it has not dynamic information and we can neglect it.

Equation (138) is called the Codazzi equation and shows the relationship between the scalar of curvature of the hypersurface and the scalar of curvature of space-time.

6.4 Hamiltonian Formalism

From equation (117) we see that the volume element is given by

$$\sqrt{-g}d^4x = N\sqrt{\det(h)}dtd^3x,$$

from where it can be found by comparison $\sqrt{-g} = N\sqrt{\det(h)}$.

Taking into account the Codazzi equation we can rewrite the action for the gravitational field in the form

$$S[g_{ab}, N, N^a] = \int dt \int d^3x N\sqrt{\det(h)} \times \left(^{(3)}R - K^2 + K_{\mu\nu}K^{\mu\nu} - \Lambda\right), \tag{139}$$

where

$$\mathscr{L}_G = N\sqrt{\det(h)}\left(^{(3)}R - K^2 + K_{\mu\nu}K^{\mu\nu} - \Lambda\right). \tag{140}$$

So far we have rewritten the action of the gravitational field so that we can find the field equations with cosmological constant, taking the variation of the action and setting it equal to zero ($\delta S = 0$)

$$0 = \int dt \int d^3x \left(\frac{\delta\mathscr{L}_G}{\delta\dot{h}_{ab}}\delta\dot{h}_{ab} + \frac{\delta\mathscr{L}_G}{\delta\dot{N}^a}\delta\dot{N}^a + \frac{\delta\mathscr{L}_G}{\delta\dot{N}}\delta\dot{N}\right),$$

from where the conjugated moments are

$$\pi^{ab} = \frac{\delta\mathscr{L}_G}{\delta\dot{h}_{ab}} = \sqrt{\det(h)}\left(K^{ab} - Kq^{ab}\right), \tag{141}$$

$$\pi^a = \frac{\delta\mathscr{L}_G}{\delta\dot{N}_a} = 0, \tag{142}$$

$$\pi = \frac{\delta\mathscr{L}_G}{\delta\dot{N}} = 0. \tag{143}$$

In equations (141), (142) and (143) we have taken into account that the action (139) depends on $\frac{\partial h_{ab}}{\partial t} = \dot{h}_{ab}$ and not from $\dot{N}, \dot{N}^a$.

Equations (141), (142) and (143) describe a singular system with the primary constraints

$$C_a = \pi_a \approx 0, \quad C = \pi \approx 0.$$

Now we are able to write the Hamiltonian, using the Legendre transformation. Constructing a Hamiltonian for a system with constraints [40, 41] such as our case $H + \lambda^j C_j$ and introducing the Lagrange multipliers λ y λ^a and with the help of the transformation from Legendre and the action for the gravitational field [42–51] we write the canonical form of the action

$$S[g_{ab}, N, N^a] = -\int dt \int d^3x$$

$$\left[\dot{h}_{ab}\pi^{ab} + \lambda C + \lambda^a C_a - N\sqrt{\det(h)}\left(^{(3)}R - K^2 + K^{ab}K_{ab} - \Lambda\right)\right]$$

$$+ \int dt \int d^3x \left(\dot{h}_{ab}\pi^{ab} + \dot{N}^a\pi_a + \dot{N}\pi\right).$$

We have, by taking the Lie derivative of the projecting tensor h_{ab} in the direction of N

$$\mathfrak{L}_N h_{ab} = N^c\nabla_c h_{ab} + h_{cb}\nabla_a N^c + h_{ac}\nabla_b N^c$$

and with the help of equation (115)

$$\mathfrak{L}_N h_{ab} = \mathfrak{L}_t h_{ab} - N\mathfrak{L}_n h_{ab}. \tag{144}$$

Let us introduce the equation

$$\pi^{ab}\pi_{ab} - \frac{1}{2}\left(\pi_a^a\right)^2 = \det(h)\left(K^{ab}K_{ab} - K^2\right), \tag{145}$$

equation in which we show the relationship between the conjugate moments π^{ab} and the extrinsic curvature K^{ab}.

From equation (144) and considering equation (126) the result is obtained

$$\dot{h}_{ab}\pi^{ab} = \mathfrak{L}_t h_{ab}\pi^{ab} = \mathfrak{L}_N h_{ab}\pi^{ab} + 2N\pi^{ab}K_{ab},$$

which, when taking into account equation (141), becomes

$$\mathfrak{L}_t h_{ab}\pi^{ab} = \mathfrak{L}_N h_{ab}\pi^{ab} + 2N\sqrt{\det(h)}\left(K^{ab}K_{ab} - K^2\right). \tag{146}$$

With the results (144) and (145) we transform the canonical form of the action to the equation:

$$S[g_{ab}, N, N^a] = \int dt \int d^3x \left(\dot{h}_{ab}\pi^{ab} + \dot{N}^a\pi_a + \dot{N}\pi - \lambda C - \lambda^a C_a - \mathfrak{L}_N h_{ab}\pi^{ab}\right)$$

$$+ \int dt \int d^3x N\left[\sqrt{\det(h)}\left(^{(3)}R - \Lambda\right) - \frac{1}{\sqrt{\det(h)}}\left(\pi^{ab}\pi_{ab} - \frac{1}{2}\left(\pi_a^a\right)^2\right)\right].$$

Once again we use the Lie derivative to find the equation

$$\mathfrak{L}_N h_{ab}\pi^{ab} = 2h_{ab}\nabla_b N^c \pi^{ab},$$

if we partially integrate this equation by parts

$$\int dt \int d^3x \mathfrak{L}_N h_{ab}\pi^{ab} = 2\int dt \int d^3x$$

$$\left[D_b\left(h_{ac}N^c\right)\pi^{ab} - h_{ac}N^c D_b \pi^{ab}\right]$$

and using Gauss's theorem, we arrive at

$$\mathfrak{L}_N h_{ab}\pi^{ab} = -2h_{ac}N^c D_b \pi^{ab}. \tag{147}$$

Therefore the action is expressed by

$$S[g_{ab}, N, N^a] = \int dt \int d^3x$$

$$\left\{\dot{h}_{ab}\pi^{ab} + \dot{N}^a\pi_a + \dot{N}\pi - \lambda C - \lambda^a C_a - N^a \mathcal{H}_a - N\mathcal{H}\right\} \tag{148}$$

where

$$\mathcal{H} = \frac{1}{\sqrt{\det(h)}}\left(\pi^{ab}\pi_{ab} - \frac{1}{2}\left(\pi^a_a\right)^2\right) - \sqrt{\det(h)}\left({}^{(3)}R - \Lambda\right), \tag{149}$$

$$\mathcal{H}_a = -2h_{ac}D_b\pi^{bc}. \tag{150}$$

We can then write the first order Lagrangian density from equation (148)

$$\mathscr{L}\left(h_{ab}, \frac{\partial h_{ab}}{\partial t}, \pi^{ab}, N, N^a\right) = \pi^{ab}\frac{\partial h_{ab}}{\partial t}$$

$$+\dot{N}^a\pi_a + \dot{N}\pi - \lambda C - \lambda^a C_a - N\mathcal{H} - N^a\mathcal{H}_a.$$

The action integral from this Lagrangian density will be functional of the metric h_{ab} as well as the moments π, π^a and π^{ab} and the functions N, N^a. It is important to note that in this Lagrangian density the functions N^μ could play the role of Lagrange multipliers and the action is no longer written in parameterized form, that is, the Lagrangian density is in canonical form. The analysis of each Bianchi model presented in this article, in accordance with the formalism presented in this appendix, can be extended to the case where matter, cosmological constant and a scalar field are considered (to analysis of some Bianchi's models, see [52–60].

For arbitrary functions, we can write the functions

$$C(f) = \int d^3x f C, \tag{151}$$

$$C(f) = \int d^3x f^a C_a, \tag{152}$$

$$\mathcal{H}(f) = \int d^3x f \mathcal{H}, \tag{153}$$

$$\mathcal{H}(f) = \int d^3x f^a \mathcal{H}_a, \tag{154}$$

and therefore the Hamiltonian takes the form

$$H = C(\lambda) + C(\lambda^a) + \mathcal{H}(N) + \mathcal{H}(N^a). \tag{155}$$

The consideration of the previous integrals has been carried out as a consequence of having the principle of gravitational action in the ADM variables, that is, equation (148) and with this we would avoid working with the Hamiltonian densities $\mathcal{H}, \mathcal{H}_a$ and the primary constraints C, C_a, but we would work with the Hamiltonian of equation (155) and consequently, we can note that the Lagrangian function can be defined in terms of the Hamiltonian of the equation (155) using the Legendre transformation as in Classical Mechanics when the Hamilton principle is defined and one wants to change from the Lagrangian to the Hamiltonian formalism and viceversa.

The phase space is structured by the local configurations of the fields $\left(h_{ab}, \pi, \pi^a, \pi^{ab}, N, N^a\right)$ for t fixed and Poisson brackets

$$\begin{aligned} \left\{h_{ab}(t,x), \pi^{cd}(t,x')\right\} &= \delta_a^{(c}\delta_b^{d)}\delta(x-x'), \\ \left\{N^a(t,x), \pi_b(t,x)\right\} &= \delta_b^a\delta(x-x'), \\ \left\{N(t,x), \pi(t,x')\right\} &= \delta(x-x'). \end{aligned} \tag{156}$$

Intuitively, the constraints $C = \pi \approx 0$ and $C_a = \pi_a \approx 0$ are preserved under the evolution of the system, and therefore we impose consistency conditions

$$\frac{d}{dt}C(f) = 0 = \{H, C(f)\} = \mathcal{H}(f), \tag{157}$$

$$\frac{d}{dt}C(f^a) = 0 = \{H, C(f^a)\} = \mathcal{H}(f^a), \tag{158}$$

where the total derivatives are derivatives with respect to the parameter t (proper time), and since these relations cancel out independently of the values of the fields f and f^a we obtain the secondary constraints

$$\mathcal{H} \approx 0; \quad \mathcal{H}_a \approx 0. \tag{159}$$

The constriction $\mathcal{H}_a$ results from the arbitrariness in the way we slice our space-time into hypersurfaces.

The Poisson parentheses that satisfy the Hamiltonian and the secondary constraints satisfy the Dirac algebra:

$$\begin{aligned} \{\mathcal{H}_a(t,x), \mathcal{H}_b(t,x')\} &= \mathcal{H}_b(t,x)\tfrac{\partial}{\partial x^a}\delta(x-x') \\ \{\mathcal{H}(t,x), \mathcal{H}(t,x')\} &= g^{ab}\mathcal{H}_a(t,x)\tfrac{\partial}{\partial x^b}\delta(x-x'), \\ \{\mathcal{H}_a(t,x), \mathcal{H}(t,x')\} &= \mathcal{H}(t,x)\tfrac{\partial}{\partial x^a}\delta(x-x'). \end{aligned} \tag{160}$$

The structure of this algebra together with the equations $C = \pi \approx 0$ and $C_a = \pi_a \approx 0$, guarantees that the evolution of the canonical coordinates in the phase space assigned to equations (157) and (158) is independent of how the spatial hypersurface is deformed and the coordinates of the initial and final configurations for the lapse and shift functions.

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