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The symmetry S_3 in the dark matter

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Abstract
Experimental evidence so far suggests that there are only three generations of quarks and leptons. Before electroweak symmetry breaking, the three families of quarks and leptons are indistinguishable, so they are invariant under transformations of the S_3 group. Using the symmetry S_3 we have at our disposal 3 irreducible representations, $\mathbf{2}$, $\mathbf{1}_s$, $\mathbf{1}_a$, where we can accommodate up to 4 Higgses doublets in a model that occupies all the irreducible representations of the group S_3. This model with four Higgses doublets (4HDM) is of great interest, thanks to the fact that we can take the fourth Higgs doublet as a stable particle without interaction with fermions so that it becomes a candidate for dark matter, while with the remaining three Higgses the properties obtained are maintained. An important condition for having a viable dark matter candidate is its stability, i. e., it does not decay into Standard Model particles. The simplest way to establish the stability of a particle is imposing a discrete symmetry Z_2, so that all the fields are transformed in the form $\Psi \longrightarrow \Psi$, while the dark matter candidates are transformed as $\chi \longrightarrow -\chi$, this way we make sure we don't have terms denoting decays of χ. This method will be used in 4HDM. Another imposition required to propose the candidacy of a field of the doublet H_a, is that its corresponding Vacuum Expectation Value (VEV) is equal to zero, $v_a = 0$.
Keywords: Four Higgses doublet model, flavor symmetry, dark matter candidate.

La simetría S_3 en la materia oscura

Resumen
La evidencia experimental hasta ahora sugiere que sólo existen tres generaciones de quarks y leptones. Antes de que se rompa la simetría electrodébil, las tres familias de quarks y leptones son indistinguibles, por lo que son invariantes ante transformaciones del grupo S_3. Usando la simetría S_3 tenemos a nuestra disposición 3 representaciones irreductibles, $\mathbf{2}$, $\mathbf{1}_s$, $\mathbf{1}_a$, donde podemos acomodar hasta cuatro dobletes de Higgs en un modelo que ocupa todas las representaciones irreductibles del grupo S_3. Este modelo con cuatro dobletes de Higgs (4HDM) es de gran interés, gracias a que podemos tomar el cuarto doblete de Higgs como una partícula estable sin interacción con los fermiones por lo que se convierte en candidato a materia oscura, mientras que con los tres restantes las propiedades obtenidas se mantienen. Una condición importante para tener un candidato viable a materia oscura es su estabilidad, es decir, no se desintegra en partículas del modelo estándar. La forma más sencilla de establecer la estabilidad de una partícula es imponiendo una simetría discreta Z_2, de modo que todos los campos se transformen en la forma $\Psi \longrightarrow \Psi$, mientras que las candidatas a materia oscura se transformen como $\chi \longrightarrow -\chi$, de esta manera nos aseguramos de que no tengamos términos que denoten desintegraciones de χ. Este método se utilizará en 4HDM. Otra imposición requerida para proponer la candidatura de un campo del doblete H_a, es que su Valor de Espectación del Vacío (VEV) correspondiente sea igual a cero, $v_a = 0$.
Palabras clave: Modelo de cuatro dobletes de Higgs, simetría del sabor, candidato a materia oscura.

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1 Introduction

The symmetric group S_n is the group of bijections of $\{1, 2, ..., n\}$ to itself, also called the permutation group of n objects. It is a finite group of order $n!$, that is, there are $n!$ ways to swap n objects. The group that interests us is S_3.

The group S_1, comprises the permutations of a single object, and its only element is the identity $\{E\}$. The group S_2 comprises the permutations of two objects f_1 and f_2. This group has $2! = 2$ elements $\{E, A\}$, where E is the identity that produces the trivial transformation, $f_2 \longrightarrow f_2$ and A produces the transformation $A : f_1 \longrightarrow f_2,\, f_2 \longrightarrow f_1$. Note that the symmetry groups are abelian.

The group S_3 comprises the permutations of three objects f_1, f_2 and f_3. This group has $3! = 6$ elements $\{E, A_1, A_2, A_3, A_4, A_5\}$, where E as always, is the identity, A_1, A_2, A_3 transform two elements and leave one fixed (for example A_2 produce la transfomación $A_2 : f_1 \longrightarrow f_3, f_2 \longrightarrow f_2, f_3 \longrightarrow f_1$) and A_4, A_5 produce a permutation of all objects (for example A_4 : $f_1 \longrightarrow f_2, f_2 \longrightarrow f_3, f_3 \longrightarrow f_1$). Let us now note that $A_1 A_4 \neq A_4 A_1$, so the symmetry group S_3 is non-abelian. In fact, since a group S_m, where $m < n$, is a subgroup of S_n. The groups S_n with $n \geq 3$ are non-abelian.

Experimental evidence so far suggests that there are only three generations of quarks and leptons. Before electroweak symmetry breaking, the three families of quarks and leptons are indistinguishable, so they are invariant under transformations of the S_3 group. This tells us that the S_3 symmetry [1–17] is convenient. The fermions in the irreducible representation of the doublet are denoted as $\psi_{D(L,R)}$, where

$$\psi_D\,(L,R) \equiv \begin{pmatrix} \psi_{1(L,R)} \\ \psi_{2(L,R)} \end{pmatrix} \sim \mathbf{2} \tag{1}$$

and those found in the symmetric singlet representation as

$$\psi_{S(L,R)} \equiv \psi_{3(L,R)} \sim \mathbf{1}_s \tag{2}$$

where 1, 2, 3 represent the index of each family of the left (L) or right (R) fermionic field. For quarks we have:

$$\psi_{3,L} = (b_L, t_L)\,, \quad \psi_{3,R} = t_R, \quad \psi_{3,R} = b_R \tag{3}$$

$$\begin{pmatrix} \psi_{1,L} \\ \psi_{2,L} \end{pmatrix} = \begin{pmatrix} (u_L, d_L) \\ (c_L, s_L) \end{pmatrix}, \tag{4}$$

$$\begin{pmatrix} \psi_{1,R} \\ \psi_{2,R} \end{pmatrix}_{\psi=u} = \begin{pmatrix} u_R \\ c_R \end{pmatrix}, \tag{5}$$

$$\begin{pmatrix} \psi_{1,R} \\ \psi_{2,R} \end{pmatrix}_{\psi=d} = \begin{pmatrix} d_R \\ s_R \end{pmatrix}, \tag{6}$$

where (u_L, d_L) and (c_L, s_L) are doublets $SU\,(2)_L$, while u_R, c_R, d_R y s_R are singlets $SU\,(2)_L$.

The Higgs fields are then denoted as:

$$H_D \equiv \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \sim \mathbf{2}, \quad H_s \sim \mathbf{1}_s, \quad H_a \sim \mathbf{1}_a. \tag{7}$$

We accommodate four $SU\,(2)$ doublets into the irreducible representations of the permutation group S_3, denoting the symmetric and antisymmetric scalars by H_s and H_a respectively, while the remaining two doublet H_1 and H_2 are arranged in a column vector conforming the S_3 doublet, i. e.

$$H_s = \begin{pmatrix} h_s^c \\ h_s^n + v_0 + i h_s^p \end{pmatrix} \quad H_a = \begin{pmatrix} h_a^c \\ h_a^n + v_a + i h_a^p \end{pmatrix}$$
$$H_1 = \begin{pmatrix} h_1^c \\ h_1^n + v_1 + i h_1^p \end{pmatrix} \quad H_2 = \begin{pmatrix} h_2^c \\ h_2^n + v_2 + i h_2^p \end{pmatrix}.$$

2 The symmetry group S_3

The group S_3 is defined as the group of permutations of three objects [18]. This is a non-abelian group and is formed by three even and three odd permutations of three objects, which can be labeled as follows (f_1, f_2, f_3). The elements of the group are

$$S_3 = \{E, A_1, A_2, A_3, A_4, A_5\} \tag{8}$$

where E is the identity element of the group and the A_i, with $i = 1, 2, 3, 4, 5$, label the permutations as follows

$$\begin{aligned} E &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \\ A_3 &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}. \end{aligned} \tag{9}$$

The notation used refers to the exchange of the subscripts of f_1, f_2, f_3, for example A_1 acts as follows:

$$\begin{aligned} f_1 &\longrightarrow f_2 \\ f_2 &\longrightarrow f_1 \\ f_3 &\longrightarrow f_3 \end{aligned}\,.$$

In general, for the permutation group S_n the product between two of its elements is simply their successive application and this product is not commutative. Thus, for example, by multiplying A_2 with A_5 we obtain that:

$$A_2A_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = A_3,$$

in terms of f_1, f_2 and f_3

$$\begin{array}{c} f_1 \longrightarrow f_3 \longrightarrow f_1, \\ f_2 \longrightarrow f_1 \longrightarrow f_3, \\ f_3 \longrightarrow f_2 \longrightarrow f_2. \end{array}$$

The products between the remaining elements of group S_3 are carried out in an analogous manner to the previous example. Thus, the multiplication table of the group is given in table 2.

Now in general, if

$$A = \begin{pmatrix} 1 & 2 & . & . & . & n \\ i_1 & i_2 & . & . & . & i_n \end{pmatrix} \in S_n \Longrightarrow A^{-1} = \begin{pmatrix} i_1 & i_2 & . & . & . & i_n \\ 1 & 2 & . & . & . & n \end{pmatrix} \in S_n,$$

where A^{-1} is the inverse element. Therefore, the condition is satisfied

$$A^{-1}A = AA^{-1} = E.$$

As a particular example in the case of S_3, the inverse of A_1 is

$$A_1^{-1} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = A_1,$$

analogously, we have the inverses of the other elements

$$\begin{array}{c} E^{-1} = E, \quad A_1^{-1} = A_1, \quad A_2^{-1} = A_2, \\ A_3^{-1} = A_3, \quad A_4^{-1} = A_5, \quad A_5^{-1} = A_4. \end{array}$$

Elements	E	A_1	A_2	A_3	A_4	A_5
E	E	A_1	A_2	A_3	A_4	A_5
A_1	A_1	E	A_5	A_4	A_3	A_2
A_2	A_2	A_4	E	A_5	A_1	A_3
A_3	A_3	A_5	A_4	E	A_2	A_1
A_4	A_4	A_2	A_3	A_1	A_5	E
A_5	A_5	A_3	A_1	A_2	E	A_4

Table 1: Multiplication table of group S_3.

2.1 Conjugate Classes

A conjugate class for a group G of order g is defined as

$$(a) \equiv \left\{a, b/u^{-1}bu = a, u \in G\right\}.$$

The identity element E by itself forms a conjugate class, the remaining classes are determined with the help of the multiplication table of the group (see table 1). Thus, for the conjugate class whose representative is A_1 and which is denoted as k_1, we have:

$$\begin{array}{c} A_4^{-1}A_1A_4 = A_4^{-1}A_2 = A_1 \\ A_2^{-1}A_3A_2 = A_2^{-1}A_4 = A_1 \\ A_3^{-1}A_2A_3 = A_3^{-1}A_5 = A_1. \end{array}$$

Denoting with k_2 the conjugate class with representative A_4, for this we have

$$\begin{array}{c} A_5^{-1}A_4A_5 = A_5^{-1}E = A_4, \\ A_1^{-1}A_5A_1 = A_1^{-1}A_3 = A_4. \end{array}$$

Therefore, the group S_3 has three conjugate classes that are denoted as E, k_1 and k_2, and have the form:

$$E = \{E\}, \quad k_1 = \{A_1, A_2, A_3\}, \quad k_2 = \{A_4, A_5\}. \tag{10}$$

The conjugate classes of the group are useful for constructing the so-called class operators and in turn constructing the projectors of the group.

2.2 Representations of S_3

In this section we will construct a pair of matrix representations of the group S_3, we will begin by considering the three-dimensional vector space generated by the basis vectors

$$\left\{ \ \vec{e}_1 = (1,0,0), \quad \vec{e}_2 = (0,1,0), \quad \vec{e}_3 = (0,0,1) \ \right\}. \tag{11}$$

The way in which the element E acts on the base vectors is evident, so we are tempted to represent element in matrix form in the following form

$$D(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For a more illustrative example, we analyze the effect of applying A_5 to the basis vectors,

$$\begin{array}{c} A_5\vec{e}_1 = \vec{e}_3, \\ A_5\vec{e}_2 = \vec{e}_1, \\ A_5\vec{e}_3 = \vec{e}_2, \end{array}$$

thus the matrix representation of A_5 is

$$D(A_5) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

analogously for the other elements of the group S_3, we then obtain the so-called representation S_3

$$D(E)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad D(A_1)=\begin{pmatrix}0&1&0\\1&0&0\\0&1&0\end{pmatrix},\quad D(A_2)=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix},$$
$$D(A_3)=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad D(A_4)=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\quad D(A_5)=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}. \tag{12}$$

The real representation is a three-dimensional representation.

We will now construct one more representation, taking advantage of the fact that the group S_3 is isomorphic to the symmetry group of a rectangle triangle [19], elements A_1, A_2 and A_3 are associated with reflections on the indicated axes of symmetry and elements A_4 and A_5 with rotations corresponding to angles of $2\pi/3$ and $4\pi/3$ respectively, around the z axis. Let us consider again the three-dimensional vector space generated by the base vectors (11), as an example, let's see how A_4 (rotation by $2\pi/3$ around z) acts on the base vectors,

$$A_1\vec{e}_1=\cos\left(\tfrac{2\pi}{3}\right)\vec{e}_1+\sin\left(\tfrac{2\pi}{3}\right)\vec{e}_2=-\tfrac{1}{2}\vec{e}_1+\tfrac{\sqrt{3}}{2}\vec{e}_2,$$
$$A_4\vec{e}_2=-\sin\left(\tfrac{2\pi}{3}\right)\vec{e}_1+\cos\left(\tfrac{2\pi}{3}\right)\vec{e}_2=-\tfrac{\sqrt{3}}{2}\vec{e}_1-\tfrac{1}{2}\vec{e}_2,$$
$$A_4\vec{e}_3=\vec{e}_3,$$

so we represent element A_4 as follows:

$$D(A_5)=\begin{pmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}&0\\-\frac{\sqrt{3}}{2}&-\frac{1}{2}&0\\0&0&1\end{pmatrix}.$$

Proceeding in a similar way with the other elements

$$D(E)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad D(A_1)=\begin{pmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}&0\\-\frac{\sqrt{3}}{2}&-\frac{1}{2}&0\\0&0&-1\end{pmatrix},\quad D(A_2)=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix},$$
$$D(A_3)=\begin{pmatrix}\frac{1}{2}&\frac{\sqrt{3}}{2}&0\\\frac{\sqrt{3}}{2}&-\frac{1}{2}&0\\0&0&-1\end{pmatrix},\quad D(A_4)=\begin{pmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}&0\\-\frac{\sqrt{3}}{2}&-\frac{1}{2}&0\\0&0&1\end{pmatrix},\quad D(A_5)=\begin{pmatrix}-\frac{1}{2}&-\frac{\sqrt{3}}{2}&0\\\frac{\sqrt{3}}{2}&-\frac{1}{2}&0\\0&0&1\end{pmatrix}. \tag{13}$$

The matrices of this representation are unitary, therefore the representation we have just constructed is a unitary representation. From now on we will work specifically on this representation.

Now, from group theory, we know that the number of non-equivalent irreducible representations of a group G of order g is equal to the number of conjugate classes in the group, so in our case, because the number of conjugate classes of S_3 is 3, we can say that the number of irreducible representations of S_3 is 3. On the other hand, we can also determine the dimensions of the irreducible representations of S_3 by the following relation

$$\sum_{\mu=1}^{3}n_\nu^2=n_1^2+n_2^2+n_3^2=6$$

by simple inspection we can notice that the values for which this equation holds are $n_1=n_2=1$ and $n_3=2$, so the irreducible representations of S_3 have dimensions 1 and 2 respectively; that is, we can decompose the group into two singlets and a doublet.

We see that all the matrices in the representation (13) have a block structure, so it is evident that they are reducible and therefore we can decompose them into two matrices of dimensions (2×2) and (1×1), but we also know that there exists a trivial irreducible representation for each group, called the identical representation and which is characterized by $D^{(1)}(R_i)=1$, for all $R_i\in G$. All this is summarized in Table 2.

Using the equation to obtain the characters of a group [19]

$$\chi(R)=\sum_{i=1}^{4}D_{ii}(R) \tag{14}$$

and using table 2 we can build the respective character table 3 for S_3.

Element	$D^{(1)}$	$D^{(2)}$	$D^{(3)}$
E	1	1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
A_1	1	-1	$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
A_2	1	-1	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
A_3	1	-1	$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
A_4	1	1	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
A_5	-1	1	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

Table 2: Representations of the elements of S_3. $D^{(1)}$ and $D^{(2)}$ correspond to one-dimensional representations, while $D^{(3)}$ is a two-dimensional representation.

Classes/Representation	E	K_1^3	K_2^2
$D^{(1)}$	1	1	1
$D^{(2)}$	1	-1	1
$D^{(3)}$	2	0	-1

Table 3: Characters of S_3, where k_i^l $(i = 1, 2)$ corresponds to the ith class of the group and the superscript l indicates the number of elements in the class.

We can express the representations in terms of their irreducible components, that is

$$D = a_1 D^{(1)} \oplus a_2 D^{(2)} \oplus a_3 D^{(3)}, \tag{15}$$

where $\oplus$ denotes the direct sum and the a_ν can be determined according to the equation

$$a^\nu = \frac{1}{g} \sum_{i=1} c_i \chi_i^{\nu *} \chi_i, \tag{16}$$

where g is the order of the group, c_i is the number of elements of class K_i, $\chi_i^{(\nu)}$ is the primitive character of class K_i, and χ_i is the composite character of the class K_i. We compute a_1

$$a_1 = \frac{1}{g} \left[c_1 \chi_1^{(1)} \chi_1 + c_2 \chi_2^{(1)} \chi_2 + c_3 \chi_3^{(1)} \chi_3 \right]$$

$$= \frac{1}{6} \left[(1)(1)(3) + (3)(1)(-1) + (2)(1)(0) \right]$$

consequently $a_1 = 0$, similarly, we find that

$$a_1 = 0, \quad a_2 = 1, \quad a_3 = 1,$$

which means that the representation $D^{(2)}$ and $D^{(3)}$ are included in D, but $D^{(1)}$ is not here.

The matrix representation of equation (13) in terms of its irreducible components is expressed as the direct sum of a single singlet and a doublet, that is,

$$D = D^{(2)} \oplus D^{(3)}. \tag{17}$$

We can obtain a representation of S_3 by direct product of the irreducible representations, in particular we construct the direct product of each element of S_3 with itself in the representation $D^{(3)}$, this is:

$$\begin{aligned} E \otimes E &= \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}, & A_1 \otimes A_1 &= \begin{pmatrix} \frac{1}{2} A_1 & -\frac{\sqrt{3}}{2} A_1 \\ -\frac{\sqrt{3}}{2} A_1 & -\frac{1}{2} \end{pmatrix}, \\ A_2 \otimes A_2 &= \begin{pmatrix} -A_2 & 0 \\ 0 & A_2 \end{pmatrix}, & A_3 \otimes A_3 &= \begin{pmatrix} \frac{1}{2} A_3 & \frac{\sqrt{3}}{2} A_3 \\ \frac{\sqrt{3}}{2} A_3 & -\frac{1}{2} A_3 \end{pmatrix}, \\ A_4 \otimes A_4 &= \begin{pmatrix} -\frac{1}{2} A_4 & \frac{\sqrt{3}}{2} A_4 \\ -\frac{\sqrt{3}}{2} A_4 & -\frac{1}{2} A_4 \end{pmatrix}, & A_5 \otimes A_5 &= \begin{pmatrix} -\frac{1}{2} A_5 & -\frac{\sqrt{3}}{2} A_5 \\ \frac{\sqrt{3}}{2} A_5 & -\frac{1}{2} A_5 \end{pmatrix}, \end{aligned} \tag{18}$$

then we get

$$D^{(2)} \otimes D^{(3)} = D^{(1)} \oplus D^{(2)} \oplus D^{(3)}, \tag{19}$$

which is the direct sum of a pair of singlets and a doublet.

2.3 Projectors

The projectors of the group we can obtain the functions adapted to the S_3 symmetry (invariants). For the projectors we use the equation

$$P^\nu = \frac{n_\nu}{g} \sum_{i=1} \chi_i^{\nu *} C_i, \tag{20}$$

where C_i are the class operators, obtained from the direct product of each representation element $D^{(3)}$; this is,

$$\begin{aligned} C_0 &= E \otimes E, & C_1 &= A_1 \otimes A_1, \\ C_2 &= A_2 \otimes A_2, & C_3 &= A_3 \otimes A_3, \\ C_4 &= A_4 \otimes A_4, & C_5 &= A_5 \otimes A_5 \end{aligned} \tag{21}$$

Then, the form of the projection operator on the symmetric singlet is given by:

$$P_{\mathbf{1}_S} = \frac{1}{6}\left[1C_0 + 1\left(C_1 + C_2 + C_3\right) + 1\left(C_4 + C_5\right)\right]$$

$$P_{\mathbf{1}_S} = \frac{1}{6}\begin{pmatrix} E - A_2 + \frac{1}{2}\left(A_1 + A_3 - A_4 - A_5\right) & -\frac{\sqrt{3}}{2}\left(A_1 - A_3 - A_4 + A_5\right) \\ -\frac{\sqrt{3}}{2}\left(A_1 - A_3 + A_4 - A_5\right) & E + A_1 - \frac{1}{2}\left(A_1 + A_3 + A_4 + A_5\right) \end{pmatrix}$$

$$P_{\mathbf{1}_S} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Similarly, we find $P_{1_{\mathbf{A}}}$, the projector on the antisymmetric singlet $\left(P^2\right)$

$$P_{\mathbf{1}_A} = \frac{1}{6}\left[1C_0 - 1\left(C_1 + C_2 + C_3\right) + 1\left(C_4 + C_5\right)\right] P_{\mathbf{1_A}}$$

$$P_{\mathbf{1}_A} = \frac{1}{6}\begin{pmatrix} E + A_2 - \frac{1}{2}\left(A_1 + A_4 + A_5\right) & -\frac{\sqrt{3}}{2}\left(A_1 - A_3 + A_4 - A_5\right) \\ -\frac{\sqrt{3}}{2}\left(A_1 - A_3 - A_4 + A_5\right) & E - A_2 - \frac{1}{2}\left(A_1 + A_3 - A_4 - A_5\right) \end{pmatrix}$$

$$P_{\mathbf{1}_A} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

And finally we find P_2, the projector on the doublet $\left(P^3\right)$,

$$P_{\mathbf{2}} = \frac{2}{6}\left[2C_0 - 0\left(C_1 + C_2 + C_3\right) - 1\left(C_4 + C_5\right)\right] P_{\mathbf{A}}$$

$$P_{\mathbf{2}} = \mathbb{I}_{4\times 4} - P_{\mathbf{1}_S} - P_{\mathbf{1}_A}$$

$$P_{\mathbf{2}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

The projector P_2 can be decomposed into two terms, each of which is a tensor of rank one, that is,

$$P_2 = P_2^{(1)} + P_2^{(2)},$$

where

$$P_2^{(1)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & -1 \end{pmatrix},$$

and

$$P_2^{(1)} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of the elementary projectors will allow the construction of the matrices $\mathcal{U}$ that diagonalize the product by blocks $D\left(R\right) \times D\left(R\right)$.

With the help of the projectors, the direct product $\mathbf{X} \otimes \mathbf{Y}$ is decomposed into a direct sum of singlets and doublets. Then, applying each of the projectors obtained previously to said direct product, we obtain:

$$P_{\mathbf{1}_S}\left(\mathbf{X}\otimes\mathbf{Y}\right)=\frac{1}{2}\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix}\begin{pmatrix}x_1y_1\\x_1y_2\\x_2y_1\\x_2y_2\end{pmatrix}=\frac{1}{2}\begin{pmatrix}x_1y_1+x_2y_2\\0\\0\\x_1y_1+x_2y_2\end{pmatrix}, \tag{22}$$

$$P_{\mathbf{1}_A}\left(\mathbf{X}\otimes\mathbf{Y}\right)=\frac{1}{2}\begin{pmatrix}0&0&0&0\\0&1&-1&0\\0&-1&1&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}x_1y_1\\x_1y_2\\x_2y_1\\x_2y_2\end{pmatrix}=\frac{1}{2}\begin{pmatrix}0\\x_1y_2-x_2y_1\\-x_1y_2+x_2y_1\\0\end{pmatrix}, \tag{23}$$

$$P_{\mathbf{2}}\left(\mathbf{X}\otimes\mathbf{Y}\right)=\frac{1}{2}\begin{pmatrix}1&0&0&-1\\0&1&1&0\\0&1&1&0\\-1&0&0&1\end{pmatrix}\begin{pmatrix}x_1y_1\\x_1y_2\\x_2y_1\\x_2y_2\end{pmatrix}=\frac{1}{2}\begin{pmatrix}x_1y_1+x_2y_2\\x_1y_2+x_2y_1\\x_1y_2+x_2y_1\\-x_1y_1+x_2y_2\end{pmatrix}. \tag{24}$$

Or

$$P_{\mathbf{1}_S}\left(\mathbf{X}\otimes\mathbf{Y}\right)=\frac{1}{\sqrt{2}}\left(x_1y_1+x_2y_2\right)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \tag{25}$$

$$P_{\mathbf{1}_A}\left(\mathbf{X}\otimes\mathbf{Y}\right)=\frac{1}{\sqrt{2}}\left(x_1y_2-x_2y_1\right)\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \tag{26}$$

$$P_{\mathbf{2}}\left(\mathbf{X}\otimes\mathbf{Y}\right)=\frac{1}{\sqrt{2}}\left(x_1y_1-x_2y_2\right)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}+\frac{1}{\sqrt{2}}\left(x_1y_2+x_2y_1\right)\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}. \tag{27}$$

The coefficients of the eigenvectors are the functions adapted to symmetry. If (x_1,x_2) and (y_1,y_2) are the components of two doublets of S_3, the Kronecker product $(x_1,x_2)\otimes(y_1,y_2)$ contains the following components:

- A symmetrical singlet:

  $$\frac{1}{\sqrt{2}}\left(x_1y_1+x_2y_2\right),$$

  which is invariant under the group S_3.

- An antisymmetric singlet:

  $$\frac{1}{\sqrt{2}}\left(x_1y_2-x_2y_1\right),$$

  which is not invariant under the group S_3.

- A doublet whose components are:

  $$\left(\frac{1}{\sqrt{2}}\left(x_1y_1-x_2y_2\right),\frac{1}{\sqrt{2}}\left(x_1y_2+x_2y_1\right)\right).$$

3 Mass matrices

The Yukawa Lagrangian density for three families of quarks and leptons is written as follows

$$\begin{aligned}\mathscr{L}_Y=&\textstyle\sum_{i=1}^{3}\sum_{j=1}^{3}\left\{\overline{Q}_{ij}Y_{ij}^{u}H\frac{1+\sigma_3}{2}Q_3+\overline{Q}_iY_{ij}^{d}H\frac{1-\sigma_3}{2}Q_j\right\}\\&+\textstyle\sum_{i=1}^{3}\sum_{j=1}^{3}\left\{\bar{l}_{Li}Y_{ij}^{\nu}H\frac{1+\sigma_3}{2}l_{Rj}+\bar{l}_{Li}Y_{ij}^{l}H\frac{1-\sigma_3}{2}l_{Rj}\right\},\end{aligned} \tag{28}$$

where Q_i and L_i denote the weak isospin doublets of quarks and leptons, respectively; H is the Higgs field.

The weak isospin doublets are expressed as follows:

$$\begin{aligned}Q_1(x)&=\begin{pmatrix}u(x)\\d(x)\end{pmatrix}=\begin{pmatrix}\psi_1^u(x)\\\psi_1^d(x)\end{pmatrix}, & l_1(x)&=\begin{pmatrix}\nu_e(x)\\e(x)\end{pmatrix}=\begin{pmatrix}\psi_1^{\nu_l}(x)\\\psi_1^{l}(x)\end{pmatrix},\\Q_2(x)&=\begin{pmatrix}c(x)\\s(x)\end{pmatrix}=\begin{pmatrix}\psi_2^u(x)\\\psi_2^d(x)\end{pmatrix}, & l_2(x)&=\begin{pmatrix}\nu_\mu(x)\\\mu(x)\end{pmatrix}=\begin{pmatrix}\psi_2^{\nu_l}(x)\\\psi_2^{l}(x)\end{pmatrix},\\Q_3(x)&=\begin{pmatrix}t(x)\\b(x)\end{pmatrix}=\begin{pmatrix}\psi_3^u(x)\\\psi_3^d(x)\end{pmatrix}, & l_3(x)&=\begin{pmatrix}\nu_\tau(x)\\\tau(x)\end{pmatrix}=\begin{pmatrix}\psi_3^{\nu_l}(x)\\\psi_3^{l}(x)\end{pmatrix}.\end{aligned} \tag{29}$$

It is convenient to rearrange the terms of equation (28) and write the Lagrangian density $\mathscr{L}_Y$ as a function of the spinors ψ^q and ψ^l, whose components are defined in the space of families like:

$$\begin{array}{ll} \psi^u(x) = \begin{pmatrix} u(x) \\ c(x) \\ t(x) \end{pmatrix} = \begin{pmatrix} \psi_1^u(x) \\ \psi_2^u(x) \\ \psi_3^u(x) \end{pmatrix}, & \psi^{\nu_1}(x) = \begin{pmatrix} \nu_e(x) \\ \nu_\mu(x) \\ \nu_\tau(x) \end{pmatrix} = \begin{pmatrix} \psi_1^{\nu_l}(x) \\ \psi_2^{\nu_l}(x) \\ \psi_3^{\nu_l}(x) \end{pmatrix}, \\ \psi^d = \begin{pmatrix} d(x) \\ s(x) \\ b(x) \end{pmatrix} = \begin{pmatrix} \psi_1^d(x) \\ \psi_2^d(x) \\ \psi_3^d(x) \end{pmatrix}, & \psi^l(x) = \begin{pmatrix} e(x) \\ \mu(x) \\ \tau(x) \end{pmatrix} = \begin{pmatrix} \psi_1^l(x) \\ \psi_2^l(x) \\ \psi_3^l(x) \end{pmatrix}. \end{array} \quad (30)$$

The Yukawa Lagrangian density in family notation, equation (28), is

$$\begin{aligned} \mathscr{L}_Y &= \textstyle\sum_{i=1}^3 \sum_{j=1}^3 \left\{ \overline{\psi}_{Li} Y_{ij}^u H \psi_{Rj}^u + \overline{\psi}_{Rj}^d Y_{ij}^d H \psi_{Li}^d \right\} \\ &+ \textstyle\sum_{i=1}^3 \sum_{j=1}^3 \left\{ \overline{\psi}_{Li}^{\nu_l} Y_{ij}^{\nu_l} H \psi_{Rj}^{\nu_l} + \overline{\psi}_{Rj}^l Y_{ij}^l H \psi_{Li}^l \right\}. \end{aligned} \quad (31)$$

Before the breaking of gauge symmetry, quarks and leptons have no mass and the theory is chiral. Therefore, the left and right spinors transform independently, that is:

$$\psi_L^q(x) \to \psi'^q_L(x) = \mathbf{g} \begin{pmatrix} \psi_{1L}^q(x) \\ \psi_{2L}^q(x) \\ \psi_{3L}^q(x) \end{pmatrix}, \quad (32)$$

$$\psi_L^l(x) \to \psi'^l_L(x) = \mathbf{g} \begin{pmatrix} \psi_{1L}^l(x) \\ \psi_{2L}^l(x) \\ \psi_{3L}^l(x) \end{pmatrix}, \quad (33)$$

$$\psi_R^q(x) \to \psi'^q_R(x) = \widetilde{\mathbf{g}} \begin{pmatrix} \psi_{1R}^q(x) \\ \psi_{2R}^q(x) \\ \psi_{3R}^q(x) \end{pmatrix}, \quad (34)$$

$$\psi_R^l(x) \to \psi'^l_R(x) = \widetilde{\mathbf{g}} \begin{pmatrix} \psi_{1R}^l(x) \\ \psi_{2R}^l(x) \\ \psi_{3R}^l(x) \end{pmatrix}. \quad (35)$$

with $q = u, d$ and $l = \nu_l, l$; where $\mathbf{g} \in S_{3L}$ acts on the left spinors, and $\widetilde{\mathbf{g}} \in S_{3R}$ on the right spinors.

When the gauge symmetry is spontaneously broken, the fermions acquire mass. Therefore, the fields of quarks are transformed as follows:

$$\psi^q(x) \to \psi'^q(x) = \mathbf{g} \begin{pmatrix} \psi_{1L}^q(x) \\ \psi_{2L}^q(x) \\ \psi_{3L}^q(x) \end{pmatrix} + \mathbf{g} \begin{pmatrix} \psi_{1R}^q(x) \\ \psi_{2R}^q(x) \\ \psi_{3R}^q(x) \end{pmatrix}. \quad (36)$$

In the same way, the fields of charged leptons and neutrinos transform as:

$$\psi^q(x) \to \psi'^q(x) = \mathbf{g} \begin{pmatrix} \psi_{1L}^l(x) \\ \psi_{2L}^l(x) \\ \psi_{3L}^l(x) \end{pmatrix} + \mathbf{g} \begin{pmatrix} \psi_{1R}^l(x) \\ \psi_{2R}^l(x) \\ \psi_{3R}^l(x) \end{pmatrix}. \quad (37)$$

The left and right chirality components of the same field are transformed with the same group element. The flavor symmetry group of the bilinear forms in equations (36) and (37) is the group S_3^{diag} whose elements are the pairs $(\mathbf{g}, \widetilde{\mathbf{g}})$, with $\mathbf{g} \in S_{3L}$ and $\widetilde{\mathbf{g}} \in S_{3R}$ and $\mathbf{g} = \mathbf{g}'$. Clearly, $S_3^{diag} \subset S_{3L} \times S_{3R}$.

The mass term from the Yukawa coupling is transformed as follows:

$$\mathscr{L}_Y \to \mathscr{L}'_Y = \overline{\psi}'^u_L M_u \psi'^u_R + \overline{\psi}'^d_R M_d \psi'^d_L + \overline{\psi}'^{\nu_l}_L M_{\nu_l} \psi'^{\nu_l}_R + \overline{\psi}'^l_R M_l \psi'^l_L + h.c., \quad (38)$$

substituting the expressions for $\psi'^q, \overline{\psi}'^q$, $\psi'^l, \overline{\psi}'^l$, we obtain

$$\mathscr{L}'_Y = \overline{\psi}^u_L \mathbf{g}^T M_u \mathbf{g} \psi^u_R + \overline{\psi}^d_R \mathbf{g}^T M_d \mathbf{g} \psi^d_L + \overline{\psi}^{\nu_l}_L \mathbf{g}^T M_{\nu_l} \mathbf{g} \psi^{\nu_l}_R + \overline{\psi}^l_R \mathbf{g}^T M_l \mathbf{g} \psi^l_L + h.c.. \quad (39)$$

Therefore, under the action of the flavor group S_3^{diag}, the mass matrices M_q and M_l are transformed according to the following rule:

$$\begin{aligned} M'_u &= \mathbf{g}^T M_u \mathbf{g}, \\ M'_d &= \mathbf{g}^T M_d \mathbf{g}, \\ M'_{\nu_l} &= \mathbf{g}^T M_{\nu_l} \mathbf{g}, \\ M'_l &= \mathbf{g}^T M_l \mathbf{g}. \end{aligned} \quad (40)$$

If the Yukawa sector is invariant to the family group S_3^{diag}, it must be true that

$$M'_q = M_q, \quad M'_l = M_l, \quad (41)$$

The Yukawa sector of the Standard Model has family symmetry if the mass matrix commutes with all elements of the group S_3, that is

$$\begin{aligned}[M_{u,S},\mathbf{g}]=0,\\ [M_{d,S},\mathbf{g}]=0,\\ [M_{\nu_l,S},\mathbf{g}]=0,\\ [M_{l,S},\mathbf{g}]=0.\end{aligned} \tag{42}$$

4 Model with 4 Higgs doublets and symmetry S_3

Using the symmetry S_3 we have at our disposal 3 irreducible representations, $\mathbf{2}$, $\mathbf{1}_s$, $\mathbf{1}_a$, where we can accommodate up to 4 Higgses in a model that occupies all the irreducible representations of the group S_3. This model S_3-4H is of great interest, thanks to the fact that we can take the fourth Higgs as a stable particle without interaction with fermions so that it becomes a candidate for dark matter, while with the remaining three Higgses the properties obtained are maintained. in a model with four doublets. It is worth mentioning that the possibility of obtaining dark matter candidates with only three doublets was explored [20–22], but no way was found without altering the results obtained for this model.

The Higgs potential has quadratic and quartic terms. This means that we need to find the invariants of S_3 made with two and four fields that are irreducible representations of S_3. The invariants of S_3 for the quadratic and quartic tensor product are, respectively,

$$\mathbf{1_s}\otimes\mathbf{1_s},\quad \mathbf{1_a}\otimes\mathbf{1_a},\quad [\mathbf{2}\otimes\mathbf{2}]_\mathbf{s}$$

and

$$\begin{array}{ccc}
\mathbf{1_s}\otimes\mathbf{1_s}\otimes\mathbf{1_s}\otimes\mathbf{1_s}, & \mathbf{1_a}\otimes\mathbf{1_a}\otimes\mathbf{1_a}\otimes\mathbf{1_a}, & [\mathbf{2}\otimes\mathbf{2}]_\mathbf{s}\otimes[\mathbf{2}\otimes\mathbf{2}]_\mathbf{s}, \\
\mathbf{1_s}\otimes\mathbf{1_s}\otimes\mathbf{1_a}\otimes\mathbf{1_a}, & \mathbf{1_s}\otimes\mathbf{1_s}\otimes[\mathbf{2}\otimes\mathbf{2}]_\mathbf{s}, & \mathbf{1_a}\otimes\mathbf{1_a}\otimes[\mathbf{2}\otimes\mathbf{2}]_\mathbf{s}, \\
\mathbf{1_s}\otimes\mathbf{1_a}\otimes[\mathbf{2}\otimes\mathbf{2}]_\mathbf{a}, & \mathbf{1_s}\otimes\left[\mathbf{2}\otimes[\mathbf{2}\otimes\mathbf{2}]_\mathbf{2}\right]_\mathbf{s}, & \mathbf{1_a}\otimes\left[\mathbf{2}\otimes[\mathbf{2}\otimes\mathbf{2}]_\mathbf{2}\right]_\mathbf{a}, \\
[\mathbf{2}\otimes\mathbf{2}]_\mathbf{a}\otimes[\mathbf{2}\otimes\mathbf{2}]_\mathbf{a}, & \text{y} & \left[[\mathbf{2}\otimes\mathbf{2}]_\mathbf{2}\otimes[\mathbf{2}\otimes\mathbf{2}]_\mathbf{2}\right]_\mathbf{s}.
\end{array}$$

For the construction of the terms in the potential, we need to consider the weak index of the fields. We make the corresponding projections to generate the invariants of S_3 and for this purpose we use the projections of the four-dimensional real basis of S_3.

- Invariants with two fields:

(a)

$$H_s^\dagger\otimes H_s$$

(b)

$$H_a^\dagger\otimes H_a$$

(c)

$$\mathcal{W}_s\left(H_D^\dagger\otimes H_D\right)=\mathcal{W}_s\begin{pmatrix}H_1^\dagger\\ H_2^\dagger\end{pmatrix}\otimes\begin{pmatrix}H_1\\ H_2\end{pmatrix}=$$

$$\mathcal{W}_s\begin{pmatrix}H_1^\dagger H_1\\ H_1^\dagger H_2\\ H_2^\dagger H_1\\ H_2^\dagger H_2\end{pmatrix}=\frac{1}{\sqrt{2}}\left(H_1^\dagger H_1+H_2^\dagger H_2\right)|\mathbf{1_S}\rangle\,.$$

- Invariants with four fields:

H_D+H_s :

(a)

$$\left(H_s^\dagger\otimes H_s\right)\otimes\left(H_s^\dagger\otimes H_s\right)$$

(b)

$$\left(H_s^\dagger\otimes H_s\right)\mathcal{W}_s\left(H_D^\dagger\otimes H_D\right)=\frac{1}{\sqrt{2}}\left(H_s^\dagger\otimes H_s\right)\left(H_1^\dagger H_1+H_2^\dagger H_2\right)|\mathbf{1_s}\rangle$$

(c)

$$\mathcal{W}_s(H_D^\dagger H_1\otimes H_s^\dagger H_D)+h.c.=\frac{1}{\sqrt{2}}\left[\left(H_s^\dagger H_1\right)^2+\left(H_s^\dagger H_2\right)^2+\left(H_1^\dagger H_s\right)^2+\left(H_2^\dagger H_s\right)^2\right]|\mathbf{1_s}\rangle$$

(d)

$$\mathcal{W}_s(H_D^\dagger H_1 \otimes H_s^\dagger H_D) = \frac{1}{\sqrt{2}}\left[\left(H_s^\dagger H_1\right)\left(H_1^\dagger H_s\right) + \left(H_2^\dagger H_s\right)\left(H_2^\dagger H_s\right)\right]|\mathbf{1_s}\rangle$$

(e)

$$\mathcal{W}_s\left[H_s^\dagger H_D \otimes \mathcal{W}_s\left(H_D^\dagger \otimes H_D\right)\right] + h.c. = \tfrac{1}{\sqrt{2}}\mathcal{W}_s\left[\left(H_s^\dagger H_D\right) \otimes \begin{pmatrix} H_1^\dagger H_2 + H_2^\dagger H_1 \\ H_1^\dagger H_1 - H_2^\dagger H_2 \end{pmatrix}\right] + h.c.$$
$$= \tfrac{1}{2}\left[\left(H_s^\dagger H_1 + H_1^\dagger H_s\right)\left(H_1^\dagger H_2 + H_2^\dagger H_1\right) + \left(H_s^\dagger H_2 + H_2^\dagger H_s\right)\left(H_1^\dagger H_1 - H_2^\dagger H_2\right)\right]|\mathbf{1_s}\rangle$$

(f)

$$\mathcal{W}_s\left[\mathcal{W}_s\left(H_D^\dagger \otimes H_D\right) \otimes \mathcal{W}_s\left(H_D^\dagger \otimes H_D\right)\right] = \tfrac{1}{2}\mathcal{W}_s\left[\begin{pmatrix} H_1^\dagger H_2 + H_2^\dagger H_1 \\ H_1^\dagger H_1 - H_2^\dagger H_2 \end{pmatrix} \otimes \begin{pmatrix} H_1^\dagger H_2 + H_2^\dagger H_1 \\ H_1^\dagger H_1 - H_2^\dagger H_2 \end{pmatrix}\right]$$
$$= \tfrac{1}{2\sqrt{2}}\left[\left(H_1^\dagger H_2 + H_2^\dagger H_1\right)\left(H_1^\dagger H_2 + H_2^\dagger H_1\right) + \left(H_1^\dagger H_1 - H_2^\dagger H_2\right)\left(H_1^\dagger H_1 - H_2^\dagger H_2\right)\right]|\mathbf{1_s}\rangle$$

(g)

$$\mathcal{W}_s\left(H_D^\dagger \otimes H_D\right) \otimes \mathcal{W}_s\left(H_D^\dagger \otimes H_D\right)$$
$$= \tfrac{1}{2}\left[\left(H_1^\dagger H_2 + H_2^\dagger H_1\right)^2 + \left(H_1^\dagger H_1 - H_2^\dagger H_2\right)^2\right]|\mathbf{1_s}\rangle$$

(h)

$$\mathcal{W}_a\left(H_D^\dagger \otimes H_D\right) \otimes \mathcal{W}_a\left(H_D^\dagger \otimes H_D\right)$$
$$= \tfrac{1}{2}\left(H_1^\dagger H_1 - H_2^\dagger H_2\right)^2|\mathbf{1_s}\rangle$$

$H_D + H_s + H_a$:

(a)

$$\left(H_a^\dagger \otimes H_a\right) \otimes \left(H_s^\dagger \otimes H_s\right)$$

(b)

$$\left(H_a^\dagger \otimes H_a\right) \otimes \left(H_a^\dagger \otimes H_a\right)$$

(c)

$$\left(H_a^\dagger \otimes H_s\right) \otimes \left(H_s^\dagger \otimes H_a\right)$$

(d)

$$\left(H_a^\dagger \otimes H_s\right) \otimes \left(H_a^\dagger \otimes H_s\right) + h.c.$$

(e)

$$\left(H_a^\dagger \otimes H_a\right)\mathcal{W}_s\left(H_D^\dagger \otimes H_D\right) = \frac{1}{\sqrt{2}}\left(H_a^\dagger H_a\right)\left(H_1^\dagger H_1 + H_2^\dagger H_2\right)|\mathbf{1_s}\rangle$$

(f)

$$\mathcal{W}_s\left(H_D^\dagger H_a \otimes H_D^\dagger H_a\right) + h.c. = \mathcal{W}_s\left[\begin{pmatrix} -H_2^\dagger H_a \\ H_1^\dagger H_a \end{pmatrix} \otimes \begin{pmatrix} -H_2^\dagger H_a \\ H_1^\dagger H_a \end{pmatrix}\right] + h.c.$$
$$= \tfrac{1}{\sqrt{2}}\left[\left(H_1^\dagger H_a\right)^2 + \left(H_2^\dagger H_a\right)^2 + \left(H_a^\dagger H_1\right)^2 + \left(H_a^\dagger H_2\right)^2\right]|\mathbf{1_s}\rangle$$

(g)

$$\mathcal{W}_s\left(H_D^\dagger H_a\otimes H_a^\dagger H_D\right)=\mathcal{W}_s\left[\left(\begin{array}{c}-H_2^\dagger H_a\\ H_1^\dagger H_a\end{array}\right)\otimes\left(\begin{array}{c}-H_a^\dagger H_2\\ H_a^\dagger H_1\end{array}\right)\right]+h.c.$$
$$=\tfrac{1}{\sqrt{2}}\left[\left(H_1^\dagger H_a\right)\left(H_a^\dagger H_1\right)+\left(H_2^\dagger H_a\right)\left(H_a^\dagger H_2\right)\right]|\mathbf{1_s}\rangle$$

(h)

$$\left(H_s^\dagger\otimes H_a\right)\mathcal{W}_a\left(H_D^\dagger\otimes H_D\right)+h.c.=\frac{1}{\sqrt{2}}\left(H_s^\dagger H_a-H_a^\dagger H_s\right)\left(H_1^\dagger H_2-H_2^\dagger H_1\right)|\mathbf{1_s}\rangle$$

(i)

$$\mathcal{W}_s\left(H_D^\dagger H_s\otimes H_D^\dagger H_a\right)+h.c.=\mathcal{W}_s\left[\left(\begin{array}{c}-H_1^\dagger H_s\\ H_2^\dagger H_s\end{array}\right)\otimes\left(\begin{array}{c}-H_2^\dagger H_a\\ H_1^\dagger H_a\end{array}\right)\right]$$
$$=\tfrac{1}{\sqrt{2}}\left[\left(H_2^\dagger H_s\right)\left(H_1^\dagger H_a\right)-\left(H_1^\dagger H_s\right)\left(H_2^\dagger H_a\right)+h.c.\right]|\mathbf{1_s}\rangle$$

(j)

$$\mathcal{W}_s\left(H_D^\dagger H_s\otimes H_a^\dagger H_D\right)+h.c.=\mathcal{W}_s\left[\left(\begin{array}{c}-H_1^\dagger H_s\\ H_2^\dagger H_s\end{array}\right)\otimes\left(\begin{array}{c}-H_a^\dagger H_2\\ H_a^\dagger H_1\end{array}\right)\right]$$
$$=\tfrac{1}{\sqrt{2}}\left[\left(H_2^\dagger H_s\right)\left(H_a^\dagger H_1\right)-\left(H_1^\dagger H_s\right)\left(H_a^\dagger H_2\right)+h.c.\right]|\mathbf{1_s}\rangle$$

(k)

$$\mathcal{W}_s\left[H_a^\dagger H_D\otimes\mathcal{W}_D\left(H_D^\dagger\otimes H_D\right)\right]+h.c.=\tfrac{1}{\sqrt{2}}\mathcal{W}_s\left[\left(\begin{array}{c}-H_a^\dagger H_2\\ H_a^\dagger H_1\end{array}\right)\otimes\left(\begin{array}{c}H_1^\dagger H_2+H_2^\dagger H_1\\ H_1^\dagger H_1-H_2^\dagger H_2\end{array}\right)\right]$$
$$=\tfrac{1}{2}\left[-\left(H_a^\dagger H_2+H_2^\dagger H_a\right)\left(H_1^\dagger H_2+H_2^\dagger H_1\right)+\left(H_a^\dagger H_1+H_1^\dagger H_a\right)\left(H_1^\dagger H_1-H_2^\dagger H_2\right)\right]|\mathbf{1_s}\rangle\,.$$

To simplify the calculations, we introduce the following variables as in [23, 24]

$$\begin{array}{llll} x_1=H_1^\dagger H_1, & x_5=\mathbb{R}\left(H_2^\dagger H_s\right), & x_9=\mathbb{I}\left(H_2^\dagger H_s\right), & y_4=\mathbb{R}\left(H_s^\dagger H_a\right),\\ x_2=H_2^\dagger H_2, & x_6=\mathbb{R}\left(H_1^\dagger H_s\right), & y_1=H_a^\dagger H_a, & y_5=\mathbb{I}\left(H_1^\dagger H_a\right),\\ x_3=H_s^\dagger H_s, & x_7=\mathbb{I}\left(H_1^\dagger H_2\right), & y_2=\mathbb{R}\left(H_1^\dagger H_a\right), & y_6=\mathbb{I}\left(H_2^\dagger H_a\right),\\ x_4=\mathbb{R}\left(H_1H_2\right), & x_8=\mathbb{I}\left(H_1^\dagger H_s\right), & y_3=\mathbb{R}\left(H_2H_a\right), & y_7=\mathbb{I}\left(H_s^\dagger H_a\right). \end{array}\tag{43}$$

Using the above and taking the assignment of the self-coupling coefficients in the order in which the constructed invariant terms are listed, we obtain:

$$V_{4H}=V_{H_s\otimes H_a}+V_{H_a},\tag{44}$$

where

$$\begin{array}{c} V_{H_s\otimes H_a}=\mu_0^2x_3+\mu_1^2\left(x_1+x_2\right)+ax_3^2+bx_3\left(x_1+x_2\right)+2c\left(x_5^2+x_6^2-x_8^2-x_9^2\right)\\ +d\left(x_5^2+x_6^2+x_8^2+x_9^2\right)+2e\left[\left(x_1-x_2\right)x_5+2x_4x_6\right]+f\left[\left(x_1-x_2\right)^2+4x_4^2\right]\\ +g\left(x_1+x_2\right)^2-4hx_7^2, \end{array}\tag{45}$$

$$\begin{array}{c} V_{H_a}=\mu_2^2y_1+\alpha_1x_3y_1+\alpha_2y_1^2+\alpha_3\left(y_4^2+y_7^2\right)+2\alpha_4\left(y_4^2-y_7^2\right)+\alpha_5y_1\left(x_1+x_2\right)\\ +2\alpha_6\left(y_2^2+y_3^2-y_5^2-y_6^2\right)+\alpha_7\left(y_2^2+y_3^2+y_5^2+y_6^2\right)-4\alpha_8x_7y_7\\ +2\alpha_9\left(x_5y_2+x_8y_6-x_6y_3-x_9y_5\right)+2\alpha_{10}\left(x_5y_2-x_8y_6-x_6y_3+x_9y_5\right)\\ +2\alpha_{11}\left[y_2\left(x_1-x_2\right)-2x_4y_3\right]. \end{array}\tag{46}$$

At the normal minimum, the VEV's of the Higgs fields are considered real. So, once the fields acquire VEV's, we can relate them to the new variables (43) like

$$\begin{array}{cccc} \langle x_1\rangle=v_1^2, & \langle x_2\rangle=v_2^2, & \langle x_3\rangle=v_s^2, & \langle x_4\rangle=v_1v_2,\\ \langle x_5\rangle=v_2v_s, & \langle x_6\rangle=v_1v_s, & \langle x_7\rangle=\langle x_8\rangle=\langle x_9\rangle=0, & \langle y_1\rangle=v_a^2,\\ \langle y_2\rangle=v_1v_a, & \langle y_3\rangle=v_2v_a, & \langle y_4\rangle=v_sv_a, & \langle y_5\rangle=\langle y_6\rangle=\langle y_7\rangle=0. \end{array}\tag{47}$$

4.1 Normal minimum

The minimization equations of potential are defined as:

$$\frac{\partial V_4}{\partial v_i} = \frac{\partial V_4}{\partial x_j}\frac{\partial x_j}{\partial v_i} = 0. \tag{48}$$

with $i = 1, 2, 3$ and $j = 1, 2, ...15$.

First of all, we calculate the different terms individually

$$\tfrac{\partial x_1}{\partial v_1} = 2v_1, \quad \tfrac{\partial x_4}{\partial v_1} = v_2, \quad \tfrac{\partial x_6}{\partial v_1} = v_s, \quad \tfrac{\partial y_2}{\partial v_1} = v_a \quad y \quad \tfrac{\partial x_j}{\partial v_1} = 0, \quad \text{for the rest.}$$

$$\tfrac{\partial x_2}{\partial v_2} = 2v_2, \quad \tfrac{\partial x_4}{\partial v_2} = v_1, \quad \tfrac{\partial x_5}{\partial v_2} = v_s, \quad \tfrac{\partial y_3}{\partial v_2} = v_a \quad y \quad \tfrac{\partial x_j}{\partial v_2} = 0, \quad \text{for the rest.}$$

$$\tfrac{\partial x_3}{\partial v_s} = 2v_s, \quad \tfrac{\partial x_5}{\partial v_s} = v_2, \quad \tfrac{\partial x_6}{\partial v_s} = v_1, \quad \tfrac{\partial y_4}{\partial v_s} = v_a \quad y \quad \tfrac{\partial x_j}{\partial v_s} = 0, \quad \text{for the rest.}$$

$$\tfrac{\partial y_1}{\partial v_a} = 2v_a, \quad \tfrac{\partial y_2}{\partial v_s} = v_1, \quad \tfrac{\partial y_3}{\partial v_a} = v_2, \quad \tfrac{\partial y_4}{\partial v_a} = v_s \quad y \quad \tfrac{\partial x_j}{\partial v_a} = 0, \quad \text{for the rest.}$$

After

$$\frac{\partial V}{\partial x_1} = \mu_1^2 + bx_3 + 2ex_5 + 2f\left(x_1 - x_2\right) + 2g\left(x_1 + x_2\right) + \alpha_5 y_1 + 2\alpha_{11} y_2,$$

$$\frac{\partial V}{\partial x_2} = \mu_1^2 + bx_3 - 2ex_5 - 2f\left(x_1 - x_2\right) + 2g\left(x_1 + x_2\right) + \alpha_5 y_1 - 2\alpha_{11} y_2,$$

$$\frac{\partial V}{\partial x_3} = \mu_1^2 + 2ax_3 + b\left(x_1 + x_2\right) + \alpha_1 y_1,$$

$$\frac{\partial V}{\partial x_4} = 4ex_6 + 8fx_4 - 4\alpha_{11} y_3,$$

$$\frac{\partial V}{\partial x_5} = 2\left(2c + d\right)x_5 + 2e\left(x_1 - x_2\right) + 2\left(\alpha_9 + \alpha_{10}\right)y_2,$$

$$\frac{\partial V}{\partial x_6} = 2\left(2c + d\right)x_6 + 4ex_4 - 2\left(\alpha_9 + \alpha_{10}\right)y_3,$$

$$\frac{\partial V}{\partial x_7} = -8hx_7 - 4\alpha_8 y_7,$$

$$\frac{\partial V}{\partial x_8} = -2\left(2c - d\right)x_8 + 2\left(\alpha_9 - \alpha_{10}\right)y_6,$$

$$\frac{\partial V}{\partial x_9} = -2\left(2c - d\right)x_9 + 2\left(\alpha_9 - \alpha_{10}\right)y_5,$$

$$\frac{\partial V}{\partial y_1} = \mu_2^2 + \alpha_1 x_3 + 2\alpha_2 y_1 + \alpha_5\left(x_1 + x_2\right),$$

$$\frac{\partial V}{\partial y_2} = 2\left(2\alpha_6 + \alpha_7\right)y_2 + 2\left(\alpha_9 + \alpha_{10}\right)x_5 + 2\alpha_{11}\left(x_1 - x_2\right),$$

$$\frac{\partial V}{\partial y_3} = 2\left(2\alpha_6 + \alpha_7\right)y_3 - 2\left(\alpha_9 + \alpha_{10}\right)x_6 - 4\alpha_{11} x_4,$$

$$\frac{\partial V}{\partial y_4} = 2\left(\alpha_3 + 2\alpha_4\right)y_4,$$

$$\frac{\partial V}{\partial y_5} = 2\left(\alpha_7 - 2\alpha_6\right)y_5 + 2\left(\alpha_{10} - \alpha_9\right)x_9,$$

$$\frac{\partial V}{\partial y_6} = 2\left(\alpha_7 - 2\alpha_6\right)y_6 + 2\left(\alpha_9 - \alpha_{10}\right)x_8,$$

$$\frac{\partial V}{\partial y_7} = 2\left(\alpha_3 - 2\alpha_4\right)y_7 - 4\alpha_8 x_7.$$

Evaluating all partial derivatives at the minimum:

$$\left.\frac{\partial V}{\partial x_1}\right|_{min} = \mu_1^2 + bv_s^2 + 2ev_2v_s + 2f\left(v_1^2 - v_2^2\right) + 2g\left(v_1^2 + v_2^2\right) + \alpha_5 v_a^2 + 2\alpha_{11}v_1v_a,$$
$$\left.\frac{\partial V}{\partial x_2}\right|_{min} = \mu_1^2 + bv_s^2 - 2ev_2v_s - 2f\left(v_1^2 - v_2^2\right) + 2g\left(v_1^2 + v_2^2\right) + \alpha_5 v_a^2 - 2\alpha_{11}v_1v_a,$$
$$\left.\frac{\partial V}{\partial x_3}\right|_{min} = \mu_1^2 + 2ax_3 + b\left(x_1 + x_2\right) + \alpha_1 y_1,$$
$$\left.\frac{\partial V}{\partial x_4}\right|_{min} = 4ev_1v_s + 8fv_1v_2 - 4\alpha_{11}v_2v_a,$$
$$\left.\frac{\partial V}{\partial x_5}\right|_{min} = 2\left(2c + d\right)v_2v_s + 2e\left(v_1^2 - v_2^2\right) + 2\left(\alpha_9 + \alpha_{10}\right)v_1v_a,$$
$$\left.\frac{\partial V}{\partial x_6}\right|_{min} = 2\left(2c + d\right)v_1v_s + 4ev_1v_2 - 2\left(\alpha_9 + \alpha_{10}\right)v_2v_a,$$
$$\left.\frac{\partial V}{\partial x_7}\right|_{min} = 0,$$
$$\left.\frac{\partial V}{\partial x_8}\right|_{min} = 0,$$
$$\left.\frac{\partial V}{\partial x_9}\right|_{min} = 0,$$
$$\left.\frac{\partial V}{\partial y_1}\right|_{min} = \mu_2^2 + \alpha_1 v_s^2 + 2\alpha_2 v_a^2 + \alpha_5\left(v_1^2 + v_2^2\right),$$
$$\left.\frac{\partial V}{\partial y_2}\right|_{min} = 2\left(2\alpha_6 + \alpha_7\right)v_1v_a + 2\left(\alpha_9 + \alpha_{10}\right)v_2v_s + 2\alpha_{11}\left(v_1^2 - v_2^2\right),$$
$$\left.\frac{\partial V}{\partial y_3}\right|_{min} = 2\left(2\alpha_6 + \alpha_7\right)v_2v_a - 2\left(\alpha_9 + \alpha_{10}\right)v_1v_s - 4\alpha_{11}v_1v_2,$$
$$\left.\frac{\partial V}{\partial y_4}\right|_{min} = 2\left(\alpha_3 + 2\alpha_4\right)v_sv_a,$$
$$\left.\frac{\partial V}{\partial y_5}\right|_{min} = 0,$$
$$\left.\frac{\partial V}{\partial y_6}\right|_{min} = 0,$$
$$\left.\frac{\partial V}{\partial y_7}\right|_{min} = 0.$$

Then, we can write the system of equations of the minimum:

$$v_1\left[\mu_1^2 + \left(b + d + 2c\right)v_s^2 + \left(\alpha_5 + \alpha_7 + 2\alpha_6\right)v_a^2 + 2\left(f + g\right)\left(v_1^2 + v_2^2\right) + 6ev_2v_s\right] = 3\alpha_{11}v_a\left(v_2^2 - v_1^2\right),$$
$$v_2\left[\mu_1^2 + \left(b + d + 2c\right)v_s^2 + \left(\alpha_5 + \alpha_7 + 2\alpha_6\right)v_a^2 + 2\left(f + g\right)\left(v_1^2 + v_2^2\right) - 6\alpha_{11}v_1v_a\right] = 3ev_s\left(v_2^2 - v_1^2\right),$$
$$v_s\left[\mu_0^2 + 2av_s^2 + \left(\alpha_1 + \alpha_3 + 2\alpha_4\right)v_a^2 + 2\left(b + d + 2c\right)\left(v_1^2 + v_2^2\right)\right] = ev_2\left(v_2^2 - 3v_1^2\right),$$
$$v_a\left[\mu_2^2 + 2\alpha_2 v_a^2 + \left(\alpha_1 + \alpha_3 + 2\alpha_4\right)v_s^2 + 2\left(\alpha_5 + \alpha_7 + 2\alpha_6\right)\left(v_1^2 + v_2^2\right)\right] = \alpha_{11}v_1\left(3v_2^2 - v_1^2\right).$$

Using the potential (54), the equations of the minimum are

$$v_1\left[\mu_1^2 + \left(\lambda_5 + \lambda_6 + 2\lambda_7\right)v_s^2 + \left(\lambda_{10} + \lambda_{12} + 2\lambda_{15}\right)v_a^2 + 2\left(\lambda_1 + \lambda_2\right)\left(v_1^2 + v_2^2\right) + 6\lambda_4 v_2v_s\right] = 3\lambda_9 v_a\left(v_2^2 - v_1^2\right), \tag{49}$$

$$v_2\left[\mu_1^2 + \left(\lambda_5 + \lambda_6 + 2\lambda_7\right)v_s^2 + \left(\lambda_{10} + \lambda_{12} + 2\lambda_{15}\right)v_a^2 + 2\left(\lambda_1 + \lambda_2\right)\left(v_1^2 + v_2^2\right) - 6\lambda_9 v_1v_s\right] = 3\lambda_4 v_0\left(v_2^2 - v_1^2\right), \tag{50}$$
$$v_s\left[\mu_0^2 + 2\lambda_8 v_s^2 + \lambda_{14}v_a^2 + 2\left(\lambda_5 + \lambda_6 + 2\lambda_7\right)\left(v_1^2 + v_2^2\right)\right] = \lambda_4 v_2\left(v_2^2 - 3v_1^2\right), \tag{51}$$
$$v_a\left[\mu_2^2 + 2\lambda_{13}v_a^2 + \lambda_{14}v_s^2 + \left(\lambda_{10} + \lambda_{12} + 2\lambda_{15}\right)\left(v_1^2 + v_2^2\right)\right] = \lambda_9 v_1\left(3v_2^2 - v_1^2\right). \tag{52}$$

Furthermore, we obtain the equation

$$v_1 = \sqrt{3} v_2. \tag{53}$$

In this way we can describe the mass matrices in a simpler way.

4.2 $4HDM$ Model

The scalar potential (symmetry invariant and renormalizable $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \otimes S_3$) is a mixture of the potentials known from studies of the three Higgs model with permutation symmetry, and can be written as:

$$\begin{aligned}
V_4 = {} & \mu_0^2 H_s^\dagger H_s + \mu_1^2 \left(H_1^\dagger H_1 + H_2^\dagger H_2\right) + \mu_2^2 H_a^\dagger H_a \\
& + \lambda_1 \left(H_1^\dagger H_1 + H_2^\dagger H_2\right)^2 + \lambda_2 \left(H_1^\dagger H_2 - H_2^\dagger H_1\right)^2 \\
& + \lambda_3 \left[\left(H_1^\dagger H_1 - H_2^\dagger H_2\right)^2 + \left(H_1^\dagger H_2 + H_2^\dagger H_1\right)^2\right] \\
& + \lambda_4 \left[\left(H_s^\dagger H_1\right)\left(H_1^\dagger H_2 + H_2^\dagger H_1\right) + \left(H_s^\dagger H_2\right)\left(H_1^\dagger H_1 - H_2^\dagger H_2\right) + h.c.\right] \\
& + \lambda_5 \left(H_s^\dagger H_s\right)\left(H_1^\dagger H_1 + H_2^\dagger H_2\right) + \lambda_8 \left(H_s^\dagger H_s\right)^2 \\
& + \lambda_6 \left[\left(H_s^\dagger H_1\right)\left(H_1^\dagger H_s\right) + \left(H_s^\dagger H_2\right)\left(H_2^\dagger H_s\right)\right] \\
& + \lambda_7 \left[\left(H_s^\dagger H_1\right)\left(H_s^\dagger H_1\right) + \left(H_s^\dagger H_2\right)\left(H_s^\dagger H_2\right) + h.c.\right] \\
& + \lambda_9 \left[\left(H_a^\dagger H_2\right)\left(H_1^\dagger H_2 + H_2^\dagger H_1\right) - \left(H_a^\dagger H_1\right)\left(H_1^\dagger H_1 - H_2^\dagger H_2\right) + h.c.\right] \\
& + \lambda_{10} \left(H_a^\dagger H_a\right)\left(H_1^\dagger H_1 + H_2^\dagger H_2\right) \\
& + \lambda_{11} \left[\left(H_a^\dagger H_1\right)\left(H_1^\dagger H_a\right) + \left(H_a^\dagger H_2\right)\left(H_2^\dagger H_a\right)\right] \\
& + \lambda_{12} \left[\left(H_a^\dagger H_1\right)\left(H_a^\dagger H_1\right) + \left(H_a^\dagger H_2\right)\left(H_a^\dagger H_2\right) + h.c.\right] \\
& + \lambda_{13} \left(H_a^\dagger H_a\right)^2 + \lambda_{14} \left(H_s^\dagger H_a H_a^\dagger H_s\right) \\
& + \lambda_{15} \left[\left(H_1^\dagger H_a\right)\left(H_2^\dagger H_a\right) + h.c.\right].
\end{aligned} \tag{54}$$

The most general S_3-invariant Yukawa Lagrangian density for the coupling of 4-Higgs coupled Dirac fermions (see table 4), where both components of the third family are assigned to the symmetric singlet of S_3 is:

$$\begin{aligned}
-\mathscr{L}_{Y_f} = {} & Y_1 \left(\overline{\psi}_{S,L} \psi_{S,R} H_s\right) + \tfrac{1}{\sqrt{2}} Y_2 \left(\overline{\psi}_{1,L} \psi_{1,R} + \overline{\psi}_{2,L} \psi_{2,R}\right) H_s \\
& + \tfrac{1}{\sqrt{2}} Y_3 \left[\left(\overline{\psi}_{1,L} H_2 + \overline{\psi}_{2,L} H_1\right) \psi_{1,R} + \left(\overline{\psi}_{1,L} H_1 - \psi_{2,L} H_2\right) \psi_{2,R}\right] \\
& + \tfrac{1}{\sqrt{2}} Y_4 \left(\overline{\psi}_{1,L} \psi_{2,R} - \overline{\psi}_{2,L} \psi_{1,R}\right) H_a \\
& + \tfrac{1}{\sqrt{2}} Y_5 \left(\overline{\psi}_{1,L} H_1 + \overline{\psi}_{1,L} H_1 + \overline{\psi}_{2,L} H_2\right) \psi_{S,R} \\
& + \tfrac{1}{\sqrt{2}} Y_6 \overline{\psi}_{S,L} \left(H_1 \psi_{1,R} + H_2 \psi_{2,R}\right) + h.c.
\end{aligned} \tag{55}$$

Particles	$SU(3)_c \times SU(2)_L \times U(1)_Y$	S_3	Z_2
Q_1, Q_2	$(3, 2, 1/3)$	2	$+1$
Q_3	$(3, 2, 1/3)$	1	$+1$
u_{1R}, u_{2R}	$(3, 1, 4/3)$	2	$+1$
u_{3R}	$(3, 1, 4/3)$	1	$+1$
d_{1R}, d_{2R}	$(3, 1, -2/3)$	2	$+1$
d_{3R}	$(3, 1, -2/3)$	1	$+1$
L_1, L_2	$(1, 2, -1)$	2	$+1$
L_3	$(1, 2, -1)$	1	$+1$
e_{1R}, e_{2R}	$(1, 1, -2)$	2	$+1$
e_{3R}	$(1, 1, -2)$	1	$+1$
H_1, H_2	$(0, 2, 1)$	2	$+1$
H_s	$(0, 2, 1)$	1	$+1$
H_a	$(0, 2, 1)$	1	-1

Table 4: Particle spectrum of SM group and $S_3 \otimes Z_2$.

where Y_i are complex Yukawa couplings. When writing the Yukawa Lagrangian density, for up-type quarks or Dirac neutrinos, the Higgs field must be replaced by the conjugate Higgs field $H_{iw} \to i\sigma_2 H^*_{iW}$, $i = 1, 2$.

After symmetry breaking [25], the Higgs doublets $SU(2)_L$ acquire expectation values in a vacuum, which we choose real.

$$\begin{array}{cc} v_1 \equiv \langle 0| H_1 |0\rangle & v_2 \equiv \langle 0| H_2 |0\rangle \\ v_s \equiv \langle 0| H_s |0\rangle & v_a \equiv \langle 0| H_a |0\rangle , \end{array} \tag{56}$$

In this communication, the Yukawa interactions yield mass matrices, for all fermions in the theory, starting by $\mathscr{L}_{Y_D}$

$$\mathscr{L}_{Y_D} = \frac{1}{2} \left(\begin{array}{ccc} \bar{d} & \bar{s} & \bar{b} \end{array} \right) \mathbf{M}_{Y_D} \begin{pmatrix} d \\ s \\ b \end{pmatrix} = \frac{1}{2} \left(\begin{array}{ccc} \bar{d} & \bar{s} & \bar{b} \end{array} \right) \begin{pmatrix} m_1^d + m_2^d & m_4^d + m_5^d & m_6^d \\ m_4^d - m_5^d & m_1^d - m_2^d & m_7^d \\ m_8^d & m_9^d & m_3^d \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} \tag{57}$$

or

$$\mathbf{M}_{Y_D} = \begin{pmatrix} m_1^d + m_2^d & m_4^d + m_5^d & m_6^d \\ m_4^d - m_5^d & m_1^d - m_2^d & m_7^d \\ m_8^d & m_9^d & m_3^d \end{pmatrix} . \tag{58}$$

giving mass to the fermions of the Standard Model. The eigenvalues are m_i^d, $i = 1, 2, 3$. It is convenient to define the notation

$$\begin{array}{lll} m_1^d \equiv 2Y_1^d v_s, & m_2^d \equiv 2Y_2^d v_2, & m_3^d \equiv 2Y_3^d v_1, \\ m_4^d \equiv 2Y_2^d v_1, & m_5^d \equiv 2Y_4^d v_a, & m_6^d \equiv 2Y_5^d v_1, \\ m_7^d \equiv 2Y_5^d v_2, & m_8^d \equiv 2Y_6^d v_1 & m_9^d \equiv 2Y_6^d v_2, \end{array} \tag{59}$$

The rest of the matrices (and in particular $\mathbf{M}_{Y_\nu}$) have the same structure, changing only the Yukawa coupling terms, i. e.,

$$\mathscr{L}_{Y_U} = \frac{1}{2} \left(\begin{array}{ccc} \bar{u} & \bar{c} & \bar{t} \end{array} \right) \begin{pmatrix} m_1^u + m_2^u & m_4^u + m_5^u & m_6^u \\ m_4^u - m_5^u & m_1^u - m_2^u & m_7^u \\ m_8^u & m_9^u & m_3^u \end{pmatrix} \begin{pmatrix} u \\ c \\ t \end{pmatrix} , \tag{60}$$

$$\mathscr{L}_{Y_E} = \frac{1}{2} \left(\begin{array}{ccc} \bar{e} & \bar{\mu} & \bar{\tau} \end{array} \right) \begin{pmatrix} m_1^e + m_2^e & m_4^e + m_5^e & m_6^e \\ m_4^e - m_5^e & m_1^e - m_2^e & m_7^e \\ m_8^e & m_9^e & m_3^e \end{pmatrix} \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix} , \tag{61}$$

$$\mathscr{L}_{Y_\nu} = \frac{1}{2} \left(\begin{array}{ccc} \overline{\nu_e} & \overline{\nu_\mu} & \overline{\nu_\tau} \end{array} \right) \begin{pmatrix} m_1^\nu + m_2^\nu & m_4^\nu + m_5^\nu & m_6^\nu \\ m_4^\nu - m_5^\nu & m_1^\nu - m_2^\nu & m_7^\nu \\ m_8^\nu & m_9^\nu & m_3^\nu \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} , \tag{62}$$

with the mass matrices

$$\mathbf{M}_{Y_U} = \begin{pmatrix} m_1^u + m_2^u & m_4^u + m_5^u & m_6^u \\ m_4^u - m_5^u & m_1^u - m_2^u & m_7^u \\ m_8^u & m_9^u & m_3^u \end{pmatrix} , \tag{63}$$

$$\mathbf{M}_{Y_E} = \begin{pmatrix} m_1^e + m_2^e & m_4^e + m_5^e & m_6^e \\ m_4^e - m_5^e & m_1^e - m_2^e & m_7^e \\ m_8^e & m_9^e & m_3^e \end{pmatrix} , \tag{64}$$

$$\mathbf{M}_{Y_\nu} = \begin{pmatrix} m_1^\nu + m_2^\nu & m_4^\nu + m_5^\nu & m_6^\nu \\ m_4^\nu - m_5^\nu & m_1^\nu - m_2^\nu & m_7^\nu \\ m_8^\nu & m_9^\nu & m_3^\nu \end{pmatrix} . \tag{65}$$

If we use equation (53), we obtain

$$\mathbf{M}_{Y_U} = \begin{pmatrix} m_1^u + m_2^u & \sqrt{3}m_2^u + m_5^u & \sqrt{3}m_7^u \\ \sqrt{3}m_2^u - m_5^u & m_1^u - m_2^u & m_7^u \\ \sqrt{3}m_9^u & m_9^u & m_3^u \end{pmatrix} , \tag{66}$$

$$\mathbf{M}_{Y_D} = \begin{pmatrix} m_1^d + m_2^d & \sqrt{3}m_2^d + m_5^d & \sqrt{3}m_7^d \\ \sqrt{3}m_2^d - m_5^d & m_1^d - m_2^d & m_7^d \\ \sqrt{3}m_9^d & m_9^d & m_3^d \end{pmatrix} . \tag{67}$$

$$\mathbf{M}_{Y_E} = \begin{pmatrix} m_1^e + m_2^e & \sqrt{3}m_2^e + m_5^e & \sqrt{3}m_7^e \\ \sqrt{3}m_2^e - m_5^e & m_1^e - m_2^e & m_7^e \\ \sqrt{3}m_9^e & m_9^e & m_3^e \end{pmatrix} , \tag{68}$$

$$\mathbf{M}_{Y_\nu} = \begin{pmatrix} m_1^\nu + m_2^\nu & \sqrt{3}m_2^\nu + m_5^\nu & \sqrt{3}m_7^\nu \\ \sqrt{3}m_2^\nu - m_5^\nu & m_1^\nu - m_2^\nu & m_7^\nu \\ \sqrt{3}m_9^\nu & m_9^\nu & m_3^\nu \end{pmatrix}. \tag{69}$$

Models with three Higgs doublets can be obtained as special cases of models with 4 Higgs doublets, e.g. to obtain the mass matrix of a 3HDM [26–40] and the third fermion family in the singlet representation asymmetric, it is enough to take the limit when $H_a \to 0$

$$\begin{aligned} -\lim_{H_a \to 0} \mathscr{L}_{Y_f} &= Y_1 \left(\overline{\psi}_{S,L}\psi_{S,R}H_s\right) + \\ &\frac{1}{\sqrt{2}}Y_2\left(\overline{\psi}_{1,L}\psi_{1,R} + \overline{\psi}_{2,L}\psi_{2,R}\right)H_s + \\ &\frac{1}{\sqrt{2}}Y_3\left[\left(\overline{\psi}_{1,L}H_2 + \overline{\psi}_{2,L}H_1\right)\psi_{1,R} + \right. \\ &\left.\left(\overline{\psi}_{1,L}H_1 - \psi_{2,L}H_2\right)\psi_{2,R}\right. \\ &+ \cancelto{0}{\frac{1}{\sqrt{2}}Y_4\left(\overline{\psi}_{1,L}\psi_{2,R} - \overline{\psi}_{2,L}\psi_{1,R}\right)H_a} \\ &+ \frac{1}{\sqrt{2}}Y_5\left(\overline{\psi}_{1,L}H_1 + \overline{\psi}_{1,L}H_1 + \overline{\psi}_{2,L}H_2\right)\psi_{S,R} \\ &+ \frac{1}{\sqrt{2}}Y_6\overline{\psi}_{S,L}\left(H_1\psi_{1,R} + H_2\psi_{2,R}\right) + h.c. \end{aligned} \tag{70}$$

then, the mass matrices are given by:

$$\mathbf{M}_{Y_D} = \begin{pmatrix} m_1^d + m_2^d & m_4^d & m_6^d \\ m_4^d & m_1^d - m_2^d & m_7^d \\ m_8^d & m_9^d & m_3^d \end{pmatrix}. \tag{71}$$

$$\mathbf{M}_{Y_U} = \begin{pmatrix} m_1^u + m_2^u & m_4^u & m_6^u \\ m_4^u & m_1^u - m_2^u & m_7^u \\ m_8^u & m_9^u & m_3^u \end{pmatrix}, \tag{72}$$

$$\mathbf{M}_{Y_E} = \begin{pmatrix} m_1^e + m_2^e & m_4^e & m_6^e \\ m_4^e & m_1^e - m_2^e & m_7^e \\ m_8^e & m_9^e & m_3^e \end{pmatrix}, \tag{73}$$

$$\mathbf{M}_{Y_\nu} = \begin{pmatrix} m_1^\nu + m_2^\nu & m_4^\nu & m_6^\nu \\ m_4^\nu & m_1^\nu - m_2^\nu & m_7^\nu \\ m_8^\nu & m_9^\nu & m_3^\nu \end{pmatrix}. \tag{74}$$

4.3 4HDM with Z_2

This model has a dark matter candidate from a model with S_3 symmetry without interfering with the positive results obtained in [26–32].

An important condition for having a viable dark matter candidate is its stability. That is, it does not decay into Standard Model particles. The simplest way to establish the stability of a particle in a model beyond the standard, is imposing a discrete symmetry Z_2, so that all the fields are transformed in the form $\Psi \longrightarrow \Psi$, while the two dark matter candidates are transformed as $\chi \longrightarrow -\chi$, this way we make sure we don't have terms denoting decays of χ. This method has been used in numerous models, such as the scotogenic [41] and the inert scalar doublet [42]. It is worth mentioning that there are also models with more complex discrete symmetries, such as Z_3 in [43]. In general you can make models with symmetry Z_n.

In this model with symmetry Z_2, i. e. 4HDM [44–46], dark matter candidate is the Higgs boson in the anti-symmetric singlet representation H_a , so these transform under Z_2 as $H_a \longrightarrow -H_a$. So, the Lagrangian density of Yukawa is given by:

$$\begin{aligned} -\mathscr{L}_{Y_f} &= Y_1\left(\overline{\psi}_{S,L}\psi_{S,R}H_s\right) + \\ &\frac{1}{\sqrt{2}}Y_2\left(\overline{\psi}_{1,L}\psi_{1,R} + \overline{\psi}_{2,L}\psi_{2,R}\right)H_s \\ &+ \frac{1}{\sqrt{2}}Y_3\left[\left(\overline{\psi}_{1,L}H_2 + \overline{\psi}_{2,L}H_1\right)\psi_{1,R} + \left(\overline{\psi}_{1,L}H_1 - \psi_{2,L}H_2\right)\psi_{2,R}\right] \\ &+ \boldsymbol{\frac{1}{\sqrt{2}}\mathbf{Y_4}\left(\overline{\psi}_{\mathbf{1,L}}\psi_{\mathbf{2,R}} - \overline{\psi}_{\mathbf{2,L}}\psi_{\mathbf{1,R}}\right)\mathbf{H_a}} \\ &+ \frac{1}{\sqrt{2}}Y_5\left(\overline{\psi}_{1,L}H_1 + \overline{\psi}_{1,L}H_1 + \overline{\psi}_{2,L}H_2\right)\psi_{S,R} \\ &+ \frac{1}{\sqrt{2}}Y_6\overline{\psi}_{S,L}\left(H_1\psi_{1,R} + H_2\psi_{2,R}\right) + h.c. \end{aligned} \tag{75}$$

and scalar potential

$$\begin{aligned} V &= \mu_0^2 H_s^\dagger H_s + \mu_1^2\left(H_1^\dagger H_1 + H_2^\dagger H_2\right) + \mu_2^2 H_a^\dagger H_a \\ &+ \lambda_1\left(H_1^\dagger H_1 + H_2^\dagger H_2\right)^2 + \lambda_2\left(H_1^\dagger H_2 - H_2^\dagger H_1\right)^2 \\ &+ \lambda_3\left[\left(H_1^\dagger H_1 - H_2^\dagger H_2\right)^2 + \left(H_1^\dagger H_2 + H_2^\dagger H_1\right)^2\right] \\ &+ \lambda_4\left[\left(H_s^\dagger H_1\right)\left(H_1^\dagger H_2 + H_2^\dagger H_1\right) + \left(H_s^\dagger H_2\right)\left(H_1^\dagger H_1 - H_2^\dagger H_2\right) + h.c.\right] \\ &+ \lambda_5\left(H_s^\dagger H_s\right)\left(H_1^\dagger H_1 + H_2^\dagger H_2\right) + \lambda_8\left(H_s^\dagger H_s\right)^2 \\ &+ \lambda_6\left[\left(H_s^\dagger H_1\right)\left(H_1^\dagger H_s\right) + \left(H_s^\dagger H_2\right)\left(H_2^\dagger H_s\right)\right] \\ &+ \lambda_7\left[\left(H_s^\dagger H_1\right)\left(H_s^\dagger H_1\right) + \left(H_s^\dagger H_2\right)\left(H_s^\dagger H_2\right) + h.c.\right] \\ &+ \boldsymbol{\lambda_9\left[\left(\mathbf{H_a^\dagger H_2}\right)\left(\mathbf{H_1^\dagger H_2 + H_2^\dagger H_1}\right) - \left(\mathbf{H_a^\dagger H_1}\right)\left(\mathbf{H_1^\dagger H_1 - H_2^\dagger H_2}\right) + \mathbf{h.c.}\right]} \\ &+ \lambda_{10}\left(H_a^\dagger H_a\right)\left(H_1^\dagger H_1 + H_2^\dagger H_2\right) \\ &+ \lambda_{11}\left[\left(H_a^\dagger H_1\right)\left(H_1^\dagger H_a\right) + \left(H_a^\dagger H_2\right)\left(H_2^\dagger H_a\right)\right] \\ &+ \lambda_{12}\left[\left(H_a^\dagger H_1\right)\left(H_a^\dagger H_1\right) + \left(H_a^\dagger H_2\right)\left(H_a^\dagger H_2\right) + h.c.\right] \\ &+ \lambda_{13}\left(H_a^\dagger H_a\right)^2 + \lambda_{14}\left(H_s^\dagger H_a H_a^\dagger H_s\right) \\ &+ \boldsymbol{\lambda_{15}\left[\left(\mathbf{H_1^\dagger H_s}\right)\left(\mathbf{H_2^\dagger H_a}\right) + \mathbf{h.c.}\right]} \end{aligned} \tag{76}$$

with this new symmetry.

The terms highlighted in bold, correspond to those that break the Z_2 symmetry and are therefore omitted, note that H_a no longer appears in the Yukawa Lagrangian density. Another imposition required to propose the candidacy of a field of the doublet H_a, is that its corresponding VEV is equal to zero, $v_a = 0$ (For more details of this model, see [44–46]).

5 Dark matter candidate

This model is based on S_3 symmetry, which allows us to accommodate the four Higgs doublets:

$$H_1 = \begin{pmatrix} h_1^c \\ h_1^n + v_1 + ih_1^p \end{pmatrix}, \quad H_2 = \begin{pmatrix} h_2^c \\ h_2^n + v_2 + ih_2^p \end{pmatrix},$$
$$H_s = \begin{pmatrix} h_s^c \\ h_s^n + v_0 + ih_s^p \end{pmatrix}, \quad H_a = \begin{pmatrix} h_a^c \\ h_a^n + v_a + ih_a^p \end{pmatrix}. \tag{77}$$

Hence, the fourth Higgs doublet, H_a, contains four physical fields, two charged h_a^c and two neutral, the scalar h_a^n and the pseudoscalar h_a^p. Charged particles are restricted as dark matter candidates [47]. Thus, viable candidates are the antisymmetric doublet neutral Higgs fields, h_a^n and h_a^p, with masses:

$$m^2_{h_a^p} = \mu_2^2 + \lambda_{14} v_0^2 + 4\left(\lambda_{10} + \lambda_{11} - 2\lambda_{12}\right) v_2^2, \tag{78}$$

$$m^2_{h_a^n} = \mu_2^2 + \lambda_{14} v_0^2 + 4\left(\lambda_{10} + \lambda_{11} + 2\lambda_{12}\right) v_2^2, \tag{79}$$

the lightest neutral Higgs field will be the dark matter candidate resulting from the fourth Higgs doublet.

There are theoretical and experimental constraints, which apply to the analysis of the candidate to constrain the mass range of this and the rest of the Higgs fields in the model. Using the IDM [48], we have the constraints:

Theoretical restrictions

1. The potential must be bounded from below, so that it has a stable vacuum [44].

2. The quartic couplings of the Higgses must be perturbative, i.e. $\left|a_i^{\pm}\right|$, $|b_i| < 16\pi$ [44].

Experimental restrictions

1. The mass Higgs boson of the standard model is [49]:
$$m_{h_s^n} = 125.09 \pm 0.21 \text{ GeV}. \tag{80}$$

2. The upper limit of the total amplitude of the boson h_s^n [50] [51] is:
$$\Gamma < 22 \text{ MeV}. \tag{81}$$

3. The relic density obtained in the PLANCK experiment [52]
$$\Omega h^2 \leq 0.1241. \tag{82}$$

In this model, we consider the mass of the Standard Model Higgs $m_h = 125$ GeV, taking the possibility of two of the scalar neutral fields (those corresponding to the Higgs doublets H_s and H_2).

6 Concluding remarks

In this article, we study the 4HDM model in the theoretical framework of the minimum extension S_3 of the standard model. We extend the Higgs sector by adding four Higgs doublets and making the theory invariant with respect to flavor permutations. We impose Z_2 symmetry on the fourth Higgs doublet, H_a. In this model, the dark matter candidate is the Higgs doublet in the antisymmetric singlet representation H_a, so these transform under Z_2 as $H_a \longrightarrow -H_a$.

The Higgs doublets are denoted as:

$$H_D \equiv \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \sim \mathbf{2}, \quad H_s \sim \mathbf{1}_s, \quad H_a \sim \mathbf{1}_a.$$

We accommodate four $SU(2)$ doublets into the irreducible representations of the permutation group S_3, denoting the symmetric and antisymmetric scalars by H_s and H_a respectively, while the remaining two doublet H_1 and H_2 are arranged how

$$H_s = \begin{pmatrix} h_s^c \\ h_s^n + v_0 + ih_s^p \end{pmatrix} \quad H_a = \begin{pmatrix} h_a^c \\ h_a^n + v_a + ih_a^p \end{pmatrix}$$
$$H_1 = \begin{pmatrix} h_1^c \\ h_1^n + v_1 + ih_1^p \end{pmatrix} \quad H_2 = \begin{pmatrix} h_2^c \\ h_2^n + v_2 + ih_2^p \end{pmatrix}$$

In the model with four Higgs doublets (4HDM), we takes the fourth Higgs doublet as a stable particle without interaction with fermions, making it a candidate for dark matter, while with the remaining three the properties obtained are maintained. Another imposition that we impose on the doublet H_a is that its corresponding VEV is equal to zero, $v_a = 0$.

In this model, we compute the system of equations of the minimum of the potential, so we obtain the equation $v_1 = \sqrt{3} v_2$. In this way we can describe the mass matrices in a simpler way. The 3HDM models can be obtained as special cases of the 4HDM models, that is, in the limit $H_a \to 0$, we obtain the mass matrices

$$\mathrm{M}_{Y_U} = \begin{pmatrix} m_1^u + m_2^u & \sqrt{3}m_2^u & \sqrt{3}m_7^u \\ \sqrt{3}m_2^u & m_1^u - m_2^u & m_7^u \\ \sqrt{3}m_9^u & m_9^u & m_3^u \end{pmatrix},$$

$$\mathbf{M}_{Y_D} = \begin{pmatrix} m_1^d + m_2^d & \sqrt{3}m_2^d & \sqrt{3}m_7^d \\ \sqrt{3}m_2^d & m_1^d - m_2^d & m_7^d \\ \sqrt{3}m_9^d & m_9^d & m_3^d \end{pmatrix},$$

$$\mathbf{M}_{Y_E} = \begin{pmatrix} m_1^e + m_2^e & \sqrt{3}m_2^e & \sqrt{3}m_7^e \\ \sqrt{3}m_2^e & m_1^e - m_2^e & m_7^e \\ \sqrt{3}m_9^e & m_9^e & m_3^e \end{pmatrix},$$

$$\mathbf{M}_{Y_\nu} = \begin{pmatrix} m_1^\nu + m_2^\nu & \sqrt{3}m_2^\nu & \sqrt{3}m_7^\nu \\ \sqrt{3}m_2^\nu & m_1^\nu - m_2^\nu & m_7^\nu \\ \sqrt{3}m_9^\nu & m_9^\nu & m_3^\nu \end{pmatrix}.$$

Additionally, the 4HDM model can continue to be studied and understood. Then future perspectives could be to calculate radiative corrections that show corrections to the mass values. Another additional perspective of the work would be the possibility of extending the dark matter sector of the model with additional particles.

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