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Quantum Cosmology of the Universe with $S^1 \bigotimes T_g$ Spatial Sections

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Quantum creation of a universe with an $S^1 \bigotimes T_g$ spatial topology is considered. Using both the Euclidean functional integral and the tunneling approach as prescriptions applied to quantum cosmology, we calculate the wave function of such a universe with a negative cosmological constant and without matter. The anisotropic minisuperspace path integral is calculated in the semiclassical approximation, and it is shown that the wave function is proportional to the genus of the two- dimensional compact hyperbolic manifold T_g and to the absolute value of the negative cosmological constant. Thus we obtain the probability of quantum creation of a higher-genus universe, and it appears that in Hartle-Hawking approach the probability of creation increases with the genus g, while in the tunneling approach it decreases with g.
Keywords: Cosmology, Hartle-Hawking approach, tunneling approach.

Cosmología cuántica del Universo con secciones espaciales $S^1 \bigotimes T_g$

Consideramos la creación cuántica de un universo con una topología espacial $S^1 \bigotimes T_g$. Usando la integral funcional euclideana y la aproximación por tunelamiento como prescripciones aplicadas a la cosmología cuántica, calculamos la función de onda de un universo con una constante cosmológica negativa y sin materia. La integral de camino de un minisuperespacio anisotrópico es calculado en la aproximación semiclásica y muestra que la función de onda es proporcional al genus de la variedad T_g hiperbólica compacta bidimensional y al valor absoluto de la constante cosmológica negativa. De esta forma obtenemos la probabilidad de la creación cuántica de un universo de alto genus y aparece en la aproximación de Hartle-Hawking que este se incrementa con el genus g, mientras que en la aproximación por tunelamiento este decrece con g.
Palabras claves: Cosmología, aproximación de Hartle-Hawking, aproximación por tunelamiento.

In recent year there is much interest in quantum cosmology. Recently the question of topology has been added into the discussion. What is the most probable topology of the universe?. It is well known that there exist a lot more hyperbolic manifolds than elliptic and flat ones. So it is interesting to study the hyperbolic manifolds in quantum cosmology.

Here we shall study the probability of creation of a universe with the space topology $S^1 \bigotimes T_g$ in both the Hartle-Hawking (HH) boundary condition and the tunneling prescription due to Vilenkin. The anisotropic minisuperspace path integral is evaluated in the semi-classical approximation, and it is shown that the information about the space topology is encoded in the probabilities of creation.

The wave funtion of the universe due to the HH proposal is given by the path integral [1, 2]

$$\Psi(h_{ij}) = \int_{\Gamma} D(g_{\mu\nu}) \exp[-S_E(g_{\mu\nu})], \tag{1}$$

where S_E is the Euclidean action of the gravitational

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field with a cosmological constant

$$S_E = \frac{-1}{16\pi G}\int_\Omega d^4x g^{1/2}(R-2\Lambda) - \frac{1}{8\pi G}\int_{\partial\Omega} d^3x h^{1/2}K. \qquad (2)$$

As the path integral (1) has no precise definition so its qualitative behaviour can be obtained using semiclassical techniques in simple cosmological models or equivalently one restricts to minisuperspace. So the boundary conditions in the semiclassical approximation to the wave function is in the form

$$\Psi(h_{ij}) = N_0 \sum A_i \exp(-B_i)\,, \qquad (3)$$

where N_0 is a normalization constant and B_i is the action of the Euclidean classical solution. The prefactor A_i denote fluctuations about classical solutions.

The line element of the Euclidean Bianchi type III universe, is given by

$$ds^2 = N(\tau)^2 d\tau^2 + a(\tau)^2 dr^2 + b(\tau)^2 d\Omega_g^2, \qquad (4)$$

where the coordinate r is periodic with a period 2π, and the metric of the compact hyperbolic surface of genus g, T_g is $d\Omega_g^2 = d\rho^2 + \sinh^2\rho d\phi^2$.

The connections are

$$\begin{gathered}
\Gamma^\tau{}_{\tau\tau} = \frac{\dot N}{N}\,,\, \Gamma^\tau{}_{rr} = -\frac{\dot a a}{N^2}\,,\Gamma^\tau{}_{\rho\rho} = -\frac{\dot b b}{N^2}\,,\\
\Gamma^\tau{}_{\phi\phi} = -\frac{\dot b b}{N^2}\sinh^2\rho\,,\Gamma^r{}_{r\tau} = \frac{\dot a}{a}\,,\\
\Gamma^\rho{}_{\rho\tau} = \Gamma^\phi{}_{\phi\tau} = \frac{\dot b}{b} \qquad (5)\\
\Gamma^\rho{}_{\phi\phi} = -\sinh\rho\cosh\rho\,,\, \Gamma^\phi{}_{\phi\rho} = \coth\rho\,.
\end{gathered}$$

The scalar curvature becomes

$$R = -2\frac{\ddot a}{aN^2} - 4\frac{\ddot b}{bN^2} + 2\frac{\dot a\dot N}{aN^3} + 4\frac{\dot b\dot N}{bN^3} - 4\frac{\dot a\dot b}{abN^2} -2\frac{\dot b^2}{b^2N^2} - \frac{2}{b^2}. \qquad (6)$$

The volumen of space section with the topology $S^1 \bigotimes T_g$ is [3]

$$V = \int d^3x\, h^{1/2} dr d\rho d\phi = 8\pi^2(g-1)ab^2\,, \qquad (7)$$

so the Euclidean Einstein-Hilbert action is

$$S_E = \frac{\pi(g-1)}{G}\int d\tau\left[-\frac{2\dot a\dot b b}{N} - \frac{a\dot b^2}{N} + Na + Nab^2\Lambda\right]. \qquad (8)$$

Thus the information about the topology of the spatial compact section is encoded in the action. The field equations are obtained by varying the Euclidean action (8) with respect to N, b, and a. In the gauge in which $\dot N = 0$ these equations are

$$\begin{aligned}
\frac{2\dot a\dot b b}{N^2} + \frac{a\dot b^2}{N^2} + a(1+\Lambda b^2) &= 0\\
2\ddot b b + \dot b^2 + N^2(1+\Lambda b^2) &= 0 \qquad (9)\\
\ddot b a + \ddot a b + \dot a\dot b + \Lambda abN^2 &= 0.
\end{aligned}$$

With some rearrangement, the before field equations may be written as [4]

$$\begin{aligned}
\frac{2\dot a\dot b b}{N^2} + \frac{a\dot b^2}{N^2} + a(1+\Lambda b^2) &= 0\\
\frac{\dot b}{Na} &= const \qquad (10)\\
\ddot a b + 2\dot a\dot b + \Lambda abN^2 &= 0.
\end{aligned}$$

Integration of equation (9) gives

$$\dot b^2 + \frac{\Lambda}{3}N^2b^2 + N^2 - \frac{C}{b} = 0\,. \qquad (11)$$

Putting the integration constant C equal to zero we see that the cosmological constant should be negative for the existence of the solution to the field equation. Integrating once more the equation (11) and using the relation between $a(\tau)$ and $b(\tau)$ given by (10), we obtain

$$\begin{aligned}
b(\tau) &= \sqrt{\frac{3}{|\Lambda|}}\cosh\left[\sqrt{\frac{|\Lambda|}{3}}N\tau\right]\\
a(\tau) &= \sqrt{\frac{3}{|\Lambda|}}\sinh\left[\sqrt{\frac{|\Lambda|}{3}}N\tau\right]. \qquad (12)
\end{aligned}$$

From the boundary condition we know that the classical solution must be everywhere regular, the boundary term at $\tau = 0$ should vanish, and the quantum 4-metric has a vanishing 3-volume at the bottom. From (12) we see that

$$a(0) = 0\,,\, \frac{1}{N}\frac{da(0)}{d\tau} = 1\,, \qquad (13)$$

which is consistent with above requirement.

Wave function in HH approach and comparison with tunneling wave function

We now turn to the calculation of the wave function for the solution given above. Using the constraint equation, the action (8) can be rewriting as

$$S_E = -\frac{2\pi(g-1)}{G}\int_0^1 d\tau\left[\frac{2\dot a\dot b b}{N} + \frac{a\dot b^2}{N}\right]. \qquad (14)$$

Taking the field equation (10) or $\dot a = NHb(\tau)$ and the boundary conditions $b(0) = 0$, $b(1) = b_0$ we obtain

$$S_E = -\frac{2\pi(g-1)}{G}\int_0^{b_0} db\left[2Hb^2 + \frac{\dot b^2}{NH}\right], \qquad (15)$$

where $H = \sqrt{|\Lambda|/3}$. Thus, for our case the final form of the action is

$$S_E(a,b) = -\frac{2\pi(g-1)}{G} ab \left[H^2b^2 - 1\right]^{1/2}. \quad (16)$$

The semiclassical approximation of the HH wave function (3) for $b < H^{-1}$ is therefore

$$\Psi(a,b) = N_0 \exp\left[\frac{2\pi(g-1)}{G} ab \left[H^2b^2 - 1\right]^{1/2}\right], \quad (17)$$

whereas for the $b > H^{-1}$ case, the wave function can be obtained by analytical continuation

$$\Psi(a,b) = N'_0 \cos\left[\frac{2\pi(g-1)}{G} ab \left[1 - H^2b^2\right]^{1/2} + \delta\right]. \quad (18)$$

The phase shift constant δ must be calculated by using the K-representation of the HH [1] which is under our investigation.

Thus the unnormalized probability of creation of a universe with the topology taken into consideration for $b < H^{-1}$ case is

$$|\Psi_{HH}|^2 = {N_0}^2 \exp\left[\frac{4\pi(g-1)}{G} ab \left[H^2b^2 - 1\right]^{1/2}\right]. \quad (19)$$

From the other hand, Vilenkin [5] observed that the full Wheller-De Witt equation is invariant under the transformation

$$h_{ij} \to \exp[i\pi] h_{ij}\,, V(\phi) \to \exp[-i\pi] V(\phi). \quad (20)$$

These means that there is a relation between tunneling and HH wave funtions: Ψ_{HH} and Ψ_T are related by an analytical continuation.

Using these relations for our case

$$\Psi_T = \Psi_{HH}$$

$$\begin{aligned} H^2 &\to \exp[i\pi] H^2 \\ a &\to \exp[-i\pi/2] a \\ b &\to \exp[-i\pi/2] b \end{aligned} \quad (21)$$

we obtain the tunneling wave funtion for $b < H^{-1}$

$$\Psi_T(a,b) = N_1 \exp\left[-\frac{2\pi(g-1)}{G} ab \left[H^2b^2 - 1\right]^{1/2}\right] \quad (22)$$

where the coefficients N_0 and N_1 are different since they correspond to different boundary conditions, and their exact values should be evaluated using the semiclassical saddle point method in the K-representation.

For the $b > H^{-1}$ case we have

$$\Psi_T(a,b) = N'_1 \exp\left[-\frac{2i\pi(g-1)}{G} ab \left[1 - H^2b^2\right]^{1/2}\right] \quad (23)$$

The unnormalized probability of creation of a universe in the tunneling approach for $b < H^{-1}$ is given by

$$|\Psi_T|^2 = {N_1}^2 \exp\left[-\frac{4\pi(g-1)}{G} ab \left[H^2b^2 - 1\right]^{1/2}\right]. \quad (24)$$

Thus we obtain the probability of quantum creation of a higher-genus universe, and note that the wave functions are proportional to the genus of the two-dimensional compact hyperbolic manifold T_g and to the absolute value of the negative cosmological constant.

Final remarks

In this paper we have considered the probability of creation of a universe with the space topology $S^1 \bigotimes T_g$ in both the Hartle-Hawking boundary condition and the tunneling prescription due to Vilenkin. In order to calculate the anisotropic minisuperspace path integral it is used the semiclassical approximation, and we show that the wave function is proportional to the genus of the two-dimensional compact hyperbolic manifold and to the absolute value of the negative cosmological constant. Comparing equations (19) and (24), it appears that in HH approach the probability of creation increases with the genus g, while in the tunneling approach it decreases with g.

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