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GLOBAL SOLUTIONS AND DECAY OF A NON LINEAR COUPLED SYSTEM WITH THERMO-ELASTIC

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Abstract: In this present work, the authors prove the existence of global solutions and the decay of nonlinear wave equation with thermo-elastic coupling give by the system of equation:

$$u''(x,t) - \mu(t)\Delta u(x,t) + \sum_{i=1}^{n} \frac{\partial \theta}{\partial x_i}(x,t) + F\left(u(x,t)\right) = 0 \text{ in } Q = \Omega \times (0,\infty)$$

$$\theta'(x,t) - \Delta\theta(x,t) + \sum_{i=1}^{n} \frac{\partial u'}{\partial x_i}(x,t) = 0 \text{ in } Q = \Omega \times (0,\infty),$$

where u is displacement, θ is absolute temperature, Δ denotes the Laplace operator, μ is a positive real function of t, $F : \mathbb{R} \to \mathbb{R}$ is continuous function such that $s \cdot F(s) \geq 0$, Ω is a smooth bounded open set in $\mathbb{R}^n$ with boundary Γ.

Keywords: Weak solutions; Strauss approximation; asymptotic behavior.

1. Introduction

The nonlinear wave equation with thermos-elastic coupling is given by the system of equation

$$u''(x,t) - \mu(t)\Delta u(x,t) + \sum_{i=1}^{n} \frac{\partial \theta}{\partial x_i}(x,t) + F\left(u(x,t)\right) = 0 \tag{1}$$

$$\theta'(x,t) - \Delta\theta(x,t) + \sum_{i=1}^{n} \frac{\partial u'}{\partial x_i}(x,t) = 0 \tag{2}$$

with initial and boundary conditions

$$u(x,0) = u^0(x), \quad u'(x,0) = u^1(x), \quad \theta(x,0) = \theta^0(x) \text{ in } \Omega, \tag{3}$$

$$u(x,t) = 0, \quad \theta(x,t) = 0 \text{ in } \Gamma \times (0,\infty), \tag{4}$$

where u is displacement, θ is absolute temperature, Δ denotes the Laplace operator, μ is a positive real function of t, F is a function such that $s \cdot F(s) \geq 0$, the temporal partial derivative is represented by $\dfrac{\partial u}{\partial t} = u'$, is Ω is a smooth bounded open set in $\mathbb{R}^n$ with C^2 boundary Γ, and $Q = \Omega \times (0,\infty)$.
The non linearity $F(v) = |v|^{\rho} v$ usually appears in relativistic quantum mechanic (see Segal [13] or Schiff [12]), and has been considered by various authors for hyperbolic, parabolic and elliptic equations. Lions [6] studied the wave equation with the same non linearity, i.e., $|v|^{\rho} v$, in

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a smooth bounded open domain Ω of $\mathbb{R}^n$ and proved existence and uniqueness of solution using both Faedo-Galerkin's and Compactnesss' methods.
In [1] investigated the system (1) - (4) with $F(v) = |v|^\rho v$. They established global existence and strong and weak solutions by Faedo-Galerkin's method using a basis of the space $H_0^1(\Omega) \cap H^2(\Omega)$ and the exponential stability of total energy associated to the weak solution using Komornik-Zuazua's method [4].
Based in the theory developed in the paper [1] and Strauss approximations of F [15], we will prove that the system (1) - (4) has a unique global strong solution, a unique global weak solution, and the total energy associated to these solutions is asymptotically stable.
The outline of this chapter is as follows. In Section 2, the basic theory is laid out and global existence of strong and weak solutions are issued for the Lipschitzian case and general case, whilst exponential decay in Section 3.

2. Existence of Solution

To obtain the existence and uniqueness of global solution of the mixed problem (1) - (4) we suppose the additional hypotheses about μ and the function F :

$$\mu \in W^{1,1}_{loc}(0,\infty), \quad \mu(t) \geq \mu_0, \ \forall t \geq 0 \text{ and } \ \mu'(t) \leq 0 \text{ a. e. in } (0,\infty) \tag{5}$$

$$\text{F is continuous and } \ s \cdot F(s) \geq 0 \text{ for all } s \in \mathbb{R}$$

Let us represent by G the function

$$G(s) = \int_0^s F(r)dr$$

Now we can present the existence results of the initial and boundary value problem (1) - (4).

Theorem 2.1 (Case: F Lipchitzian) *Let $F : \mathbb{R} \to \mathbb{R}$, be such that $s \cdot F(s) \geq 0$, F is Lipschitzian and derivable except a finite number of points and μ be the function defined by above hypothesis (5). Given*

$$u_0, \theta_0 \in H_0^1(\Omega) \cap H^2(\Omega) \quad , u_1 \in H_0^1(\Omega)$$

then the system (1) - (4) has a unique strong solution $\{u, \theta\}$ such that

$$u, \theta \in L^\infty(0,\infty; H_0^1(\Omega))$$

$$u' \in L^\infty(0,\infty; L^2(\Omega)) \cap L^\infty_{loc}(0,\infty; H_0^1(\Omega)), \quad \theta' \in L^2_{loc}(0,\infty; H_0^1(\Omega))$$

$$u'' \in L^\infty_{loc}(0,\infty; L^2(\Omega))$$

Proof: Existence. To show global existence of solution we will use both the Faedo-Galerkin's and Compactness' methods. We consider $(w_j)_{j\in\mathbb{N}}$ an orthonormal basis of $H_0^1(\Omega) \cap H^2(\Omega)$, and denote by $V_m = [w_1, w_2, ..., w_m]$ the subspace of $H_0^1(\Omega) \cap H^2(\Omega)$ spanned by the m first vectors of $(w_j)_{j\in\mathbb{N}}$.
In these conditions, the approximated system associated to system (1) - (2) is given by

$$u_m(t) = \sum_{i=1}^m g_{jm}(t) w_j, \quad \theta_m(t) = \sum_{i=1}^m h_{jm}(t) w_j$$

$$(u_m''(t), v) + \mu(t)((u_m(t), v)) + \sum_{i=1}^n \left(\frac{\partial \theta_m}{\partial x_i}(t), v \right) + (F(u_m(t)), v) = 0 \tag{6}$$

$$(\theta'_m(t), w) + ((\theta_m(t), w)) + \sum_{i=1}^{n} \left(\frac{\partial u'_m}{\partial x_i}(t), w \right) = 0 \tag{7}$$

where v and w belong to V_m.
Let $u_m(0) = u_{0m},\ u'_m(0) = u_{1m}$ and $\theta_m(0) = \theta_{0m}$ be. Hence u_{0m}, u_{1m} and θ_{0m} belong to V_m and satisfy

$$u_{0m} \to u_0 \ \text{ strongly in } \ H_0^1(\Omega) \cap H^2(\Omega) \tag{8}$$

$$\theta_{0m} \to \theta_0 \ \text{ strongly in } \ H_0^1(\Omega) \cap H^2(\Omega) \tag{9}$$

$$u_{1m} \to u_1 \ \text{ strongly in } \ H_0^1(\Omega) \tag{10}$$

Under these conditions, the system (6) - (7) has a local solution $\{u_m(t), \theta_m(t)\}$ over the interval $[0, t_m)$. This interval will be extended to any interval $[0, \infty)$ thanks to the first estimate below.
Estimate I. Substituting v by $2u'_m(t)$ and w by $2\theta_m(t)$ in (6) and (7) respectively, using Green's formula in the term $\sum_{i=1}^{n} \left(\frac{\partial u'_m}{\partial x_i}(t), \theta \right)$ and integrating over $[0, t), 0 \leq t \leq t_m$, we get

$$2(u''_m(t), u'_m(t)) + 2\mu(t)((u_m(t), u'_m(t))) + 2\sum_{i=1}^{n} \left(\frac{\partial \theta_m}{\partial x_i}(t), u'_m(t) \right)$$

$$+2(F\left(u_m(t)\right), u'_m(t)) = 0$$

$$2(\theta'_m(t), \theta_m(t)) + 2((\theta_m(t), \theta_m(t))) + 2\sum_{i=1}^{n} \left(\frac{\partial u'_m}{\partial x_i}(t), \theta_m(t) \right) = 0$$

We obtain

$$\frac{d}{dt} \left[\left| u'_m(t) \right|^2 + |\theta_m(t)|^2 + \mu(t) \left\| u_m(t) \right\|^2 \right] +$$

$$+2 \int_\Omega F\left(u_m(t)\right) . u'_m(t) dx + 2 \left\| \theta_m(t) \right\|^2 = \mu'(t) \left\| u_m(t) \right\|^2$$

We denote

$$E_{1m}(t) = \left| u'_m(t) \right|^2 + |\theta_m(t)|^2 + \mu(t) \left\| u_m(t) \right\|^2 + 2 \int_\Omega G(u_m(t)) dx$$

Integrating de 0 a $t < t_m$, we obtain:

$$E_{1m}(t) + 2 \int_0^t \left\| \theta_m(s) \right\|^2 ds = E_{1m}(0) + 2 \int_0^t \mu'(s) \left\| u_m(s) \right\|^2 ds$$

We know that

$$u_{0m} \to u_0 \ \text{ in } H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega)$$

Then we have

$$\int_\Omega G(u_{0m}) \, dx \ \to \ \int_\Omega G(u_0) \, dx$$

By convergence there is a positive constant K_1, independent of m such that

$$E_{1m}(t) + 2 \int_0^t \left\| \theta_m(s) \right\|^2 ds \leq K_1 \quad \text{for all } \ t \geq 0$$

Hence, we can extent the approximate solutions $\{u_m(t), \theta_m(t)\}$ on the whole interval $[0, \infty)$ independent of m and t.
Estimative II. Make sense take the first derivative of the approximated equation because the existence theorem implied that u''_m is absolutely continuous on $[0, T]$. Derive both sides with respect to t

$$(u'''_m(t), v) + \mu(t)((u'_m(t), v)) + \sum_{i=1}^{n} \left(\frac{\partial \theta'_m}{\partial x_i}(t), v \right) +$$

$$+(F'(u_m(t))\,u'_m(t),v) = -\mu'(t)((u_m(t),v))$$

$$(\theta''_m(t),w) + ((\theta'_m(t),w)) + \sum_{i=1}^{n}\left(\frac{\partial u''_m}{\partial x_i}(t),w\right) = 0$$

We denote

$$E_{2m}(t) = \left|u''_m(t)\right|^2 + \mu(t)\left\|u'_m(t)\right\|^2 + \left|\theta'_m(t)\right|^2$$

Take $v = 2u''_m(t)$ and $w = 2\theta'(t)$, we get

$$\frac{d}{dt}E_{2m}(t) + 2\,\left\|\theta'_m(t)\right\|^2 \le 2\,\beta\left|u'_m(t)\right|.\left|u''_m(t)\right| + 2\frac{\mu'(t)}{\mu(t)}\left|u''_m(t)\right|^2$$

$$2\,\frac{\mu'(t)}{\mu(t)}\sum_{i=1}^{n}\int_\Omega \frac{\partial\theta_m}{\partial x_i}(t)u''_m(t)dx + 2\frac{\mu'(t)}{\mu(t)}\int_\Omega F(u_m(t)).u''_m(t)\,dx \tag{11}$$

By hypotheses of the μ, and usual inequalities yields applying the Gronwall's inequality we obtain and

$$E_{2m}(t) + 2\int_0^t \left\|\theta'_m(s)\right\|^2 ds \le K_2 \quad \text{for all } t \ge 0 \tag{12}$$

The regularity for u and θ is guaranteed by results on elliptic regularity (see, for instance, Medeiros and Milla Miranda [9] or Nirenberg [11].

Theorem 2.2 (Weak Solution) *Given $u_0, \theta_0 \in H_0^1(\Omega), G(u_0) \in L^1(\Omega)$ and $u_1 \in L^2(\Omega)$ then system (1.1)-(1.4) has a unique weak solution $\{u,\theta\}$ such that*

$$u,\theta \in L^\infty(0,\infty;H_0^1(\Omega))$$

$$u' \in L^\infty(0,\infty;L^2(\Omega)),\quad \theta' \in L^2(0,\infty;L^2(\Omega))$$

and equations (1.1) and (1.2) are given in the sense of $L^\infty(0,\infty;L^2(\Omega))$.

Proof: We use arguments of density.

Theorem 2.3 (General Case) *Let $F:\mathbb{R}\to\mathbb{R}$, be continuous such that $s\cdot F(s)\ge 0$. Consider*

$$G(s) = \int_0^s F(\sigma)\,d\sigma$$

Given

$$u_0,\theta_0 \in H_0^1(\Omega),\quad G(u_0)\in L^1(\Omega) \text{ and } u_1 \in L^2(\Omega)$$

then there exists $\{u,\theta\}: Q \to \mathbb{R}$ such that:

$$u,\theta \in L^\infty(0,\infty;H_0^1(\Omega))$$

$$u' \in L^\infty(0,\infty;L^2(\Omega)),\quad \theta' \in L^2(0,\infty;L^2(\Omega))$$

and $\{u,\theta\}$ satisfies the equations

$$u''(x,t) - \mu(t)\Delta u(x,t) + \sum_{i=1}^{n}\frac{\partial\theta}{\partial x_i}(x,t) + F\left(u(x,t)\right) = 0,\quad in\ L^\infty_{loc}(0,\infty,L^2(\Omega))$$

$$\theta'(x,t) - \Delta\theta(x,t) + \sum_{i=1}^{n}\frac{\partial u'}{\partial x_i}(x,t) = 0,\quad in\ L^\infty_{loc}(0,\infty,L^2(\Omega))$$

and initial conditions

$$u(0)=u_0,\quad u'(0)=u_1,\quad \theta(0)=\theta_0$$

Proof: We first approximate u_0 by a sequence of functions $(u_{0j})_{j\in\mathbb{N}}$ of $H_0^1(\Omega)\cap L^\infty(\Omega)$.
In fact, let us consider

$$\beta_j(s)=\begin{cases} s, & \text{if } |s|\le j\\ j, & \text{if } s>j\\ -j, & \text{if } s<-j\end{cases}$$

Use the notation $\beta_j(u_0)=u_{0j}$. Then $u_{0j}\in H_0^1(\Omega)$ (see Brezis-Cazanave [2]) and $u_{0j}\to u_0\in H_0^1(\Omega)$.
Let F and G be as above and represent by F_k the Strauss approximation of F, that is, F_k , $k\in\mathbb{N}$, is a continuous function defined by:

$$\begin{cases} F_k(s)=-(k)\left[G\left(s-\dfrac{1}{k}\right)-G(s)\right] & \text{if } \quad -k\le s\le -\dfrac{1}{k}\\ F_k(s)=k\left[G\left(s+\dfrac{1}{k}\right)-G(s)\right] & \text{if } \quad \dfrac{1}{k}\le s\le k\\ F_k(s) \text{ is linear by parts} & \text{if } \quad -\dfrac{1}{k}<s\le\dfrac{1}{k} \text{ with } F_k(0)=0\\ F_k(s) \text{ appropriate constants} & \text{for } \quad |s|>k.\end{cases} \tag{13}$$

It follows by Strauss [15], that F_k is Lipschitz for each k,
$s\,F_k(s)\ge 0$ and (F_k) converges to F uniformly on the compacts subsets of $\mathbb{R}$.
Represent by

$$G_k(s)=\int_0^s F_k(r)\,dr,\quad G_k(0)=F_k(0)=0,\quad s.G_k(s)\ge 0$$

for all $k\in\mathbb{N}$.
Approximations of u_{0j},θ_0 and u_1 by elements of $\mathfrak{D}(\Omega)$.
Let $\varphi_{\nu j}\in\mathfrak{D}(\Omega)$, $\chi_\nu\in\mathfrak{D}(\Omega)$ and $\psi_\nu\in\mathfrak{D}(\Omega)$ such that

$$\varphi_{\nu j}\to u_{0j} \text{ in } H_0^1(\Omega)$$

$$\chi_\nu\to\theta_0 \text{ in } H_0^1(\Omega)$$

$$\psi_\nu\to u_1 \text{ in } L^2(\Omega)$$

Then, by Theorem above, Lipschitz case, there exists a unique $\{u_{\nu jk},\theta_{\nu k}\}$ in the conditions:

$$u_{\nu jk}\in L^\infty(0,\infty;H_0^1(\Omega))\cap L^\infty_{loc}(0,\infty,H_0^1(\Omega)\cap H^2(\Omega))$$

$$\theta_{\nu k}\in L^\infty(0,\infty;L^2(\Omega))\cap L^2(0,\infty,H_0^1(\Omega))$$

$$u'_{\nu jk}\in L^\infty(0,\infty;L^2(\Omega))\cap L^\infty_{loc}(0,\infty,H_0^1(\Omega))$$

$$\theta'_{\nu k}\in L^\infty_{loc}(0,\infty;L^2(\Omega))\cap L^2_{loc}(0,\infty,H_0^1(\Omega))$$

$$u''_{\nu jk}\in L^\infty_{loc}(0,\infty;L^2(\Omega))$$

$$(u''_{\nu jk}(t),v)+\mu(t)a(u_{\nu jk}(t),v)+\sum_{i=1}^n(\frac{\partial\theta_{\nu k}}{\partial x_i}(x,t),v)+(F\left(u_{\nu jk}(t)\right),v)=0$$

$$(\theta'_{\nu k}(t),w)+a(\theta_{\nu k}(t),w)+\sum_{i=1}^n(\frac{\partial u'_{\nu jk}}{\partial x_i}(t),w)=0$$

$$u_{\nu jk}(0)=\varphi_{\nu j},\quad u'_{\nu jk}(0)=\psi_\nu,\quad \theta_{\nu k}(0)=\chi_\nu.$$

By similar arguments used to obtain estimates, we find

$$\left|u'_{\nu jk}(t)\right|^2+|\theta_{\nu k}(t)|^2+\mu(t)\left\|u_{\nu jk}(t)\right\|^2+2\int_\Omega G_k(u_{\nu jk}(t))dx+2\int_0^t\left\|\theta_{\nu k}(s)\right\|^2ds\le$$

$$\leq |\psi_\nu|^2 + |\chi_\nu|^2 + \mu(0) \|\varphi_{\nu j}\|^2 + 2\int_\Omega G_k(\varphi_{\nu j})dx$$

By the above convergences of $(\psi_\nu), (\varphi_{\nu j})$ and (χ_ν) we obtain that the second member of the preceding inequality is bounded except $\int_\Omega G_k(\varphi_{\nu j})dx$.

Therefore the behavior of Estimative I depends on the behavior of the term $\int_\Omega G_k(\varphi_{\nu j})\, dx$ as $\nu \to \infty$. We must estimate independent of j and k too.
We divide the proof in three parts.

<u>First Part</u> We prove that

$$\int_\Omega G_k(\varphi_{\nu j})dx \to \int_\Omega G_k(u_{0j}(x))dx, \quad \nu \to \infty$$

By convergence there exists a subsequence of $(\varphi_{\nu j})$, still denoted by $(\varphi_{\nu j})$, and a function $v_j \in L^2(\Omega)$ such that

$$\varphi_{\nu j}(x) \to u_{0j}(x) \text{ a.e. in } \Omega, \ \nu \to \infty$$
$$|\varphi_{\nu j}(x)| \leq v_j(x) \text{ a.e. in } \Omega,$$

By continuity of $G_k(s)$, we obtain

$$G_k(\varphi_{\nu j}(x)) \to G_k(u_{0j}(x)), \ \nu \to \infty$$

Let c_k be the Lipschitz constant of $F_k(s)$. We have

$$0 \leq G_k(\varphi_{\nu j}(x)) \leq c_k\, |\varphi_{\nu j}(x)| \leq c_k v_j(x) \text{ a.e. in } \Omega$$

The above two convergences and Lebesgue Theorem of Dominated Convergence imply first part.

<u>Second Part</u> We show that

$$\int_\Omega G(u_{0j}(x))dx \to \int_\Omega G(u_0(x))dx, \quad j \to \infty$$

In fact, by the continuity of $G(s)$ and convergence,

$$G(u_{0j}(x)) \to G(u_0(x)) \text{ a.e. in } \Omega, \quad j \to \infty$$

By construction of u_{0j} we find

$$G(u_{0j}(x)) \leq G(u_0(x)) \text{ a.e. in } \Omega, \quad \forall j$$

Noting that $G(u_0) \in L^1(\Omega)$, the last two expressions and Lebesgue Theorem of Dominated Convergence, give convergence of second part.

<u>Third Part</u> We prove that

$$\int_\Omega G_k(u_{0j}(x))dx \to \int_\Omega G(u_{0j}(x))dx, \quad k \to \infty$$

In fact, for j fixed we obtain

$$|u_{0j}(x)| \leq j, \quad \text{a.e. in } \Omega$$

We note that

$$G'_k(s) = F_k(s) \to G'(s) = F(s) \text{ uniformly in } [-j, j]$$

Also

$$G_k(0) = 0 \to G(0) = 0$$

Then

$$G_k(s) \to G(s) \text{ uniformly in } [-j, j]$$

In particular

$$G_k(u_{0j}(x)) \to G(u_{0j}(x)) \text{ uniformly in } \Omega,\ k \to \infty$$

This implies convergence third part.
Uniqueness: We use arguments of density, Strauss approximation and energy inequality.

3. Asymptotic Behavior of Solutions

In order to obtain the decay of solutions we introduce an internal damping in the problem, more precisely, we consider the following system:

$$\begin{aligned} & u''(x,t) - \mu(t)\Delta u(x,t) + \sum_{i=1}^{n} \frac{\partial \theta}{\partial x_i}(x,t) + F\left(u(x,t)\right) + \gamma u'(x,t) = 0, && \text{in } Q = \Omega \times (0,\infty) \\ & \theta'(x,t) - \Delta\theta(x,t) + \sum_{i=1}^{n} \frac{\partial u'}{\partial x_i}(x,t) = 0, && \text{in } Q = \Omega \times (0,\infty) \end{aligned} \tag{14}$$

where γ is a positive constant.
We make the supplementary hypothesis

$$sF(s) \geq C_0 G(s), \quad \forall s \in \mathbb{R}(C_0 \text{ positive constant}) \tag{15}$$

With the same hypothesis on F(s) and initial data u_0, u_1, θ_0 and by similar arguments used to obtain Theorem (3.1) and Theorem (4.1) .
We get, respectively, strong solutions and weak solutions of the mixed problem for system (14).
Consider the energy

$$E(t) = \left|u'(t)\right|^2 + |\theta(t)|^2 + \mu(t)\left\|u(t)\right\|^2 + 2\int_{\Omega} G(u(t))dx,\ t \geq 0 \tag{16}$$

associated to system (14).
By applying a Lyapunov functional, we obtain the follwing result:

Theorem 3.1 (Asymptotic Behavior) *Assume hypotheses of Theorem (4.1) and hypotheses (15). Let $\{u, \theta\}$ be the solution given by Theorem (4.1) for system (14). Then*

$$E(t) \leq 3E(0)e^{-\eta t}, \quad t \geq 0 \tag{17}$$

where $\eta = \text{mín}\left\{\epsilon_1, \epsilon_3\right\} > 0$

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