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Facultad de Ciencias Matemáticas – UNMSM

Nonlinear Elliptic Equations with Maximal Growth Range

Yony Raúl Santaria Leuyacc [1]

Abstract: In this work we are interested in studying the existence of nontrivial weak solutions for a class of nonlinear elliptic equations defined in a bounded domain in dimension two, where the nonlinearities possess maximal exponential growth range motivated by Trudinger-Moser inequalities in Lorentz-Sobolev spaces. In order to study the solvability we use a variational approach. More specifically, we use mountain pass theorem combined with Trudinger-Moser type inequalities.

Keywords: Nonlinear elliptic equations; exponential growth; mountain pass theorem.

Ecuaciones Elípticas no Lineales con Rango de Crecimiento Máximo

Resumen: En este trabajo nos interesa estudiar la existencia de soluciones débiles no triviales para una clase de ecuaciones elípticas no lineales definidas en un dominio limitado en dimensión dos, donde las no linealidades poseen un rango de crecimiento exponencial máximo motivado por las desigualdades de Trudinger-Moser en espacios de Lorentz-Sobolev. Para estudiar la solubilidad se utiliza un enfoque variacional. Más específicamente, usamos el teorema del paso de montaña junto con desigualdades de tipo Trudinger-Moser.

Palabras clave: Ecuaciones elípticas no lineales; crecimiento exponencial; teorema del paso de montaña.

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[1]Universidade de São Paulo, ICMC, e-mail: yonyraul@alumni.usp.br

1 Introduction

The aim of this paper is study the following nonlinear elliptic equation:

$$\begin{cases} -\Delta u \;=\; f(u), & x \in \Omega, \\ u \in W_0^1 L^{2,r}(\Omega), & 1 < r \leq 2, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ and f has maximal growth range.

In order to study the maximal growth of f in the elliptic problem $-\Delta u = f(u)$, we recall some properties of $W_0^{1,2}(\Omega)$ where Ω is a bounded domain in $\mathbb{R}^N$ with $N \geq 3$. The classical Sobolev theorem asserts that the following embedding is continuous: $W_0^{1,2}(\Omega) \subset L^q(\Omega)$ for all $1 \leq q \leq 2^* = 2N/(N-2)$. Thus, using variational methods, the maximal growth of the function f in $W_0^{1,2}(\Omega)$ is of type:

$$|f(s)| \sim |s|^{2^*-1}.$$

In dimension $N = 2$ one has $W_0^{1,2}(\Omega) \subset L^q(\Omega)$ for all $q \geq 1$ and $W_0^{1,2}(\Omega) \not\subseteq L^\infty(\Omega)$. In this situation another kind of maximal growth were established independently by Trudinger [17] and Pohožaev [15]. The authors proved that the maximal growth allow us to consider in $W_0^{1,2}(\Omega)$ is of type:

$$|f(s)| \sim e^{|s|^2}. \tag{2}$$

Adimurthi and Yadava [4], Adimurthi et al. [3] and Figueiredo et al. [8] studied the following type of elliptic equations

$$\begin{cases} -\Delta u \;=\; f(u), & x \in \Omega, \\ \quad u \in W_0^{1,2}(\Omega), \end{cases} \tag{3}$$

where the function f was of type (2).

In this work, in order to improve the growth given by (2), we consider Lorentz-Sobolev spaces $W_0^1 L^{2,p'}(\Omega)$ with $p' = p/(p-1)$ which represent a generalization of Sobolev space $W_0^{1,2}(\Omega)$ (see Seccion 2 for more details), in these spaces the growth of the nonlinearities can be considered such as:

$$f(s) \sim e^{|s|^p}, \quad p > 1. \tag{4}$$

In our work, we study elliptic equations where the nonlinearity f is of type (4), for $p \geq 2$.

We suppose the following assumptions on the function f:

(A_1) f is a continuous function and $f(s) = o(s)$ near the origin.

(A_2) There exist constants $\mu > 2$ and $s_0 > 0$ such that

$$0 < \mu F(s) \leq s f(s), \quad \text{for all} \quad |s| > s_0,$$

where $F(s) = \displaystyle\int_0^s f(t)\,dt$.

(A_3) There exist constants $M > 0$ and $s_1 > 0$ such that

$$0 < F(s) \leq M|f(s)|, \quad \text{for all} \quad |s| > s_1.$$

(A_4) There exist $\alpha_0 > 0$ and $p \geq 2$ such that

$$\lim_{|s|\to\infty} \frac{f(s)}{e^{\alpha|s|^p}} = \begin{cases} 0, & \alpha > \alpha_0, \\ +\infty, & \alpha < \alpha_0. \end{cases}$$

(A_5) There exist constants $\theta > 2$ and $C_\theta > 0$ such that

$$F(s) \geq C_\theta |s|^\theta, \quad \text{for all} \quad s \in \mathbb{R},$$

where

$$C_\theta > \left[\frac{\alpha_0(\theta-2)}{4\pi}\right]^{(\theta-2)/2} \left(\frac{S_{\theta,p}}{\theta}\right)^\theta$$

and

$$S_{\theta,p} := \inf_{0\neq u \in W_0^1 L^{2,p'}(\Omega)} \frac{\|u\|_{W_0^1 L^{2,p'}}}{\left(\int_\Omega |u|^\theta \, dx\right)^{1/\theta}}. \tag{5}$$

In the literature, condition (A_4) says that f has critical growth in the Trudinger-Moser sense (see [2] and also [8]).

Example 1.1 *Let $p > 2$, $A > 0$ and consider the following continuous function defined on $\mathbb{R}$.*

$$f(s) = A|s|^{p-2}s + p|s|^{p-2}se^{|s|^p}.$$

Therefore,

$$F(s) = \int_0^s f(t)\, dt = \frac{A}{p}|s|^p + e^{|s|^p} - 1.$$

The function f satisfies conditions $(A_1) - (A_5)$ for A sufficiently large.

(a) The following limit holds:

$$\lim_{|s|\to 0} \frac{f(s)}{s} = \lim_{|s|\to 0} A|s|^{p-2} + p|s|^{p-2}e^{|s|^p} = 0.$$

Thus, f satisfies condition (A_1).

(b) Observe that

$$sf(s) - pF(s) = pe^{|s|^p}\left(|s|^p - 1\right) + p > 0, \quad \textit{for all} \quad |s| > 0.$$

Thus, f satisfies condition (A_2) with $\mu = p > 2$.

(c) Since,

$$\lim_{|s|\to\infty} \frac{F(s)}{|f(s)|} = \lim_{|s|\to\infty} \frac{\dfrac{A|s|^p}{p} + e^{|s|^p} - 1}{A|s|^{p-1} + p|s|^{p-1}e^{|s|^p}} = 0.$$

Then, condition (A_3) follows.

(d) Note that,

$$\lim_{|s|\to\infty} \frac{|f(s)|}{e^{\alpha|s|^p}} = \lim_{|s|\to\infty} \frac{A|s|^{p-1} + p|s|^{p-1}e^{|s|^p}}{e^{\alpha|s|^p}} = \begin{cases} 0, & \alpha > 1, \\ +\infty, & \alpha < 1. \end{cases}$$

That is, f satisfies condition (A_4) with $\alpha_0 = 1$.

(e) Since $e^{|s|^p} - 1 \geq 0$*, we have*

$$F(s) = \frac{A}{p}|s|^p + e^{|s|^p} - 1 \geq \frac{A}{p}|s|^p, \quad \text{for all} \quad s \in \mathbb{R}.$$

Thus, taking A *sufficiently large* f*, satisfies condition* (A_5)*.*

The following theorem contains our main result.

Theorem 1.2 *Suppose* $(A_1) - (A_5)$*. Then, the equation* (1) *possesses a nontrivial weak solution.*

Observe that, in the case $p = 2$, we have $p' = 2$ and $W_0^1 L^{2,2}(\Omega) = W_0^{1,2}(\Omega)$. Thus, the equation of our study considered in (1) represents an extension of the equation (3). In order to find solutions of the equation (1) we use variational methods.

The paper is organized as follows: Section 2 contains some preliminaries results. In Section 3, we set up the framework to treat equation (1) variationally. In section 4, we show that the energy functional associated has the pass mountain geometry. In section 5, we estimate Palais-Smale sequences and minimax levels. Finally, in Sections 6, we present the proof of our main result.

2 Preliminaries

In this section, we present some preliminaries results which will be used throughout this paper.

2.1 Lorentz-Sobolev spaces

We start setting some previous definitions in order to present the Lorentz spaces which were introduced by G. Lorentz in [13].

Let (Ω, m) be a measure space, where Ω is a measurable subset in $\mathbb{R}^N$ and m the Lebesque measure, denote by $\mathcal{M}(\Omega, \overline{\mathbb{R}})$ the collection of all extended real-valued measurable functions on Ω and $\mathcal{M}_0(\Omega, \mathbb{R})$ the class of functions in $\mathcal{M}(\Omega, \overline{\mathbb{R}})$ that are finite almost everywhere in Ω. As usual, any two functions coinciding almost everywhere in Ω will be identified.

Definition 2.1 *The distribution function* μ_ϕ *of a function* $\phi \in \mathcal{M}_0(\Omega, \mathbb{R})$ *is defined by*

$$\mu_\phi(t) := m\big(\{x \in \Omega : |\phi(x)| > t\}\big), \quad \text{for} \quad t \geq 0,$$

where $m(\Omega')$ *denote the Lebesgue measure of a set* $\Omega' \subset \Omega$*.*

Definition 2.2 *The decreasing rearrangement of* $\phi \in \mathcal{M}_0(\Omega, \mathbb{R})$ *is defined by*

$$\phi^*(s) := \inf\{t \geq 0 : \mu_\phi(t) \leq s\}, \quad \text{for} \quad s \geq 0.$$

Lemma 2.1 *Let* $\phi \in \mathcal{M}_0(\Omega, \mathbb{R})$ *and* $G : [0, +\infty) \to [0, +\infty)$ *be a nondecreasing function such that* $G(|\phi|) \in L^1(\Omega)$ *and* $G(0) = 0$*. Then,* $G(\phi^*) \in L^1([0, +\infty))$ *and*

$$\int_0^{+\infty} G(\phi^*(s))\, ds = \int_\Omega G(|\phi(x)|)\, dx.$$

Proof. See [11, pp. 260–261].

Definition 2.3 *Let $1 < p < +\infty$, $1 \leq q \leq +\infty$. The Lorentz space $L^{p,q}(\Omega)$ is the collection of all functions $\phi \in \mathcal{M}_0(\Omega, \mathbb{R})$ such that $\|\phi\|_{p,q} < +\infty$, where*

$$\|\phi\|_{p,q} = \begin{cases} \Big(\displaystyle\int_0^{+\infty} \big[\phi^*(t) t^{1/p}\big]^q \, \frac{dt}{t} \Big)^{1/q}, & \text{if} \quad 1 \leq q < +\infty, \\ \\ \displaystyle\sup_{t>0} t^{1/p} \phi^*(t), & \text{if} \quad q = +\infty. \end{cases} \tag{6}$$

For a function $f = (f_1, \cdots, f_N) : \Omega \to \mathbb{R}^N$, with $f_i \in \mathcal{M}_0(\Omega, \mathbb{R})$, we say that $f \in L^{p,q}(\Omega)$ if and only if $f_i \in L^{p,q}(\Omega)$ for $1 \leq i \leq N$. In this case, we set

$$\|f\|_{p,q} := \Big(\sum_{i=1}^{N} \|f_i\|_{p,q}^2 \Big)^{1/2}.$$

Basic properties of distribution functions, decreasing rearrangements and Lorentz spaces can be found in [10, 6, 1, 12].

Proposition 2.4 *Let $1 < p < +\infty$ and $1 \leq q \leq +\infty$. Then, the map $\| \cdot \|$ given by (6) is a quasinorm and $L^{p,q}(\Omega)$ is a vector space.*

Proof. See [12, Proposition 2.12].

Remark 2.5 *Using Lemma 2.1 for $G(s) = s^p$ with $p > 1$, we have*

$$\|\phi\|_{p,p} = \Big(\int_0^{+\infty} [\phi^*(t)]^p \, dt \Big)^{1/p} = \Big(\int_\Omega |\phi(x)|^p \, dx \Big)^{1/p} = \|\phi\|_p,$$

this implies that

$$L^{p,p}(\Omega) = L^p(\Omega).$$

Thus, Lorentz spaces are a generalization of L^p-spaces.

Lemma 2.2 *Let $1 \leq q_1 \leq q_2 \leq +\infty$ and $p > 1$. Then, the following embedding is continuous*

$$L^{p,q_1}(\Omega) \hookrightarrow L^{p,q_2}(\Omega).$$

Proof. See [10, pp. 254–255].

Proposition 2.6 *Let Ω an open subset in $\mathbb{R}^N$. Then, the following results holds:*

(i) Let $1 < p < +\infty$. Then, the dual space of $L^{p,1}(\Omega)$ is given by $L^{p',\infty}(\Omega)$ where $1/p + 1/p' = 1$.

(ii) Let $1 < p < +\infty$ and $1 < q < +\infty$. Then, the dual space of $L^{p,q}(\Omega)$ is given by $L^{p',q'}(\Omega)$ where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. Moreover, these spaces are reflexive.

Proof. See [10, pp. 262–263].

Proposition 2.7 *Let Ω be an open subset in $\mathbb{R}^N$, $1 < p < +\infty$ and $1 < q < +\infty$. Then, the Lorentz space $L^{p,q}(\Omega)$ is uniformly convex.*

Proof. See [9, pp. 198–203].

Definition 2.8 *Let Ω be a bounded domain in $\mathbb{R}^N$, assume that $1 < p < +\infty$, $1 < q \leq +\infty$, we define*

$$W_0^1 L^{p,q}(\Omega) := cl\{u \in \mathcal{C}_0^\infty(\Omega) : \|\nabla u\|_{p,q} < +\infty\}$$

with respect to the quasinorm

$$\|u\|_{1,(p,q)} := \|\nabla u\|_{p,q}, \tag{7}$$

where $\nabla u = (D_1 u, \cdots, D_N u)$ and D_i is the weak derivative with respect to x_i for $1 \leq i \leq N$.

Proposition 2.9 *Let Ω be an open domain in $\mathbb{R}^N$, assume that $1 < p, q < +\infty$. Then, $W_0^1 L^{p,q}(\Omega)$ endowed with the quasinorm defined by (7) is a reflexive, uniformly convex quasi-Banach space.*

Proof. To prove that $W_0^1 L^{p,q}(\Omega)$ is a quasi-Banach space we can proceed as in [1, Theorem 3.3] where is applied Hölder's inequality in Lorentz spaces (see [12, Lemma 2.18]). On the other hand, let consider the following isometry

$$\begin{array}{rcl} J \quad : W_0^1 L^{p,q}(\Omega) & \to & L^{p,q}(\Omega)^N \\ u & \mapsto & \nabla u. \end{array}$$

Since, $W_0^1 L^{p,q}(\Omega)$ is a quasi-Banach space, $J\big(W_0^1 L^{p,q}(\Omega)\big)$ is a closed subset in $L^{p,q}(\Omega)^N$. By Propositions 2.6 and 2.7, we get that $J\big(W_0 L^{p,q}(\Omega)\big)$ is a uniformly convex and reflexive space. Finally, since $J\big(W_0^1 L^{p,q}(\Omega)\big)$ and $W_0^1 L^{p,q}(\Omega)$ are isometrically isomorphic the same properties holds for $W_0^1 L^{p,q}(\Omega)$. ■

Lemma 2.3 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $1 \leq q < +\infty$. Then, the following embeddings are compact*

$$W_0^1 L^{N,q}(\Omega) \hookrightarrow L^r(\Omega), \quad \textit{for all} \quad r \geq 1.$$

Proof. See [12, Lemma 2.38].

Proposition 2.10 *Let (u_n) be a sequence in $W_0^1 L^{p,q}(\Omega)$ and $u \in W_0^1 L^{p,q}(\Omega)$ such that*

$$u_n \to u \quad \textit{in} \quad W_0^1 L^{p,q}(\Omega).$$

Then, there exists a subsequence (u_{n_k}) and a function $h \in W_0^1 L^{p,q}(\Omega)$ such that

$$|u_{n_k}(x)| \leq h(x), \quad \textit{for all} \quad k \geq 1 \quad \textit{and almost everywhere in} \quad \Omega.$$

Proof. See [12, Proposition 2.44].

2.2 Trudinger-Moser inequalities for Lorentz-Sobolev spaces

Let Ω be a bounded domain in $\mathbb{R}^2$. A famous result obtained independently by Pohožaev [15] and Trudinger [17] states that $e^{\alpha u^2} \in L^1(\Omega)$ for all $u \in H_0^1(\Omega)$ and $\alpha > 0$. Furthermore, Moser [14] showed that there exists $C = C(\alpha, \Omega) > 0$ such that

$$\sup_{u \in H_0^1(\Omega), \|\nabla u\|_2 \leq 1} \int_\Omega e^{\alpha u^2}\, dx \leq C, \quad \text{if} \quad \alpha \leq 4\pi. \tag{8}$$

Moreover, inequality (8) is sharp, in the sense that for any $\alpha > 4\pi$ the corresponding supremum become infinity.

We present the following versions of Trudinger-Moser inequalities which will be used throughout this paper.

Proposition 2.11 *Let Ω be a bounded domain in $\mathbb{R}^2$, $p > 1$ and denote $p' = p/(p-1)$. Then,*

$$\int_\Omega e^{|u|^p}\, dx < +\infty, \quad \textit{for all} \quad u \in W_0^1 L^{2,p'}(\Omega) \textit{ and for all } \quad \alpha > 0.$$

Proof. See [7, Theorem 7].

Proposition 2.12 *Let Ω be a bounded domain in $\mathbb{R}^2$, $p > 1$ and denote $p' = p/(p-1)$. Then, if $\alpha \leq \alpha_p^* = (4\pi)^{p/2}$, there exists a positive constant $C = C(\alpha, \Omega)$ such that*

$$\sup_{u \in W_0^1 L^{2,p'}(\Omega),\ \|\nabla u\|_{2,p'} \leq 1} \int_\Omega e^{|u|^p} \ dx \leq C.$$

Moreover, if $\alpha > \alpha_p^*$ the corresponding supremum become infinity.

Proof. See [5, Lemma 2.38].

3 Variational Setting

In this section, we describe the functional setting that allows us to treat (1) variationally. The natural functional associated to (1) is given by

$$\begin{array}{rcl} J \ : W_0^1 L^{2,p'}(\Omega) & \to & \mathbb{R} \\ u & \mapsto & \displaystyle\int_\Omega |\nabla u|^2\, dx - \int_\Omega F(u)\, dx. \end{array} \tag{9}$$

From now on, we use the following notation $E := W_0^1 L^{2,p'}(\Omega)$ and $\|u\| := \|\nabla u\|_{2,p'}$.

Lemma 3.1 *The functional J given by (9) is well defined. Furthermore, J belongs to the class $\mathcal{C}^1(E, \mathbb{R})$ and*

$$J'(u)\phi = \int_\Omega \nabla u \nabla \phi\, dx - \int_\Omega f(u)\phi\, dx, \quad \textit{for all} \quad u, \phi \in E.$$

Proof. Let $u \in E = W_0^1 L^{2,p'}(\Omega)$, that is, $\nabla u \in L^{2,p'}(\Omega)$, since $p \geq 2$, we have $p' = p/(p-1) \leq 2$. By Lemma 2.2, we get $L^{2,p'}(\Omega) \hookrightarrow L^{2,2}(\Omega) = L^2(\Omega)$ continuously. Consequently,

$$\int_\Omega |\nabla u|^2\, dx < +\infty. \tag{10}$$

By assumption (A_4), we have

$$\lim_{|s| \to \infty} \frac{|f(s)|}{e^{(\alpha_0+1)|s|^p}} = 0.$$

Then, there exist positives constant M_1 and $C_1 > 0$ such that $|f(s)| \leq C_1 e^{(\alpha_0+1)|s|^p}$ for all $|s| \geq M_1$. On the other hand, by continuity of f, we have $|f(s)| \leq C_2$ for all $|s| \leq M_1$. Thus,

$$|f(s)| \leq C e^{(\alpha_0+1)|s|^p}, \quad \text{for all} \quad s \in \mathbb{R}, \tag{11}$$

where $C = \max\{C_1, C_2\}$. Therefore,

$$|F(u)| \leq \int_0^{|u|} |f(s)|\, ds \leq C \int_0^{|u|} e^{(\alpha_0+1)|s|^p}\, ds \leq C|u|e^{(\alpha_0+1)|u|^p} \leq \frac{C}{2}|u|^2 + \frac{C}{2}e^{2(\alpha_0+1)|u|^p}.$$

Consequently,

$$\Big| \int_\Omega F(u)\, dx \Big| \leq \frac{C}{2}\int_\Omega |u|^2\, dx + \frac{C}{2}\int_\Omega e^{2(\alpha_0+1)|u|^p}\, dx.$$

By Lemma 2.3, we have $E \hookrightarrow L^2(\Omega)$ and using Proposition 2.11, we obtain

$$\int_\Omega F(u)\, dx < +\infty, \quad \text{for all} \quad u \in E. \tag{12}$$

Thus, combining (10) with (12), we conclude that f is well defined.

Define the functionals $J_1, J_2 : E \to \mathbb{R}$ defined by

$$J_1(u) = \int_\Omega |\nabla u|^2\, dx \quad \text{and} \quad J_2(u) = \int_\Omega F(u)\, dx.$$

Since J_1 is a quadratic form, we have that J_1 belongs to the class $\mathcal{C}^\infty(E, \mathbb{R})$ and

$$J_1'(u)(\phi) = \int_\Omega \nabla u \nabla \phi\, dx, \quad \text{for all} \quad \phi \in E. \tag{13}$$

Now, fixing u and ϕ in E, for given $x \in \Omega$ consider $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(t) = F\big(u(x) + t\phi(x)\big).$$

Let (t_n) be any sequence in $\mathbb{R}$ such that $t_n \to 0$, we can assume that $0 < |t_n| \leq 1$ for all $n \geq 1$. For any $n \geq 1$, by the Mean value theorem there exists $\theta_n = \theta(t_n, x) \in (0,1)$ such that

$$F(u + t_n\phi) - F(u) = h(t_n) - h(0) = h'(\theta_n t_n)t_n = f(u + \theta_n t_n \phi)t_n\phi.$$

Set

$$\xi_n(x) := \frac{F(u + t_n\phi) - F(u)}{t_n} = f(u + \theta_n t_n\phi)\phi.$$

Since f is continuous and $t_n \to 0$, we have

$$\lim_{n\to\infty} \xi_n(x) = f(u)\phi, \quad \text{for all} \quad x \in \Omega.$$

Note that $|u + \theta_n t_n \phi| \leq |u| + |\phi| = w \in E$, from (11), we have

$$\begin{aligned} |\xi_n(x)| &= |f(u + \theta_n t_n\phi)\phi| \\ &\leq Ce^{(\alpha_0+1)|u+\theta_n t_n\phi|^p}|\phi| \\ &\leq Ce^{(\alpha_0+1)|w|^p}|\phi| \\ &\leq \frac{C}{2}|\phi|^2 + \frac{C}{2}e^{2(\alpha_0+1)|w|^p}. \end{aligned}$$

By Lemma 2.3 and Proposition 2.11, we get

$$\frac{C}{2}|\phi|^2 + \frac{C}{2}e^{2(\alpha_0+1)|w|^p} \in L^1(\Omega).$$

Using Dominated convergence theorem, we obtain

$$\begin{aligned} J_2'(u)\phi &= \lim_{n\to+\infty} \frac{J_2(u+t_n\phi)-J_2(u)}{t_n} \\ &= \lim_{n\to+\infty} \int_\Omega \frac{F(u+t_n\phi)-F(u)}{t_n}\,dx \\ &= \lim_{n\to+\infty} \int_\Omega \xi_n(x)\,dx \\ &= \int_\Omega f(u)\phi\,dx. \end{aligned}$$

Now, we prove the continuity of the derivative. Let (u_n) be a sequence in E such that $u_n \to u$ in E. By Proposition 2.10, there exists a subsequence (not renamed) (u_n) and $\widehat{u} \in E$ such that

$$|u_n(x)| \leq \widehat{u}(x) \quad \text{almost everywhere in} \quad \Omega$$

and

$$u_n(x) \to u(x) \quad \text{almost everywhere in} \quad \Omega. \tag{14}$$

Thus,

$$\begin{aligned} |f(u_n)-f(u)|^2 &\leq 2|f(u_n)|^2 + 2|f(u)|^2 \\ &\leq 2Ce^{2(\alpha_0+1)|u_n|^p} + 2Ce^{2(\alpha_0+1)|u|^p} \\ &\leq 2Ce^{2(\alpha_0+1)|\widehat{u}|^p} + 2Ce^{2(\alpha_0+1)|u|^p}. \end{aligned}$$

By Proposition 2.11, we obtain

$$2Ce^{2(\alpha_0+1)|\widehat{u}|^p} + 2Ce^{2(\alpha_0+1)|u|^p} \in L^1(\Omega).$$

Moreover, using (14) and the continuity of f, we have

$$|f(u_n)-f(u)|^2 \to 0, \quad \text{almost everywhere in} \quad \Omega.$$

By Dominated convergence theorem, we get

$$\|f(u_n)-f(u)\|_2 \to 0, \tag{15}$$

which implies

$$\begin{aligned} |\langle J_2'(u_n)-J_2'(u),\phi\rangle| &\leq \int_\Omega |f(u_n)-f(u)\phi|\,dx \\ &\leq \|f(u_n)-f(u)\|_2\|\phi\|_2 \\ &\leq C\|f(u_n)-f(u)\|_2\|\nabla\phi\|_{2,p'}. \end{aligned}$$

Using (15), we obtain

$$\sup_{\|\nabla\phi\|_{2,p'}\leq 1} |\langle J_2'(u_n)-J_2'(u),\phi\rangle| \leq C\|f(u_n)-f(u)\|_2 \to 0.$$

That is, J_2 belongs to $\mathcal{C}^1(E,\mathbb{R})$. Consequently, $J \in \mathcal{C}^1(E,\mathbb{R})$. ■

We say that $u \in E$ is a weak solution of the equation (1) if

$$\int_\Omega \nabla u \nabla\phi\,dx = \int_\Omega f(u)\phi\,dx, \quad \text{for all} \quad \phi \in E.$$

Hence, critical points of the functional J correspond to the weak solutions of the equation (1).

4 The geometry of Pass Mountain

This section is devoted to set the geometry of the pass mountain theorem of the functional J given by (9).

Lemma 4.1 *Suppose (A_1) and (A_4) holds. Then, there exist $\sigma > 0$ and $\rho > 0$ such that $J(u) \geq \sigma$ for all $u \in E$, satisfying $\|u\| = \rho$.*

Proof. From (A_1), we have $f(s) = o(s)$. Thus, given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(s)| \leq \epsilon |s|, \quad \text{for all} \quad |s| < \delta.$$

By (A_4), there exist constants $C_1 > 0$ and $M > 0$ such that

$$|f(s)| \leq C|s|^2 e^{2\alpha_0 |s|^p}, \quad \text{for all} \quad |s| \geq M.$$

Note also that

$$|f(s)| \leq \frac{\max\limits_{\delta \leq |s| \leq M} |f(s)|}{|\delta|^2 e^{2\alpha_0 |\delta|^p}} |s|^2 e^{2\alpha_0 |s|^p}, \quad \text{for all} \quad \delta \leq |s| \leq M.$$

From these estimates, we get a constant $C > 0$ such that

$$|f(s)| \leq \epsilon |s| + C|s|^2 e^{2\alpha_0 |s|^p}, \quad \text{for all} \quad s \in \mathbb{R}.$$

Then,

$$|F(s)| \leq \epsilon |s|^2 + C|s|^3 e^{2\alpha_0 |s|^p}, \quad \text{for all} \quad s \in \mathbb{R}.$$

By Hölder's inequality and Proposition 2.12, we obtain

$$\begin{aligned}
\int_\Omega |u|^3 e^{2\alpha_0 |u|^p} \, dx &\leq \Big(\int_\Omega |u|^6 \, dx \Big)^{1/2} \Big(\int_\Omega (e^{4\alpha_0 |u|^p} \, dx \Big)^{1/2} \\
&\leq C \|u\|_6^3 \Big(\int_\Omega (e^{4\alpha_0 |u|^p} \, dx \Big)^{1/2} \\
&\leq C \|u\|_6^3,
\end{aligned}$$

provided that $\|u\| \leq \rho_1$ for some $\rho_1 > 0$ such that $4\alpha_0 \rho_1^p < \alpha_p^*$. Thus,

$$\int_\Omega F(u) \, dx \leq \epsilon \|u\|_2^2 + C\|u\|_6^3.$$

Using Lemma 2.3, we obtain a positive constant C such that

$$\int_\Omega F(u) \, dx \leq \epsilon C \|u\|^2 + C\|u\|^3.$$

Hence,

$$J(u) \geq \|u\|^2 - \int_\Omega F(u) \, dx \geq (1 - \epsilon C)\|u\|^2 - C\|u\|^3.$$

Then,

$$J(u) \geq \big(1 - \epsilon C - C\rho\big)\rho^2, \quad \text{if} \quad \|u\| = \rho.$$

Therefore, taking $\epsilon > 0$ and $\rho > 0$ sufficiently small, such that $1 - \epsilon C - C\rho \geq 1/2$, we obtain

$$J(u) \geq \frac{\rho^2}{2} = \sigma, \quad \text{for all} \quad u \in E, \quad \|u\| = \rho.$$

■

Lemma 4.2 *For each $\theta > 2$ and $p > 1$, the positive constant $S_{\theta,p}$ defined by (5) is attained for a function $u_\theta \in E\backslash\{0\}$.*

Proof. Observe that,

$$S_{\theta,p} = \inf_{0 \neq u \in E} \frac{\|u\|}{\|u\|_\theta} = \inf_{\substack{0 \neq u \in E \\ \|u\|_\theta = 1}} \|u\|.$$

Let (u_n) be a sequence in $E\backslash\{0\}$ such that $\|u_n\|_\theta = 1$ for all $n \in \mathbb{N}$, and

$$\|u_n\| \to S_{\theta,p}.$$

In particular (u_n) is a bounded sequence in E and using the fact that this space is reflexive, we can suppose that there exists $u_\theta \in E\backslash\{0\}$ such that

$$u_n \rightharpoonup u_\theta \quad \text{weakly in} \quad E, \tag{16}$$

and

$$u_n \to u_\theta \quad \text{in} \quad L^\theta(\Omega) \tag{17}$$

By (17), we have

$$1 = \int_\Omega |u_n|^\theta \, dx \to \int_\Omega |u_\theta|^\theta \, dx$$

Thus, $\|u_\theta\| = 1$, which implies

$$S_{\theta,p} \leq \|u_\theta\|. \tag{18}$$

From (16), we have

$$\|u_\theta\| \leq \liminf_{n\to\infty} \|u_n\| = S_{\theta,p}. \tag{19}$$

Combining (18) with (19), the lemma follows. ■

Lemma 4.3 *Suppose that $(A_1) - (A_2)$ hold. Then, there exists $e \in E$ such that*

$$J(e) < 0 \quad \textit{and} \quad \|e\| > \rho,$$

where $\rho > 0$ is given by Lemma 4.1.

Proof. It follows from [16, Remark 2.13] that the assumption (A_2) guarantees the existence of positive constants a and b such that

$$F(s) \geq a|s|^\mu - b, \quad \text{for all} \quad s \in \mathbb{R}.$$

Now, taking $0 \neq u_\theta \in E$ given by Lemma 4.2, we obtain

$$\begin{aligned} J(tu_\theta) &= t^2\|u_\theta\|^2 - \int_\Omega F(tu_\theta)\, dx \\ &\leq t^2\|u_\theta\|^2 - \int_\Omega \big(a|tu_\theta|^\mu - b\big)\, dx \\ &\leq t^2\|u_\theta\|^2 - at^\mu\|u_\theta\|_\mu^\mu + b|\Omega|. \end{aligned}$$

Since, $\mu > 2$, we get $J(tu_\theta) \to -\infty$ as $t \to +\infty$. Therefore, we can take $e = t_0 u_\theta$ with $t_0 > 0$ sufficiently large such that $J(e) < 0$ and $\|e\| > \rho$. ■

5 On Palais-Smale sequences

By Lemmas 4.1 and 4.3 in Pass mountain theorem (see [16, Theorem 2.2]), there exists a Palais-Smale sequence at level $c \geq \sigma$, where σ is given by Lemma 4.1, that is, there exists a sequence $(u_n) \subset E$ such that

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0, \tag{20}$$

and $c > 0$ can be characterized as

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \tag{21}$$

where

$$\Gamma = \{\gamma \in \mathcal{C}([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}.$$

Lemma 5.1 *Let (u_n) be a Palais-Smale sequence given by (20). Then, $\|u_n\| \leq C$, for every $n \in \mathbb{N}$ and for some positive constant C.*

Proof. Observe that

$$J(u_n) - \frac{1}{\mu} J'(u_n)u_n = \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|u_n\|^2 - \frac{1}{\mu}\int_\Omega \Big(\mu F(u_n) - f(u_n)u_n\Big)\,dx. \tag{22}$$

Since f and F are continuous functions, there exists $K > 0$ such that

$$K = \max_{|s| \leq s_0} \big|\mu F(s) - f(s)s\big|.$$

Then,

$$\Big|\int_{|u_n| \leq s_0} \Big(\mu F(u_n) - f(u_n)u_n\Big)\,dx\Big| \leq \int_{|u_n| \leq s_0} \big|\mu F(u_n) - f(u_n)u_n\big|\,dx \leq K|\Omega|. \tag{23}$$

Using (A_2) and (23), we have

$$\begin{aligned}\int_\Omega \Big(\mu F(u_n) - f(u_n)u_n\Big)\,dx &= \int_{|u_n| \leq s_0} \Big(\mu F(u_n) - f(u_n)u_n\Big)\,dx + \int_{|u_n| > s_0} \Big(\mu F(u_n) - f(u_n)u_n\Big)\,dx \\ &\leq \int_{|u_n| \leq s_0} \Big(\mu F(u_n) - f(u_n)u_n\Big)\,dx \\ &\leq K|\Omega|.\end{aligned}$$

Replacing last inequality in (22), we obtain

$$\Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|u_n\|^2 \leq \frac{1}{\mu}K|\Omega| + \Big|J(u_n) - \frac{1}{\mu}J'(u_n)u_n\Big|. \tag{24}$$

Since (u_n) satisfies (20), we have $\big|J(u_n)\big| \leq c + 1$ and $\|J'(u_n)\|_{E^{-1}} \leq 1$, for n sufficiently large. Thus,

$$\begin{aligned}\Big|J(u_n) - \frac{1}{\mu}J'(u_n)u_n\Big| &\leq |J(u_n)| + \frac{1}{\mu}|J'(u_n)u_n| \\ &\leq \big|J(u_n)\big| + \frac{1}{\mu}\|J'(u_n)\|_{E^{-1}}\|u_n\| \\ &\leq c + 1 + \frac{1}{\mu}\|u_n\|.\end{aligned} \tag{25}$$

Combining (24) with (25), we obtain

$$\Big(\frac{1}{2}-\frac{1}{\mu}\Big)\|u_n\|^2 \leq \frac{1}{\mu}K|\Omega| + d + 1 + \frac{1}{\mu}\|u_n\|, \quad \text{for } n \text{ sufficiently large,}$$

which implies that (u_n) is a bounded sequence. ∎

Lemma 5.2 *Let (u_n) be a Palais-Smale sequence satisfying* (20)*. Then,*

$$\int_\Omega f(u_n)u_n\,dx \leq C,$$

for every $n \in \mathbb{N}$ and for some positive constant C.

Proof. By Lemma 5.1, the sequence (u_n) is bounded in E. Since $J'(u_n) \to 0$ in E^{-1}, we obtain $J'(u_n)u_n \to 0$, that is,

$$J'(u_n)u_n = \|u_n\|^2 + \int_\Omega f(u_n)u_n\,dx \to 0.$$

Using again Lemma 5.1, we have that the sequence $\Big(\int_\Omega f(u_n)u_n\,dx\Big)$ is bounded. ∎

Lemma 5.3 *Let Ω be a bounded subset in $\mathbb{R}^N$, $f : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ a continuous function and (u_n) be a sequence of functions in $L^1(\Omega)$ converging to u in $L^1(\Omega)$. Assume that $f(x, u(x))$ and $f(x, u_n(x))$ are also $L^1(\Omega)$ functions. If*

$$\int_\Omega |f(x,u_n)u_n|\,dx \leq C,$$

then, $f(x, u_n)$ converges in $L^1(\Omega)$ to $f(x, u)$.

Proof. See [8, Lemma 2.1].

Lemma 5.4 *Assume $(A_1) - (A_4)$, let (u_n) be a Palais-Smale sequence and suppose there exists $u \in E$ such that $u_n \rightharpoonup u$ in E. Then, there exist a subsequence still denoted by (u_n) such that*

$$f(u_n) \to f(u) \quad \textit{and} \quad F(u_n) \to F(u), \quad \textit{in} \quad L^1(\Omega).$$

Proof. By Lemma 2.3, we can assume that there exists a subsequence still denoted by (u_n) in $E \subset L^1(\Omega)$ and $u_n \to u$ in $L^1(\Omega)$. By (A_1) and (A_4), there exists $C_0 > 0$ such that

$$|f(s)| \leq C_0 e^{(\alpha_0+1)|s|^p}, \quad \text{for all} \quad s \in \mathbb{R}.$$

Using Proposition 2.11, the sequence $(f(u_n))$ and $f(u)$ are in $L^1(\Omega)$. From (A_2), we have

$$\begin{aligned}\int_\Omega |f(u_n)u_n|\,dx &= \int_{\{x\in\Omega:|u_n(x)|\leq s_0\}} |f(u_n)u_n|\,dx + \int_{\{x\in\Omega:|u_n(x)|>s_0\}} f(u_n)u_n\,dx \\ &= \int_{\{x\in\Omega:|u_n(x)|\leq s_0\}} \Big(|f(u_n)u_n| - f(u_n)u_n\Big)\,dx + \int_\Omega f(u_n)u_n\,dx.\end{aligned} \tag{26}$$

Note that

$$\int_{\{x\in\Omega:|u_n(x)|\leq s_0\}} \Big(|f(u_n)u_n| - f(u_n)u_n\Big)\,dx \leq 2|\Omega| \sup_{|s|\leq s_0} |f(s)s|.$$

Combining last inequality with Lemma 5.2 in (26), we conclude that

$$\int_\Omega |f(u_n)u_n|\, dx \le C, \quad \text{for all} \quad n \ge 1,$$

for some $C > 0$. Consequently, by Lemma 5.3, we obtain $f(u_n) \to f(u)$ in $L^1(\Omega)$.

On the other hand, by assumption (A_3), we have

$$|F(s)| \le \max_{|s|\le s_0} |F(s)| + M|f(s)|, \quad \text{for all} \quad s \in \mathbb{R}.$$

Let $M_0 = \max_{|s|\le s_0} |F(s)|$. Then,

$$|F(u_n)| \le M_0 + M|f(u_n)|, \quad \text{for all} \quad n \in \mathbb{N},$$

where $M_0 + M|f(u_n)| \in L^1(\Omega)$. Note that, we may assume that $u_n \to u$ almost everywhere in Ω. Then, by generalized Lebesgue dominated convergence theorem, we get

$$F(u_n) \to F(u) \quad \text{in} \quad L^1(\Omega).$$

■

Now, we estimate the minimax level given by Pass mountain theorem, this estimate will be important to show that the weak solution is nontrivial.

Lemma 5.5 *If we assume*

$$C_\theta > \left[\frac{\alpha_0(\theta-2)}{4\pi}\right]^{(\theta-2)/2} \left(\frac{S_{\theta,p}}{\theta}\right)^\theta,$$

then, the minimax level given by (21) *satisfies:*

$$c < \frac{2\pi}{\alpha_0}.$$

Proof. Let u_θ be the function given by Lemma 4.2. Define $\gamma_0 : [0,1] \to E$ by $\gamma_0(t) = tt_0u_\theta$. Thus, $\gamma_0(0) = 0$ and $\gamma_0(1) = e$ which implies that $\gamma_0 \in \Gamma = \{\gamma \in \mathcal{C}([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}$. Then,

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) \le \max_{t\in[0,1]} J(\gamma_0(t)) = \max_{t\in[0,1]} J(tt_0u_\theta) \le \max_{t\ge 0} J(tu_\theta).$$

By (A_5), we have

$$c \le \max_{t\ge0} J(tu_\theta) = \max_{t\ge0}\left\{\frac{\|tu_\theta\|^2}{2} - \int_\Omega F(tu_\theta)\, dx\right\} \le \max_{t\ge0}\left\{\frac{\|tu_\theta\|^2}{2} - C_\theta\|tu_\theta\|_\theta^\theta\right\}.$$

Since $\|u_\theta\| = S_{\theta,p}$ and $\|u_\theta\|_\theta = 1$, we get

$$\begin{aligned} c &\le \max_{t\ge0}\left\{\frac{t^2S_{\theta,p}^2}{2} - C_\theta t^\theta\right\} \\ &= \left\{\frac{t^2S_{\theta,p}^2}{2} - C_\theta t^\theta\right\}_{t=t_1}, \quad \text{where} \quad t_1 = \left(\frac{S_{\theta,p}^2}{\theta C_\theta}\right)^{1/(\theta-2)} \\ &= \frac{(\theta-2)}{2\theta}\frac{S_{\theta,p}^{2\theta/(\theta-2)}}{(\theta C_\theta)^{2/(\theta-2)}}. \end{aligned}$$

Finally, by assumption on C_θ, we get $c < 2\pi/\alpha_0$. ■

6 Proof of the Theorem 1.2

Proof. Let (u_n) be a sequence the Palais-Smale given by (20), by Lemma 5.1, the sequence (u_n) is bounded in E and using the fact that E is a reflexive space, we can assume that there exists $u \in E$ such that $u_n \rightharpoonup u$ in E. Moreover, since $J'(u_n) \to 0$ for each $\phi \in \mathcal{C}_0^\infty(\Omega)$, we have

$$J'(u_n)\phi = \int_\Omega \nabla u_n \nabla \phi \, dx - \int_\Omega f(u)\phi \, dx = o_n(1), \tag{27}$$

using this together with Lemma 5.4 in (27), we obtain passing to limit

$$\int_\Omega \nabla u \nabla \phi \, dx - \int_\Omega f(u)\phi \, dx = 0.$$

Now, and using the fact that $\mathcal{C}_0^\infty(\Omega)$ is dense in E, yields

$$\int_\Omega \nabla u \nabla \phi \, dx = \int_\Omega f(u)\phi \, dx, \quad \text{for all} \quad \phi \in E.$$

Thus, $u \in E$ is a critical point of J. To conclude the proof, it only remains to prove that u is nontrivial. Suppose, by contradiction, that $u \equiv 0$. Then, we can assume that

$$u_n \to 0 \quad \text{in } L^r(\Omega), \quad \text{for all} \quad r \geq 1. \tag{28}$$

Using the fact that $J(u_n) \to c$, we have

$$J(u_n) = \frac{\|u_n\|^2}{2} - \int_\Omega F(u_n)\, dx = c + o_n(1). \tag{29}$$

Since, we suppose that $u_n \rightharpoonup 0$, by the second part of Lemma 5.4, we obtain

$$\int_\Omega F(u_n)\, dx \to \int_\Omega F(0)\, dx = 0.$$

Replacing in (29), we have

$$\frac{\|u_n\|^2}{2} = c + o_n(1).$$

Now, using Lemma 5.5, we get

$$\|u_n\|^2 = 2c + o_n(1) < \Big(\frac{\alpha_p^*}{\alpha_0}\Big)^{2/p} + o_n(1).$$

Hence, we can assume that, there exists $\delta > 0$ sufficiently small such that

$$\|u_n\|^p \leq \frac{\alpha_p^*}{\alpha_0} - \delta, \quad \text{for all } n \text{ sufficiently large.}$$

Therefore, we can find $r > 1$ sufficiently close to 1 and $\epsilon > 0$ sufficiently small such that

$$r(\alpha_0 + \epsilon)\|u_n\|^p \leq r(\alpha_0 + \epsilon)(\frac{\alpha_p^*}{\alpha_0} - \delta) < \alpha_p^*.$$

From (A_1) and (A_4), there exists $C > 0$ such that

$$|f(s)| \leq |s| + Ce^{(\alpha_0+\epsilon)|s|^p}, \quad \text{for all} \quad s \in \mathbb{R}.$$

Using Hölder's inequality, we obtain

$$\begin{aligned}
\int_\Omega f(u_n)u_n\,dx &\leq \int_\Omega |u_n|^2\,dx + C\int_\Omega e^{(\alpha_0+\epsilon)|u_n|^p}|u_n|\,dx \\
&\leq \|u_n\|_2^2 + C\|u_n\|_{r'}\Big(\int_\Omega e^{r(\alpha_0+\epsilon)|u_n|^r}\,dx\Big)^{1/r} \\
&\leq \|u_n\|_2^2 + C\|u_n\|_{r'}\Big(\int_\Omega e^{r(\alpha_0+\epsilon)\|u_n\|^p(\frac{|u_n|}{\|u_n\|})^p}\,dx\Big)^{1/r} \\
&\leq \|u_n\|_2^2 + C\|u_n\|_{r'}\Big(\int_\Omega e^{\alpha_p^*(\frac{|u_n|}{\|u_n\|})^p}\,dx\Big)^{1/r}
\end{aligned}$$

Using Proposition 2.12, for some positive constant C, we have

$$\int_\Omega f(u_n)u_n\,dx \leq \|u_n\|_2^2 + C\|u_n\|_{r'}.$$

Thus, from (28), we get

$$\int_\Omega f(u_n)u_n\,dx \to 0. \tag{30}$$

Since, (u_n) is bounded in E and $\|J'(u_n)\|_{E^{-1}} \to 0$, we obtain

$$|J'(u_n)u_n| \leq \|J'(u_n)\|_{E^{-1}}\|u_n\| \to 0. \tag{31}$$

Observe that

$$J'(u_n)u_n = \|u_n\|^2 + \int_\Omega f(u_n)u_n\,dx.$$

Combining (30) with (31), we get

$$\|u_n\|^2 = J'(u_n)u_n + \int_\Omega f(u_n)u_n\,dx \to 0.$$

From (28), we have $\|u_n\|^2 \to 2c$, which implies that $c = 0$ which is a contradiction, since $c \geq \sigma > 0$. Thus, u is a nontrivial critical point of J nontrivial and hence u is a nontrivial weak solution of the equation (1).

■

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