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GLOBAL SOLUTIONS TO THE NONLINEAR HYPERBOLIC PROBLEM

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ABSTRACT

We prove the existence and uniqueness of global regular solutions to the mixed problem for the nonlinear hyperbolic equation with nonlinear damping.

$$u_{tt} - a(u)u + |u_t|^{\rho} u_t = f(x,t) \ in \ (0,1) \, x \, (0,T) = Q,$$

$$u(0,t) = u(1,t) = 0,$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),$$

Where $a(u) \geq a_0 > 0$, $\rho > 1$. No restrictions on a size of u_0, u_1, f are imposed.

It is well-known that quasilinear hyperbolic equations, generally speaking, do not have regular solutions for all $t > 0$. Their solutions can blow up at a finite period of time. See examples of such singularities in [1, 3]. On the other hand,it was observed that adding a linear damping to the nonlinear hyperbolic equations one can expect the existence of global regular solutions provided initial conditions and right-hand side have sufficiently small appropriate norms, [2]. Moreover, in [3] was shown that the presence of the nonlinear damping allows to prove the existence of regular solutions for the equation

$$K(u)u_{tt} - \Delta u + |u_{tt}|^{\rho} u_{tt} = f, \ K > 0.$$

without restrictions on a size of the initial data and f.

Later, using the idea of [3], we proved in [4] the existence of regular solutions for the damped Carrier equation.

$$u_{tt} - M\left(|u(t)|^2\right)\Delta u + \alpha |u_t|^{\rho} u_t = f$$

without smallness conditions for the initial data and f.

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Here, we continue to exploit this idea and consider the following nonlinear mixed problem.

$$u_{tt} - a(u)\,u + |u_t|^{\rho}\,u_t = f(x,t) \;\; in \;\; Q = (0,1)\,x\,(0,T), \qquad (1)$$

$$u(0,t) = u(1,t) = 0, \qquad (2)$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad (3)$$

where $a(u)$ is a smooth positive function.

Unlike the Carrier equations. (1) has local nonlinearities: the function $a(u)$ depends on a solution; and the function $M\left(|u(t)|^2\right.$ depends on the L_2- norm of it. This difference makes study of (1)-(3) more complicated and forces us to consider only the one-dimensional case. Nevertheless, the basic technique is similar to one used in the case of the Carrier equation in [4]. Under natural conditions for $a(u)$, we prove the existence and uniqueness of regular solutions to (1)-(3) without any restrictions on a size of u_0, u_1, f.

Assumptions.

A1. $a(u) \in C^1(R);\; a(u) \geq a_0 > 0.$

A2. $|a_u| \leq Aa(u).$

A3. $1 < \rho,$

Where a_0, A are positive constants.

In the sequel, we use standard notations for functional spaces, see [5].

Theorem. Let T be an arbitrary positive number; $u_0 \in H^2(0,1) \cap H_0^1(0,1)$ and A1-A3 hold. The for any f such that $f, f_t \in L^2$, there exists a unique regular solution to (1)-(3), $u(x,t)$:

$$u \in L^{\infty}\left(0,T; H^2(0,1) \cap H_0^1(0,1)\right),$$

$$u_t \in L^{\infty}\left(0,T; H_0^1(0,1)\right),$$

$$u_{tt} \in L^{\infty}\left(0,T; L^2(0,1)\right)$$

The scheme of the proof.

The assumption A1 allows to rewrite (1) in the equivalent form

$$\frac{1}{a(u)}u_{tt} - u + \frac{|u_t|^\rho u_t}{a(u)} = \frac{f}{a(u)} \tag{4}$$

Obviously, solutions to (4), (2), (3) are also solutions to (1)-(3). Equation (4) is similar to the class of quasilinear hyperbolic equations studied in [4] with exception of the damping that can degenerate when $a(u) \to \infty$. Also, the coefficient of u_{tt}, $\left(\frac{1}{a(u)}\right)$ can be zero. It means that (4) is the degenerated hyperbolic equation. Morever, dependence of the damping term of u and u_t brings more difficulties to analysis of (4), (2), (3). Nevertheless, we can employ in our case the technique developed in [4].

Approximate solutions to (4), (2), (3) will be constructed by the Faedo-Galerkin method with the special basis. Let $w_j(x)$ be eigen-functions of the problem

$$\begin{gathered} w_{jxx} + \lambda_j w_j = 0 \ \ in \ \ (0,1), \\ w_j(0) = w_j(1) = 0. \end{gathered} \tag{5}$$

Then for $\in > 0$

$$u_\in^N(x,t) = \sum_{j=i}^{N} g_j^N(t) w_j(x),$$

where unknown functions $g_j(t)$ are solutions to the following Cauchy problem

$$\left(\left(\in + \tfrac{1}{a(u_\in^N)}\right)u_{\in tt}^N, w_j\right)(t) + \left(u_{\in x}^N, w_j\right)(t) + \left(\frac{|u_{\in t}^N|^\rho u_{\in t}^N}{a(u_\in^N)}, w_j\right)(t) = \left(\tfrac{f}{a(u_\in^N)}, w_j\right)(t), \tag{6}$$

$$g_j^N(0)=\alpha_j=(u_0, w_j),$$

$$g_{jt}^N(0)=\beta_j=(u_1, w_j),\ j=1, \ldots, N. \tag{7}$$

Here $(u,v)(t)=\int_0^1 u(x,t)\,v(x,t)\,dx$.

The system of nonlinear ordinary differential equations (6) is not solved with respect to g_{jtt}, but it can be transformed a normal system of ODE due to the fact that the matrix $\in I+\left(\sum \frac{1}{a(u\in^N)} w_j, w_i\right), i, j=1, \ldots, N,$ is positive for $\in>0$, see A1. Hence, the Cauchy problem (6), (7) has solutions g_j^N at some interval $(0, T_N)$, and we need a priori estimates in order to prolongate solutions to the interval $(0,T)$ and to pass to the limits when $\in\to 0$ and $N\to\infty$.

The First Estimate

Multiplying (6) by g_{jt}^N and using A1-A3, after some calculations we come to the inequality

$$\int_0^1\left(\frac{\left|u_{\in t}^N(x,t)\right|^2}{a\left(u_\in^N\right)}+\left|u_{\in x}^N(x,t)\right|^2\right)dx+\int_0^t\int_0^1\frac{\left|u_{\in t}^N\right|^{\rho+2}}{a\left(u_\in^N\right)}dxdr\le$$

$$C_1\left(\|u_0\|H_0^1(0,1), \|u_1\|L^3(0,1), \|f\|L^3(Q)\right) \tag{8}$$

where C does not depend on $\in, N, t$.

From here and from A1.

$$\operatorname*{Sup}_{t\in(0,T)}\ \max_{x\in(0,1)}|u(x,t)|\le C_2. \tag{9}$$

This imply

$$a_0 \le a(u) \le M < \infty \tag{10}$$

Where C_2, M do not depend on $\in, N, T$.

The Second Estimate

Taking the derivate of (6) with respect to t, multiplying the result by g_{jtt}^{N}, after standard transformations we obtain

$$\int_0^1 \left(\left| u_{\in tt}^{N}(x,t) \right|^2 + \left| u_{\in xt}^{N}(x,t) \right|^2 \right) dx \le \tag{11}$$

$$C_3 \left(\| u_0 \|_{H^2(0,1) \cap H_0^1(0,1)}, \| u_1 \|_{H_0^1(0,1)}, \| f \|_{H^1(0,T,L^3(0,1))} \right),$$

where C_3 does not depend on $\in, N, t$.

Finally, taking into account (5) and estimates (8)-(11), we get

$$\int_0^1 \left| u_{\in xx}(x,t) \right|^2 dx \le C_4 \tag{12}$$

With (8)-(12) it is easy to pass to the limits in (6) when $N \to \infty$, $\in \to 0$, hence, to prove the existence of regular solutions to (4), (2), (3) and, consequently, to (1)-(3). Uniqueness may be proved in the usual way.
Theorem is proved.

Remark. The function $a(u)$ can depend on x, t.

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