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# HOMOGENEUS MIXED PROBLEM FOR THE DAMPED CARRIER EQUATION

*L.P. San Gil Jutuca*[1],
*R.F. Apolaya*[2],
*M. Milla Miranda*[3]

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**ABSTRACT**

This paper is concerned with the existence of global solutions of an initial and homogeneous boundary problem for the damped Carrier equation

$$\frac{\partial^2 u}{\partial t^2} - M\left( \int_\Omega |u|^2 \, d\Omega \right) \Delta u + \left| \frac{\partial u}{\partial t} \right|^\rho \frac{\partial u}{\partial t} = 0 ,$$

where $M$ is a positive real function and $\rho > 1$.

## 1. Introduction

Let $\Omega$ be a bounded open set of $R^n$ with boundary $\Gamma$ of class $C^2$. We consider a partition $\{\Gamma_0, \Gamma_1\}$ of $\Gamma$ such that $\Gamma_1$ is open in $\Gamma$, $\text{mes}(\Gamma_1) > 0$, $\text{mes}(\Gamma_0) > 0$ and $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 \neq \phi$. In this paper, the authors investigate, by using Galerkin's method, the existence and uniqueness of global solutions for the following mixed problem:

$$u'' - M\left(\int_\Omega |u|^2 d\Omega\right) \Delta u + |u'|^\rho u' = 0 \;\; in \; \Omega \times [0, \infty), \tag{1.1}$$

$$u = 0 \;\; on \;\; \Gamma_0 \times [0, \infty), \tag{1.2}$$

$$\frac{\partial u}{\partial \nu} + \delta u' = 0 \;\; on \;\; \Gamma_1 \times [0, \infty), \tag{1.3}$$

$$u(x, 0) = u^0(x), \;\; u'(x, 0) = u^1(x) \;\; in \; \Omega \tag{1.4}$$

1 Dept° de Matemática e Estadística, Uní-Río – E-mail address: rsangil@iq.ufrj.br
2 Dept° de Análise, MI-UFF – E-mail adress: ganrefa@vm.uff.br
3 Instituto de Matemática, UFRJ – E-mail address: milla@dmm.im.ufrj.br

Here $M(\lambda)$ is a positive real function of class $C^1$ on $[0,\infty)$; the vector $\nu$ denotes an outward unit normal to the boundary $\Gamma$ and $\delta$ is a function in $W^{1,\infty}(\Gamma)$ such that $\delta(x)\geq 0$. By $u', u''$ we denote the time derivatives of $u$.

Global solutions for equation (1.1) with null Dirichlet boundary were obtained by C. L. Frota, A. T. Cousin and N. Larkin [3] and for the Kirchhoff-Carrier equation without damping by A. T. Cousin, C. L. Frota, N. Larkin and L. A. Medeiros [2].

## 2. Preliminaries

In order to formulate our results we consider the Hilbert space

$$V=\{v \in H^1(\Omega);\ v=0 \quad on \quad \Gamma_0$$

with inner product and norm given by:

$$((u,v)) = \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_i}(x)\,dx \quad and \quad \|u\| = \left(\sum_{i=1}^{n}\int_\Omega\left(\frac{\partial u}{\partial x_i}(x)\right)^2 dx\right)^{1/2}$$

The inner product and norm of $L^2(\Omega)$ are represented by (.,.) and $|.|$, respectively.

Let W be the space of functions $u:\ \Omega\rightarrow R$ such that $u \in V,\ \Delta u \in L^2(\Omega)$ and there is $g_u \in H^{1/2}(\Gamma)$ which satisfies $g_u \equiv 0 \ on\ \Gamma_0$ and

$$(-\Delta u, v) = ((u,v)) - (g_u, v)_{L^2(\Gamma)}, \quad for\ all \quad v \in V. \tag{2.1}$$

We remark that $g_u$ verifying (2.1) is unique. The space W is equipped with the norm

$$\|u\|_w = \left(|\Delta u|^2 + \|g_u\|^2_{H^{1/2}(\Gamma)}\right)^{1/2}$$

Then W is a separable Hilbert space and W is compactly embedding into V.

**Proposition 2.1** *The space W is dense in V.*

The proof of this Proposition, based on the density of $D(-\Delta)$ in V, is reasonably straightforward and follows arguments close to the used in [4].

**Remark 2.1** *If* $u \in W$ *then* $\frac{\partial u}{\partial \nu} \in H^{-1/2}(\Gamma)$ *and it holds that*

$$\left\langle \frac{\partial u}{\partial \nu}, v \right\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = (g_u, v)_{L^2(\Gamma)} \quad \textit{for all} \quad v \in V.$$

*This implies by taking* $v \in D(\Gamma_1)$ *that*

$$\frac{\partial u}{\partial \nu} = g_u \quad in \quad H^{1/2}(\Gamma_1)$$

## 3. Main Result

In order to obtain the existence of global solutions of Problem (1.1) – (1.4), we assumed the following supplementary assumptions on $M$:

$$M(\lambda) \geq m_0 > 0 \quad (m_0 \text{ is constant}), \tag{3.1}$$

$$\frac{|M'(\lambda)|}{M(\lambda)} \lambda^{1/2} \leq K_0, \tag{3.2}$$

where $M'(\lambda)$ denotes the derivative of $M$ with respect to $\lambda$ and $K_0$ is a constant.

**Remark 3.1** *If we consider smallness restrictions on the initial data* $u^0$ *and* $u^1$, *then above hypothesis* (3.2) *on the function M becomes unnecessary* .

We also impose on the real number $\rho$ the conditions:

$$\begin{cases} \rho > 1 & if \quad n = 2, \\ 1 < \rho \leq \dfrac{n+2}{n-2} & if \quad n \geq 3. \end{cases} \tag{3.3}$$

**Theorem 3.1** *Assume that the conditions (3.1) – (3.3) are satisfied and that* $u^0 \in V$, $u^1 \in V$ *verify* $\Delta u^0 \in L^2(\Omega)$,

$$\frac{\partial u^0}{\partial \nu} + \delta u^1 = 0 \quad in \quad H^{1/2}(\Gamma_1). \tag{3.4}$$

*Then there exists a unique function* $u : \Omega \times \left] 0, \infty \right[ \rightarrow R$ *in the class*

$$u \in L^{\infty}_{loc}\left(0, \infty; W\right), \quad u' \in L^{\infty}_{loc}\left(0, \infty; V\right), \quad u'' \in L^{\infty}_{loc}\left(0, \infty; L^2(\Omega)\right),$$
$$\delta^{1/2} u'' \in L^2_{loc}\left(0, \infty; L^2(\Gamma_1)\right)$$

*satisfying the equation*

$$u'' - M\left(\left|u(.)\right|^2\right)\Delta u + \left|u'\right|^{\rho} u' = 0 \quad in \quad L^{\infty}_{loc}\left(0, \infty; L^2(\Omega)\right)$$

*and the initial conditions*

$$u(0) = u^0, \quad u'(0) = u^1.$$

*Furthermore u verifies*

$$\frac{\partial u}{\partial \nu} + \delta\, u' = 0 \quad in \quad L^{\infty}_{loc}\left(0, \infty; H^{1/2}(\Gamma_1)\right),$$

$$\frac{\partial u'}{\partial \nu} + \delta\, u'' = 0 \quad in \quad H^{-1}_{loc}\left(0, \infty; L^2(\Gamma_1)\right),$$

**Remark 3.2** *If* $u^0$ *is in the conditions of Theorem 3.1 then* $u^0 \in W$ *and*

$$\left(-\Delta u^0, v\right) = \left(\left(u^0, v\right)\right) + \left(\delta u^1, v\right)_{L^2(\Gamma)}, \quad for\ all\ v \in V.$$

The next result has a fundamental role in the proof of Theorem 3.1.

**Lemma 3.1.** *Let us suppose that* $u^0 \in V$, $\Delta u^0 \in L^2(\Omega)$ *and* $u^1 \in V$ *with*

$$\left(-\Delta u^0, v\right) = \left(\left(u^0, v\right)\right) + \left(\delta u^1, v\right)_{L^2(\Gamma)}, \quad for\ all\ v \in V.$$

*Let* $\varepsilon > 0$. *Then there exist w and z in W such that*

$$\left(-\Delta w, v\right) = \left(\left(w, v\right)\right) + \left(\delta z, v\right)_{L^2(\Gamma)}, \quad for\ all\ v \in V,$$

$$\left\| w - u^0 \right\|_W < \varepsilon, \quad \left\| z - u^1 \right\| < \varepsilon.$$

**Proof.** Fixe $\varepsilon > 0$. By Proposition 2.1, there exists $z \in W$ such that $\left\| z - u^1 \right\| < \varepsilon$. Let $w$ be the solution of the variational problem

$$\left| \begin{array}{l} w \in V \\ \left(\left(w, v\right)\right) = \left(-\Delta u^0, v\right) + \left(\delta z, v\right)_{L^2(\Gamma)} \quad for\ all\ v \in V. \end{array} \right.$$

Then $\Delta w = \Delta u^0$; moreover

$$\left\| w - u^0 \right\|_W^2 = \left| \Delta w - \Delta u^0 \right|^2 + \left\| -\delta z + \delta u^1 \right\|_{H^{1/2}(\Gamma)}^2$$

$$\leq C \left\| z - u^1 \right\|_{H^{1/2}(\Gamma)}^2$$

$$\leq C_1 \left\| z - u^1 \right\|^2 \leq C_1 \varepsilon^2$$

where $C_1$ is a positive constant that depends only of $\delta$ and $\Omega$. Thus $w, z \in W$ and

$$(-\Delta w, v) = ((w, v)) + (\delta z, v)_{L^2(\Gamma)}, \quad \textit{for all} \quad v \in V.$$

**Proof of Theorem 3.1** From Lemma 3.1 there exist sequences $(u_\ell^0)$ and $(u^1)_\ell$ of vectors belonging to $W$ such that

$$u_\ell^0 \to u^0 \ \ \textit{strongly in } W \tag{3.5}$$

$$u_\ell^1 \to u^1 \ \ \textit{strongly in } V \tag{3.6}$$

$$(-\Delta u_\ell^0, v) = ((u_\ell^0, v)) + (\delta u_\ell^1, v)_{L^2(\Gamma)}, \ \text{for all } v \in V \tag{3.7}$$

From above sequences, for each $\ell \in N$, we construct a special basis of $W$ in the following way: first, we determine a orthonormal basis $w_k^\ell$ of the subspace of $W$ spanned by $u_\ell^0$ and $u_\ell^1$ ($\ell$ fixed). Thus $k=1$ *or* $k=1,2$. Then by the orthonormalization process, we complete $(w_k^\ell)$ just to obtain a basis of $W$. This special basis of $W$ is represented by

$$\{w_1^\ell, w_2^\ell, \ldots, w_j^\ell, \ldots\}.$$

In what follows $\ell$ is fixed, unless we mention the contrary. For $m \in N$ let us consider the subspace $W_m^\ell$ spanned by $\{w_1^\ell, w_2^\ell, \ldots, w_m^\ell\}$ and the approximate solutions $u_{lm}(t)$ of Problem (1.1) – (1.4), defined by

$$u_{lm}(t) = \sum_{j=1}^{m} g_{ljm}(t) w_j^\ell,$$

where $g_{ljm}$ are the solutions of the approximate equation

$$\begin{aligned} &\left(u''_{\ell m}(t), w\right) + M\left(|u_{\ell m}(t)|^2\right)\left(\left(u_{\ell m}(t), w\right)\right) + \\ &M\left(|u_{\ell m}(t)|^2\right) \int_{\Gamma 1} \delta u'_{\ell m}(t)\, w \, d\Gamma + \left(\left|u'_{\ell m}(t)\right|^\rho u'_{\ell m}(t), w\right) = 0, \ \textit{for all } w \in W_m^\ell \end{aligned} \tag{3.8}$$

with the initial conditions

$$u_{\ell m}(0)=u_\ell^0, \qquad u'_{\ell m}(0)=u_\ell^1. \tag{3.9}$$

Taking into account (3.2) and denoting $M\left(|u_{lm}(t)|^2\right)$ by $\mu(t)$, we rewrite (3.8) as

$$\frac{\left(u''_{\ell m}(t),w\right)}{\mu(t)}+((u_{\ell m}(t),w))+\int_{\Gamma 1}\delta u'_{\ell m}(t)\,w\,d\Gamma+\frac{\left(\left|u'_{\ell m}(t)\right|^\rho u'_{\ell m}(t),w\right)}{\mu(t)}=0. \tag{3.10}$$

Notice that the solution $u_{\ell m}$ defined on $[0,t_m[$ can be extended to the interval $[0,T]$, for any real number $T>0$, by the next first a priori estimate. We need two a priori estimates.

*First a Priori Estimate*- By choosing $w=2u'_{\ell m}(t)$ in (3.10) we obtain

$$\frac{d}{dt}\left[\frac{\left|u'_{\ell m}(t)\right|^2}{\mu(t)}+\left\|u_{\ell m}(t)\right\|^2\right]+2\int_{\Gamma 1}\delta\left(u'_{\ell m}(t)\right)^2 d\Gamma+$$

$$\frac{2}{\mu(t)}\left(\left|u'_{\ell m}(t)\right|^\rho u'_{\ell m}(t),u'_{\ell m}(t)\right)=-\frac{\mu'(t)}{\mu(t)^2}\left|u'_{\ell m}(t)\right|^2,$$

whence by using (3.1) and (3.2) it follows that

$$\frac{d}{dt}\left[\frac{\left|u'_{\ell m}(t)\right|^2}{\mu(t)}+\left\|u_{\ell m}(t)\right\|^2\right]+2\int_{\Gamma 1}\delta\left(u'_{\ell m}(t)\right)^2 d\Gamma+ \tag{3.11}$$

$$\frac{2}{\mu(t)}\left\|u'_{\ell m}(t)\right\|^{\rho+2}_{L^{\rho+2}(\Omega)}\le\frac{2K_0}{\mu(t)}\left|u'_{\ell m}(t)\right|^3,$$

Moreover, since $L^{\rho+2(\Omega)}\to L^2(\Omega)$, there exists $C_1=C_1(K_0,\Omega)$ such that

$$\frac{2K_0}{\mu(t)}\left|u'_{\ell m}(t)\right|^3\le\frac{2C_1}{\mu(t)}\left\|u'_{\ell m}(t)\right\|^3_{L^{\rho+2}(\Omega)}\text{•}$$

The use of the Young's inequality, for all $\varepsilon>0$, yields

$$2C_1\left\|u'_{\ell m}(t)\right\|^3_{L^{\rho+2}(\Omega)}\le\frac{\rho-1}{\rho+2}\frac{(2C_1)^{\frac{\rho+2}{\rho-1}}}{\varepsilon^{\frac{3}{\rho+2}}}+\frac{3\varepsilon}{\rho+2}\left\|u'_{\ell m}(t)\right\|^{\rho+2}_{L^{\rho+2}(\Omega)}$$

$$\le C_2(\varepsilon)+C_3\varepsilon\left\|u'_{\ell m}(t)\right\|^{\rho+2}_{L^{\rho+2}(\Omega)}.$$

By choosing a suitable $\varepsilon > 0$, we have

$$\frac{d}{dt}\left[\frac{\left|u'_{\ell m}(t)\right|^2}{\mu(t)} + \left\|u_{\ell m}(t)\right\|^2\right] + 2\int_{\Gamma 1}\delta\left(u'_{\ell m}(t)\right)^2 d\Gamma + \frac{C_4}{\mu(t)}\left\|u'_{\ell m}(t)\right\|^{\rho+2}_{L^{\rho+2}(\Omega)} \leq C_2(\varepsilon).$$

Integrating on $[0, t[$ with $0 < t < t_m$, by the Gronwall inequality and convergences (3.5) – (3.6), for all $0 \leq t \leq T$ and $\ell \geq \ell_0$, we obtain

$$\frac{\left|u'_{\ell m}(t)\right|^2}{\mu(t)} + \left\|u_{\ell m}(t)\right\|^2 + 2\int_0^t\int_{\Gamma 1}\delta\left(u'_{\ell m}(s)\right)^2 ds d\Gamma +$$

$$C_4\int_0^t\frac{\left\|u'_{\ell m}(s)\right\|^{\rho+2}_{L^{\rho+2}(\Omega)}}{\mu(s)}ds \leq C_2(\varepsilon)T + \frac{\left|u^1\right|^2}{\mu(0)} + \left\|u^0\right\|^2.$$

Thus for $m \in N$ and $\ell \geq \ell_0$ it follows that

$$\begin{aligned}
&(u_{\ell m}) \text{ is bounded in } L^{\infty}_{loc}(0,\infty; V),\\
&(u'_{\ell m}) \text{ is bounded in } L^{\infty}_{loc}(0,\infty; L^2(\Omega)),\\
&(\delta^{1/2} u'_{\ell m}) \text{ is bounded in } L^{2}_{loc}(0,\infty; L^2(\Gamma_1)),\\
&(u'_{\ell m}) \text{ is bounded in } L^{\rho+2}_{loc}(0,\infty; L^{\rho+2}(\Omega)),
\end{aligned} \tag{3.12}$$

Note that in the obtention of $(3.12)_2$ we have used the fact $M \in C^1([0,\infty[)$ and $(3.12)_1$.

*Second a Priori Estimate* - In order to obtain estimate for $u''_{\ell m}(t)$, we differenciate (3.10) with respect to t and then we choose $w = 2u''_{\ell m}(t)$. So, we obtain

$$\frac{d}{dt}\left[\frac{\left|u''_{\ell m}(t)\right|^2}{\mu(t)} + \left\|u'_{\ell m}(t)\right\|^2\right] + 2\int_{\Gamma 1}\delta\left(u''_{\ell m}(t)\right)^2 d\Gamma + \frac{\rho+1}{\mu(t)}\left(\left|u'_{\ell m}(t)\right|^{\rho}, \left(u''_{\ell m}(t)\right)^2\right) =$$

$$\frac{2\mu'(t)}{\mu(t)^2}\left|u''_{\ell m}(t)\right|^2 + \frac{2\mu'(t)}{\mu(t)^2}\left(\left|u'_{\ell m}(t)\right|^{\rho} u'_{\ell m}(t), u''_{\ell m}(t)\right).$$

Note that for $\varepsilon > 0$

$$\left(\left|u'_{\ell m}(t)\right|^{\rho} u''_{\ell m}(t), u''_{\ell m}(t)\right) = \left(\left|u'_{\ell m}(t)\right|^{\frac{\rho}{2}} u''_{\ell m}(t), \left|u'_{\ell m}(t)\right|^{\frac{\rho}{2}} u'_{\ell m}(t)\right)$$

$$\leq \frac{\varepsilon}{2}\left(\left|u'_{\ell m}(t)\right|^{\rho}, \left(u''_{\ell m}(t)^{2}\right)\right) + \frac{1}{2\varepsilon}\left\|u'_{\ell m}(t)\right\|^{\rho+2}_{L^{\rho+2}(\Omega)}.$$

So,

$$\frac{d}{dt}\left[\frac{\left|u''_{\ell m}(t)\right|^{2}}{\mu(t)} + \left\|u'_{\ell m}(t)\right\|^{2}\right] + 2\int_{\Gamma 1}\delta\left(u''_{\ell m}(t)\right)^{2} d\Gamma +$$

$$\left(\frac{\rho+1}{\mu(t)} - \varepsilon C\right)\left(\left|u'_{\ell m}(t)\right|^{\rho}, \left(u''_{\ell m}(t)\right)^{2}\right) \leq 2C\frac{\left|u''_{\ell m}(t)\right|^{2}}{\mu(t)} + \frac{1}{2\varepsilon}\left\|u'_{\ell m}(t)\right\|^{\rho+2}_{L^{\rho+2}(\Omega)}. \tag{3.13}$$

Taking a suitable $\varepsilon$ and integrating on $[0,t[$, for all $\ell \geq \ell_0$, we get

$$\frac{\left|u''_{\ell m}(t)\right|^{2}}{\mu(t)} + \left\|u'_{\ell m}(t)\right\|^{2} + 2\int_0^t\int_{\Gamma 1}\delta\left(u''_{\ell m}(s)\right)^{2} d\Gamma ds +$$

$$C_0\int_0^t\left(\left|u'_{\ell m}(s)\right|^{\rho}, \left(u''_{\ell m}(s)\right)^{2}\right) \leq \frac{\left|u''_{\ell m}(0)\right|^{2}}{\mu(0)} + \tag{3.14}$$

$$\left\|u^1\right\|^2 + 2C\int_0^t\frac{\left|u''_{\ell m}(s)\right|^{2}}{\mu(s)} ds + \frac{1}{2\varepsilon}\int_0^t\left\|u'_{\ell m}(s)\right\|^{\rho+2}_{L^{\rho+2}(\Omega)} ds.$$

To finish the second estimate we need to bound $\left(u''_{\ell m}(0)\right) in\, L^2(\Omega)$. In this point becomes clear the importance of the special basis that we have constructed. In fact, we make $t = 0$ in (3.10) and take $w = u''_{\ell m}(0)$. This yields

$$\frac{\left|u''_{\ell m}(0)\right|^{2}}{\mu(0)} + \left(\left(u_{\ell m}(0), u''_{\ell m}(0)\right)\right) + \int_{\Gamma 1}\delta u^1_{\ell m} u''_{\ell m}(0)\, d\Gamma + \frac{\left(\left|u^1_{\ell}\right|^{\rho} u^1_{\ell}, u''_{\ell m}(0)\right)}{\mu(0)} = 0.$$

By using Green's Theorem we obtain

$$\frac{\left|u''_{\ell m}(0)\right|^{2}}{\mu(0)} = \left(\Delta u^0_{\ell}, u''_{\ell m}(0)\right) - \int_{\Gamma 1}\left(\frac{\partial u^0_{\ell}}{\partial \nu} + \delta u^1_{\ell}\right) u''_{\ell m}(0)\, d\Gamma - \frac{\left(\left|u^1_{\ell}\right|^{\rho} u^1_{\ell}, u''_{\ell m}(0)\right)}{\mu(0)},$$

$$\frac{\left|u''_{\ell m}(0)\right|^2}{\mu(0)} = \left(\Delta u^0_\ell, u''_{\ell m}(0)\right) - \int_{\Gamma 1}\left(\frac{\partial u^0_\ell}{\partial \nu} + \delta\, u^1_\ell\right) u''_{\ell m}(0)\, d\Gamma - \frac{\left(\left|u^1_\ell\right|^\rho u^1_\ell, u''_{\ell m}(0)\right)}{\mu(0)},$$

and by using that $\frac{\partial u^0_\ell}{\partial \nu} + \delta\, u^1_\ell = 0$ on $\Gamma_1$, we get

$$\left|u''_{\ell m}(0)\right| \le \mu(0)\left[\left|\Delta u^0_\ell\right| + \left\|u^1_\ell\right\|^{\rho+1}_{L^{2(\rho+1)}(\Omega)}\right].$$

Then, taking into account (3.3) and (3.4), we have $V \hookrightarrow L^{2(\rho+1)}(\Omega)$ and therefore

$$\left|u''_{\ell m}(0)\right| \le C\, \mu(0)\left[\left|\Delta u^0_\ell\right| + \left\|u^1_\ell\right\|^{\rho+1}\right].$$

Combining the above inequality with (3.14) and (3.12)$_4$, for $m \in N$ and $\ell \ge \ell_0$, we get:

$$\begin{aligned}
&\left(u'_{\ell m}\right) \text{ is bounded in } L^\infty_{loc}(0,\infty; V),\\
&\left(u''_{\ell m}\right) \text{ is bounded in } L^\infty_{loc}\left(0,\infty; L^2(\Omega)\right),\\
&\left(\delta^{1/2} u''_{\ell m}\right) \text{ is bounded in } L^2_{loc}\left(0,\infty; L^2(\Gamma_1)\right)
\end{aligned} \tag{3.15}$$

Estimates (3.12) and (3.15) allows us, by induction and diagonal process, to obtain a subsequence $\left(u^{(\rho)}_{\ell m}\right)$ of $\left(u_{\ell m}\right)$ which will be also denoted by $\left(u_{\ell m}\right)$, and a function $u : \Omega \times \left]0,\infty\right[ \to R$ satisfying:

$$\begin{aligned}
&u_{\ell m} \to u \text{ weak star in } L^\infty_{loc}(0,\infty; V),\\
&u'_{\ell m} \to u' \text{ weak star in } L^\infty_{loc}(0,\infty; V),\\
&u'_{\ell m} \to u' \text{ weak star in } L^{\rho+2}_{loc}\left(0,\infty; L^{\rho+2}(\Omega)\right),\\
&u''_{\ell m} \to u'' \text{ weak star in } L^\infty_{loc}\left(0,\infty; L^2(\Omega)\right),\\
&\delta^{1/2} u''_{\ell m} \to X \text{ weakly in } L^2_{loc}\left(0,\infty; L^2(\Gamma_1)\right)
\end{aligned} \tag{3.16}$$

and as a consequence

$$\delta^{1/2} u'_{\ell m} \to \delta^{1/2} u' \text{ weak star in } L^{\infty}_{loc}\left(0,\infty; H^{1/2}(\Gamma_1)\right). \tag{3.17}$$

Convergences (3.16) and (3.17) allow us to pass to the limit in (3.7). Moreover, by using the regularity (3.16) of $u$, we obtain

$$u'' - M\left(|u|^2\right)\Delta u + |u'|^{\rho} u' = 0 \quad \text{in} \quad L^{\infty}_{loc}\left(0,\infty; L^2(\Omega)\right) \tag{3.18}$$

From the assumptions (3.3) and (3.4), it follows that $V \hookrightarrow L^{2(\rho+1)}(\Omega)$.. So, we take into account (3.18) to deduce that $\Delta u \in L^{\infty}_{loc}\left(0,\infty; L^2(\Omega)\right)$; and as $u \in L^{\infty}(0,\infty; V)$, we get

$$\frac{\partial u}{\partial \nu} \in L^{\infty}_{loc}\left(0,\infty; H^{-1/2}(\Gamma)\right).$$

Since $W$ is dense in $V$, after to pass to the limit in (3.8), we obtain

$$\int_0^{\infty}(u'', \upsilon)\theta\, dt + \int_0^{\infty} M\left(|u(.)|^2\right)((u,\upsilon))\theta\, dt + \int_0^{\infty} M\left(|u(.)|^2\right)\int_{\Gamma_1}\delta u' \upsilon\, \theta\, d\Gamma\, dt + \int_0^{\infty}\left(|u'|^{\rho} u', \upsilon\right)\theta\, dt = 0, \tag{3.19}$$

for all $\upsilon \in V$ and for all $\theta \in D(0,\infty)$. On the other hand, multiplying (3.18) by $\upsilon\theta$ with $\upsilon \in V$ and $\theta \in D(0,\infty)$ integrating and using Green's Theorem, we have

$$\int_0^{\infty}(u'', \upsilon)\theta\, dt + \int_0^{\infty} M\left(|u(.)|^2\right)((u,\upsilon))\theta\, dt - \int_0^{\infty} M\left(|u(.)|^2\right)\left\langle \frac{\partial u}{\partial \nu}, \upsilon\right\rangle \theta\, dt + \int_0^{\infty}\left(|u'|^{\rho} u', \upsilon\right)\theta\, dt = 0, \tag{3.20}$$

where $\langle .,. \rangle$ denotes the duality pairing between of $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. So, comparing (3.19) with (3.20) we have

$$\int_0^{\infty}\left\langle M\left(|u(.)|^2\right)\left[\frac{\partial u}{\partial \nu} + \delta \upsilon'\right], \psi\right\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} \theta\, dt = 0,$$

for all $\psi \in D(\Gamma_1)$ and $\theta \in D(]0,\infty[)$. This and regularity (3.17) imply

$$\frac{\partial u}{\partial v} + \delta \upsilon' = 0 \quad \text{in} \quad L^{\infty}_{loc}\left(0,\infty; H^{1/2}(\Gamma_1)\right). \qquad (3.21)$$

From above equality we can conclude that $g_u \equiv \delta u'$. Therefore $u \in L^{\infty}_{loc}(0,\infty; W)$. Moreover, as shown in [4], from (3.21) it follows that

$$\frac{\partial u'}{\partial v} + \delta u'' = 0 \quad \text{in} \quad H^{-1}_{loc}\left(0,\infty; H^{-1/2}(\Gamma_1)\right),$$

but by $(3.16)_6$ we have $\delta u'' \in L^2_{loc}\left(0,\infty; L^2(\Gamma_1)\right)$, therefore the last equality is verified in the space $L^2_{loc}\left(0,\infty; L^2(\Gamma_1)\right)$.

Uniqueness of solutions and the verification on the initial conditions are showed by the standard arguments.

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