PESQUIMAT, Revista de la F.C.M. de la Universidad Nacional Mayor de San Marcos
Vol. II - N°1 Pgs. 33-41 Lima – Perú Ag. 1999

A SURVEY ON SEMI-$T_{1/2}$ SPACES*

Miguel Caldas Cueva[1],
Ratnesh Kumar Saraf[2]

ABSTRACT

The goal of this survey article is to bring to your attention some of the salient features of recent research on characterizations of *Semi*-$T_{1/2}$ spaces.

1. Introduction

The concept of semi-open set in topological spaces was introduced in 1963 by N. Levine (Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70 (1963), 36-41 i.e., if (X,τ) is a topological space and $A \subset X$, then A is semi-open $(A \in SO(X,\tau))$ if there exists $0 \in \tau$ such that $0 \subseteq A \subseteq Cl(0)$, where $Cl(0)$ denotes closure of 0 in (X,τ). The complement A^c of a semi-open set A is called semi–closed and the semi-closure of a set A denoted by $_sCl(A)$ is the intersection of all semi-closed sets containing A.

After the works of N. Levine on semi-open sets, various mathematician turned their attention to the generalisations of various concepts of topology by considering semi-open sets instead of open sets. While open sets are replaced by semi–open sets, new results are obtained in some occasions and in other occasions substantial generalisations are exibited.

In this direction, in 1975, S.M.N. Maheshawari and R. Prasad (Some new separation axioms, Ann Soc. Sci. Bruxelles 89 (1975), 395-402) used semi–open sets to define and investigate three new separation axiom called Semi–to, *Semi*–T_1 (if for $x, y \in X$ such that $x = y$ there exists a semi–open set containing x but not y or (resp. and) a semi–open set containing y but not x) and *Semi*-T_2 (if for $x, y \in X$ such that $x = y$ there exist semi–open sets O_1 and O_2 such that $x \in O_1$, y $x \in O_2$, and $O_1 \cap O_2 = \emptyset$). Moreover, they have shown that the following implications hold.

---

* 1991 Math. Subject Classification. Primary: 54D30, 54^A05, Secondary: 54I05, 54G99. Keywords and phrases. Generalized closed sets, semi-open sets, Semi-$T_{1/2}$ spaces.

[1] Departamento de Matemática Aplicada, Universidad Federal Fluminense, Rua Mário Santos Braga s/nº. CEP. 24020-140, Niteroi RJ-Brasil. *Proof.* M. Caldas and J. Dontchev, paper in preparation. E-mail: gmamccs@vm.uff.br, or, caldas @nitnet.com.br

[2] Department of Mathematies, Kamala Nehru Government, Girls College, *Damoh (M.P.) – 470661 INDIA*

$$\begin{array}{ccc} T_2 & \rightarrow & Semi-T_2 \\ \downarrow & & \downarrow \\ T_1 & \rightarrow & Semi-T_1 \\ \downarrow & & \downarrow \\ T_0 & \rightarrow & Semi-T_0 \end{array}$$

Later, in 1987, P. Bhattacharyya and B.K. Lahiri (Semi-generalized closed sets in topology, Ind. Jr. Math., 29 (1987), 375-382) generalised the concept of closed sets to semi-generalised closed sets with the help of semi-openness. By definition a subset of A of (X,τ) is said to be semi-gereralised closed (written in short as sg-closed sets) in (X,τ), if ${}_sCl(A)\subset O$ whenever $A\subset O$ and O is semi-open in (X,τ). This generalisation of closed sets, introduced by P. Bhattacharyya and B.K.Lahiri, has no connection with the generalised closed sets as considered by N. Levine (Generalised closed sets in topology, Rend. Circ. Mat. Palermo 19(1970), 89-96), although both generalise the concept of closed sets, this notions are in general independent. Morover, they defined the concept of as new class of topological spaces called *Semi*-$T_{1/2}$ (i.e., the spaces where the class of semi-closed sets and the sg-closed sets coincide), and they proved that every *Semi*-T_1 space is *Semi*-$T_{1/2}$ and every *Semi*-$T_{1/2}$ space is *Semi*-T_0, although none of these aplications is reversible.

The purpose of the present survey is to give some characterizations of *Semi*-$T_{1/2}$ spaces, incluing a characterization using a new topology that M. Caldas and J. Dontchev (in preparation: $\wedge_s$-closure operator and the associeted topology $\tau^{\wedge_s}$) define as $\tau^{\wedge_s}$-topology. These characterizations are obtained mainly through the introduction of a concept of a generalized set and a new class of maps.

2. Characterizations on Semi-$T_{1/2}$ Spaces

Similar to W. Dunham (A new closure operator for non-T_1 topologies, Kyungpook Math. J. 22(1982), 55-60), P. Sundaram, H. Maki and K. Balachandram (Semi-generalized continuous maps and *Semi*-$T_{1/2}$ spaces, Bull Fukuoka Uni. Ed. Vol. 40 Part. III (1991), 33-40) characterized the *Semi*-$T_{1/2}$ spaces as follows:

Recall first, that for any subset E *of* (X,τ), ${}_sCl^*(E)=\cap\{A:E\subset A(\in {}_sD(X,\tau))\}$, where ${}_sD(X,\tau)=\{A: A\subset X \text{ and } A \text{ is } sg-closed \text{ in } (X,\tau)\}$ and $SO(X,\tau)^*=\{B: {}_sCl^*(B^c)=B^c\}$.

Theorem 2.1 A *topological space* (X,τ) is *Semi-$T_{1/2}$ spaces, if and only if* $SO(X,\tau)=SO(X,\tau)^*$ *holds.*

Proof. Necessity: Since the semi-closed sets and the sg-closed sets coincide by assumption, ${}_sCl(E)={}_sCl^*(E)$ holds for every subset *E* of (X,τ). Therefore, we have that $SO(X,\tau)=SO(X,\tau)^*$.

Sufficiency: Let *A* be a sg-closed set (X,τ). Then, we have $A={}_sCl^*(A)$ and hence $A^c\in SO(X,\tau)$. Thus *A* is semi-closed. Therefore (X,τ) is *Semi-$T_{1/2}$*.

Theorem 2.2 A space (X,τ) *is Semi-$T_{1/2}$, if and only if, for each* $x\in X$, $\{x\}$ *is semi-open or semi-closed.*

Proof. Necessity: Suppose that for some $x\in X$, $\{x\}$ is not semi-closed. Since *X* is the only semi-open set containing $\{x\}^c$, the set $\{x\}^c$ is sg-closed and so it is semi-closed in the *Semi-$T_{1/2}$* space (X,τ). Therefore $\{x\}$ is semi-open.

Sufficiency: Since $SO(X,\tau)\subseteq SO(X,\tau)^*$ holds, by Theorem 2.1, it is enough to prove that $SO(X,\tau)^*\subseteq SO(X,\tau)$. Let $E\in SO(X,\tau)^*$. Suppose that $E\notin SO(X,\tau)$. Then, ${}_sCl^*(E^c)=E^c$ and ${}_sCl(E^c)=E^c$ hold. There exists a point *x* of *X* such that $x\in {}_sCl(E^c)$ and $x\notin E^c\left(={}_sCl^*(E^c)\right)$. Since $x\notin {}_sCl^*(E^c)$ there exists a sg-closed set *A* such that $x\notin A$ and $A\supset E^c$. By the hypothesis, the singleton $\{x\}$ is semi-open or semi-closed.

Case 1. $\{x\}$ is semi-open: Since $\{x\}^c$ is semi-closed set with $E^c\subset\{x\}^c$, we have ${}_sCl(E^c)\subset\{x\}^c$, *i.e.*, $x\notin {}_sCl(E^c)$. This contradicts the fact that $x\in {}_sCl(E^c)$. Therefore $E\in SO(X,\tau)$.

Case 2. $\{x\}$ is semi-closed: Since $\{x\}^c$ is a semi-open set containing the sg-closed set $A(\supset E^c)$, we have $\{x\}^c\supset {}_sCl(A)\supset {}_sCl(E^c)$. Therefore $x\notin {}_sCl(E^c)$. This is a contradiction. Therefore $E\in SO(X,\tau)$. Hence in both cases, we have $E\in SO(X,\tau)$, i.e., $SO(X,\tau)^*\subseteq SO(X,\tau)$.

In 1994, M. Caldas (Espacios *Semi-$T_{1/2}$*, Pro-Math 8(1994), 116-121), give a different proof of Theorem 2.2 and proved also that the *Semi-T_1*, *Semi-T_0* and *Semi-$T_{1/2}$* spaces are equivalent, using semi- symmetric spaces. Next we prove these claim (Recall that, a topological space (X, τ) is called a semi-symmetric space if for x and y in X, $x \in {}_sCl(\{y\})$ implies that $y \in {}_sCl(\{y\})$.

Theorem 2.3 *Let (X, τ) be a semi-symmetric space. Then the following are equivalent.*

$(i)(X, \tau)$ *is Semi* $-T_0$

$(ii)(X, \tau)$ *is Semi* $-T_{1/2}$.

$(iii)(X, \tau)$ *is Semi* $-T_1$.

proof. It is enough to prove only the necessity of $(i) \leftrightarrow (iii)$. Let, $x = y$ and since (X, τ) *is Semi-T_0,* we may assume that $x \in O \subset \{y\}^c$ for some $O \in SO(X, \tau)$. Then $x \notin {}_sCl(\{y\})$ and hence $y \notin {}_sCl(\{x\})$. Therefore there exists $O_1 \in SO(X, \tau)$ such that $y \in O_1 \subset \{x\}^c$ and (X, τ) is a *Semi-T_1 space.*

As a consequence of Theorem 2.2, M. Caldas proved also the following characterization:

Theorem 2.4 (X, τ) *is semi-$T_{1/2}$, if and only if, every subset of X is the intersection of all semi-open sets and all semi-closed sets containing it.*

Proof. Necessity: Let (X, τ) be *Semi-$T_{1/2}$* with $B \subset X$ arbitrary. Then $B = \cap \{\{x\}^c ; x \notin B$ is an intersection of semi-open and semi-closed by Theorem 2.2. The result follows. Sufficiency: For each $x \in X$, $\{x\}^c$ is the intersection of all semi-open sets and all semi-closed sets containing it. Thus $\{x\}^c$ is either semi-open or semi-closed and hence X is *Semi-$T_{1/2}$*.

In 1995, Julian Dontchev (On point generated spaces, Questions Answers Gen. Topology 13 (1995), 63-69), proved that a topological space is *Semi-T_D* if and only if it is *Semi-$T_{1/2}$* .

First we recall, the following definitions, which are useful in the sequel.

(i) A topological space (X, τ) is *Semi-T_D* space (D. Jankovic and I. Reilly, On semiseparation properties, Indian J. Pure Appl. Math. 16 (1985), 967-964) if every singleton is either open or nowhere dense, or equivalently if the derived set $Cl(\{x\})/\{x\}$ is semi-closed for each point $x \in X$.

(ii) A subset A of a topological space (X, τ) is called an α-open set (O. Najastad, On some classes of nearly open sets, Pacific J. Math 15(1965), 961-970) if $A \subset Int(Cl(Int(A)))$ and an α-closed set if $Cl(Int(Cl(A))) \subset A$.

Note that the family τ^{α} of all α-open sets in (X, τ) forms always a topology on X, finer than τ.

Theorem 2.5 *For a topological space (X, τ) the following are equivalent:*

(i) The space (X, τ) is a Semi-T_D space.

(ii) The space (X, τ) is a Semi-$T_{1/2}$ space.

Proof. $(i) \rightarrow (ii)$. Let $x \in X$. Then $\{x\}$ is either open or nowhere dense by (i). Hence it is α-open or α-closed and thus semi-open or semi-closed. Then X is *Semi-$T_{1/2}$ space* by Theorem 2.2.

$(ii) \rightarrow (i)$. Let $x \in X$. We assume first that $\{x\}$ is not semi-closed. Then $X \setminus \{x\}$ is sg-closed. Then by (ii) it is semi-closed or equivalently $\{x\}$ is semi-open. Since every semi-open singleton is open, then $\{x\}$ is open. Next, if $\{x\}$ is semi-closed, then $Int(Cl(\{x\}) = Int(\{x\}) = \emptyset$ *if* $\{x\}$ is not open and hence $\{x\}$is either open or nowhere dense. Thus (X, τ) is *Semi-T_D* space.

Again using the semi-symmetric spaces, M. Caldas (A separation axiom between *Semi-T_0* and *Semi-T_1*, Mem. Fac. Sci. Kochi Uni. (Math.) 18 (1997), 37-42) proved the following theorem.

Theorem 2.6 *Let (X, τ) be a semi-symmetric space. Then the following statements are equivalent.*

(i) (X, τ) *is Semi-T_0.*

(ii) (X, τ) *is Semi-D_1.*

(iii) (X, τ) *is Semi-$T_{1/2}$.*

(iv) (X, τ) *is Semi-T_1.*

Where, *a topological space* (X,τ) *is said to be a Semi-D$_1$ if for x, y X such that x = y there exist an* $_sD$ *-set of X (i. e., if there are two semi-open sets* O_1, O_2 *in X such that* $O_1 = X$ *and* $S = O_1 \setminus O_2$ *) containing x but not y and an* $_sD$ *-set containing y but not x.*

A stronger class of closed set was considered by H. Maki in 1986 (Generalized $\wedge$-sets and the associeted clousure operator, The Special Issue in Commemoration of Prof. Kazuada IKEDA´s Retirement (1986), 139-196). He investigated the sets that can be represented as union (resp. intersection) of closed sets (resp. open sets).

Recently in 1998 as an analogy of H. Maki, mentioned above, M. Caldas and J. Dontchev in G. $\wedge_s$-sets and G. $\vee_s$-sets (Sem. Bras. de Anal. SBA 47(1998), 293-303) introduced the $\wedge_s$-sets (resp. $\vee_s$-sets) which are intersection of semi-open (resp. union of semi-closed) sets. In this paper they also define the concepts of g. $\wedge_s$-sets and g. $\wedge_s$-sets. In the following theorem they give an other characterization of the class of *Semi-T$_{1/2}$* spaces by using g. $\wedge_s$-sets.

Let us recall the following definitions and properties which we shall require.

In a topological space (X,τ), a subset B is called:

(i) $\wedge_s$-sets (resp. $\vee_s$-sets), if $B = B^{\vee_s}$ (resp. $B = B^{\vee_s}$), where, $B^{\wedge_s} = \cap\{O : O \supseteq B, O \in SO(X,\tau)\}$ and $B^{\vee_s} = \cup\{F : F \subseteq B, F^c \in SO(X,\tau)\}$.

(ii) Generalized $\wedge_s$-sets ($=g.\wedge_s$*-set*) of (X,τ) if $B^{\wedge_s} \subseteq F$ whenever $B \subseteq F$ and $F^c \in SO(X,\tau)$.

(iii) Generalized $\vee_s$-sets ($=g.\vee_s$*-set*) of (X,τ) if B^c is a $g.\wedge_s$*-se* of (X,τ).

By $D^{\wedge_s}$ (resp. $D^{\vee_s}$) we will denote the family of all $g.\wedge_s$*-sets* (resp. $g.\vee_s$*-sets*) of (X,τ).

Theorem 2.7 *Let* (X,τ)*.be a topological space. Then the following statements are equivalent:*

(i) (X,τ) *is a Semi-T$_{1/2}$ space.*

(ii) *Every* $g.\vee_s$*-Set* *is a* $\vee_s$*-Set.*

Proof. $(i) \to (ii)$. Suppose that there exists a $g.\vee_s$*-Set B* which is not a $\vee_s$*-Set*. Since $B^{\vee_s} \subseteq B$ $(B^{\vee_s} = B)$, then there exists a point $x \in B$ such that $x \in B^{\vee_s}$. Then the singleton $\{x\}$ is not semi-closed. Since $\{x\}^c$ is not semi-open, the space *X* itself is only semi-open set containing $\{x\}^c$. Therefore, ${}_sCl(\{x\}^c) \subset X$ holds and so $\{x\}^c$ is a sg-closed set. On the other hand, we have $\{x\}$ is not semi-open (since B is a $g.\vee_s$*-set, and* $x \notin B^{\vee_s}$). Therefore, we have that $\{x\}^c$ is not semi-closed set. This contradicts to the assumption that (X, τ) is a *Semi-T*$_{1/2}$ space.

$(ii) \rightarrow (i)$. Suppose that (X, τ) is not a *Semi-T*$_{1/2}$ space. Then, there exists a sg-closed set B which is not semi-closed. Since B is not semi-closed, there exist a point x such that $x \notin B$ and $x \in {}_sCl(B)$. It is easily to see that the singleton is a semi-open set or it is a $g.\vee_s$ *-set*. When $\{x\}$ is semi-open, we have $\{x\} \cap B = \emptyset$ because $x \in {}_sCl(B)$. This is a contradiction. Let us consider the case: $\{x\}$ is $g.\vee_s$ *-set*. If $\{x\}$ is not semi-closed, we have $\{x\}^{\vee_s} =$ and hence $\{x\}$ is not a $\vee_s$ *-set*. This contradicts (ii). Next, if $\{x\}$ is semi-closed, we have $\{x\}^c \supseteq {}_sCl(B)$ (i.e., $x \notin {}_sCl(B)$). In fact, the semi-open set $\{x\}^c$ contains the set B which is a sg-closed set. Then this also contradicts to the fact that $x \in {}_sCl(B)$. Therefore (X, τ) is a *Semi-T*$_{1/2}$ *space.*

In 1999, M. Caldas (Weak and Strong forms of Irresolute Maps, to appear in Internat. J. Math. & Math. Sci.) introduce the concept of irresoluteness called ap-irresolute maps and ap-semiclosed maps by using sg-closed sets. This definition enables us to obtain conditions under which maps and inverse maps preserve sg-closed sets. In this paper, M. Caldas characterize the class of *Semi-T*$_{1/2}$ in terms of ap-irresolute and ap-semi-closed maps, where a map f: $(X, \tau) \rightarrow (Y, \sigma)$ is said to be: (i) Approximately irresolute (or ap-irresolute) if, ${}_sCl(F) \subseteq f^{-1}(O)$ whenever O is a semi-open subset of (Y, σ), F is a sg-closed subset of (X, τ), and $F \subseteq f^{-1}(O)$; (ii) Approximately semi-closed (or ap-semi-closed) if $f(B) \subseteq {}_sInt(A)$ whenever A is a sg-open subset of (Y, σ), B is a semi-closed subset of (X, τ), and $f(B) \subseteq (A)$.

Theorem 2.8 *Let* (X, τ) *be a topological space. Then the following statements are equivalent:*

(i) (X, τ) *is a Semi-T*$_{1/2}$ *space.*

(ii) *For Every space* (Y, σ) *and every map f:* $(X, \tau) \rightarrow (Y, \sigma)$, *f is ap-irresolute.*

Proof. $(i) \rightarrow (ii)$. Let F be a sg-closed subset of (X, τ) and suppose that $F \subseteq f^{-1}(O)$ where $O \in SO(Y, \sigma)$. Since (X, τ) is a *Semi-T₁/₂* space, F is semi-closed (i.e., $F = {}_sCl(F)$). Therefore ${}_sCl(F) \subseteq f^{-1}(O)$. Then f is ap-irresolute.

$(ii) \rightarrow (i)$. Let B be a sg-closed subset of (X, τ) and let Y be the set X with the topology $\sigma = \{\emptyset, B, Y\}$. Finally let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. By assumption f is ap-irresolute. Since is sg-closed in (X, τ) and semi-open in (Y, σ) and $B \subseteq f^{-1}(B)$, it follows that ${}_sCl(B) \subseteq f^{-1}(B) = B$. Hence B is semi-closed in (X, τ) and therefore (X, τ) is a *Semi-T$_{1/2}$* space.

Theorem 2.9 *Let (X, τ) be a topological space. Then the following statements are equivalent:*

(i) (Y, σ) *is a Semi-T$_{1/2}$ space.*

(ii) *For every space (X, τ) and every map f: $(X, \tau) \rightarrow (Y, \sigma)$, f is ap-semi-closed.*

Proof. Analogous to Theorem 2.8 making the obvious changes.

Recently in 1999, M. Caldas and J. Dontchev ($\wedge_s$-closure operator and the associeted topology $\tau^{\wedge_s}$, in preparation) used the $g.\wedge_s -sets$ to define a new closure operator $C^{\wedge_s}$ and a new topology $\tau^{\wedge_s}$ on a topological space (X, τ). By definition for any subset B of (X, τ), $C^{\wedge_s}(B) = \cap\{U : B \subseteq U, U \in D^{\wedge_s}$. Then, since $C^{\wedge_s}$ is a Kuratowski closure operator on (X, τ), the topology $\tau^{\wedge_s}$ on X is generated by $C^{\wedge_s}$ in the usual manner, i.e., $\tau^{\wedge_s} = \{B : B \subseteq X, C^{\wedge_s}(B^c) = B^c$.

We can conclude our work obtain a new characterization on *Semi-T$_{1/2}$* space using the $\tau^{\wedge_s}$ topology.

Theorem 2.10 *Let (X, τ) be a topological space. Then the following condition are equivalent:*

(i) (X, τ) *is a Semi-T$_{1/2}$ space.*

(ii) *Every $\wedge_s$-open set is a $\vee_s$ $-set$.*

Proof. M. Caldas and J. Dontchev, paper in preparation.

E-mail: E-mail: gmamccs@vm.uff.br, or, caldas@nitnet.com.br