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GLOBAL EXISTENCE OF SOLUTIONS FOR THE DEGENERATE WAVE EQUATIONS OF KIRCHOFF TYPE WITH NONLINEAR DISSIPATIVE TERM OF VARIABLE COEFFICIENT

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ABSTRACT. *In this paper we investigate the global existence and decay of solutions to a degenerate wave equations with nonlinear dissipative term of variable coefficient.*

1. INTRODUCTION

The objective of this paper is to study the global existence and the decay property of the nonlinear system:

$$
(P)\quad \left|\begin{array}{lll}
u'' - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u + a(x)g(u') = 0 & \text{in} & Q = \Omega\times\,]0,T[\\
u = 0 & \text{in} & \sum = \Gamma\times\,]0,T[\\
u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x) & \text{in} & \Omega
\end{array}\right.
$$

where Ω is a bounded open domain in $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary Γ, $T > 0$, $M(s) = s$, $\forall\, s \geq 0$, Δ is the Laplace operator, g and a are functions satisfying suitables conditions.

Existence of global solutions to the system (P) has been investigated by many authors (ef. [1], [2], [4], [7], [8], etc) for different and positive constant, with a positive or non-negative function. Mochizuki [5] investigated the nondegenerate problem con dissipative term $a(x,t)u'$. Our purpose in this time is to prove the global existence and decay rate of solution for the case: $M(s) = s$ and $a(x)$ is a positive function.

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2. PRELIMINARIES

In the sequel $L^p(\Omega)$, $1 \leq p < \infty$ will denote the collection of L-functions which are pth-integrable over Ω. For $m \in \mathbb{N}$, the space $H^m(\Omega)$ is the Sobolev class of the functions of the spatial variable x wich along with their first m derivates belong to $L^2(\Omega)$ (See, for example Medeiros & Milla Miranda [3]) and the closure in $H^m(\Omega)$ of the space $D(\Omega)$ of the test functions on Ω is denoted by $H_0^m(\Omega)$ the inner product and norm of $L^2(\Omega)$ are represented by $(\cdot,\cdot)$ and $|\cdot|$ respectively.

Let X be a Banach space, $T > 0$ or $T = +\infty$ and $1 \leq p \leq \infty$. denote by $L^p(0,T;X)$ the Banach space of all measurable functions $u :]0,T[\rightarrow X$ such that $t \mapsto |u(t)|_X$ is in $L^p(0,T)$, with norm

$$|u|_{L^p(0,T;X)} = \left(\int_0^T |u(t)|_X^p dt \right)^{1/p},$$

if $1 \leq p < \infty$, and if $p = \infty$, then

$$|u|_{L^p(0,T;X)} = \text{ess sup}|u(t)|_X.$$

We use the following well-known lemmas without the proof in this paper:

Lemma 2.1. (Sobolev - Poincaré) *If $u \in H_0^1(\Omega)$ then $u \in L^q(\Omega)$ and the inequality*

$$|u|_q \leq C_q |\nabla u|$$

holds, where q is a number satisfying $1 \leq q \leq \dfrac{2N}{N-2}$ if $N > 2$ and $1 \leq q < \infty$ if $N = 2$ and $1 \leq q \leq \infty$ if $N = 1$.

Lemma 2.2. (Nakao [6]) *Let $\phi(t)$ be a nonnegative bounded function on $[0,\infty[$ satisfying*

$$\sup_{t \leq s \leq t+1} \phi(t)^{1+r} \leq k_0 \left(\phi(t) - \phi(t+1)\right)$$

for $r > 0$ and $k_0 > 0$. Then

$$\phi(t) \leq C(1+t)^{-\frac{1}{r}}, \qquad \textit{for all } t \geq 0$$

where $C > 0$ is a positive constant depending on $\phi(0)$ and other known constans.

3. THE MAIN RESULT

Theorem 3.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a non-decreasing continous function such that*

$$g(0) = 0 \tag{3.1}$$

$$g'(s) \geq \tau > 0 \tag{3.2}$$

$$|g(s)| \leq C_0 |s|^q \tag{3.3}$$

C_0 and τ are two positive constans and $q > 1$ is such that $(N-2)q \leq N+2$.

The function a satisfies

$$a \in W^{1,\infty}(\Omega) \tag{3.4}$$

$$a(x) \geq a_0 > 0, \quad \forall x \in \Omega \tag{3.5}$$

Let $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ with $u_0(x) \neq 0$, $\forall x \in \Omega$ and $u_1 \in H_0^1(\Omega) \cap L^{2q}(\Omega)$, then exist $\epsilon_0 > 0$ with the following property:

For each $\{u_0, u_1\}$ satisfying

$$2\left(\frac{|\nabla u_1|^2}{|\nabla u_0|^2} + |\triangle u_0|^2\right) < \epsilon_0 \tag{3.6}$$

then exists only one solution $u : \Omega \times]0, T[\to \mathbb{R}$ such that

$$u \in L^\infty(0, T; H_0^1 \cap H^2) \tag{3.7}$$

$$u' \in L^\infty(0, T; H_0^1) \tag{3.8}$$

$$u'' \in L^\infty(0, T; L^2) \tag{3.9}$$

$$\frac{d}{dt}(u'(t), w) - |\nabla u(t)|^2 (\triangle u(t), w) + (a(x) g(u'), w) = 0, \tag{3.10}$$

$\forall w \in H_0^1(\Omega)$, in the sense of $D'(0,T)$

$$u(0) = u_0, \quad u'(0) = u_1 \tag{3.11}$$

$$|\nabla u(t)| > 0, \quad \forall t \in [0, +, \infty[\tag{3.12}$$

Proof. We will use the Faedo-Galerkin method's.

We consider $\{w_j\}_{j\in\mathbb{N}}$ an orthonormal basis of $H_0^1(\Omega)\cap H^2(\Omega)$ and denote by $V_m=[w_1,\dots,w_m]$ the subspace of $H_0^1(\Omega)\cap H^2(\Omega)$ spanned by the first vectors of $\{w_j\}_{j\in\mathbb{N}}$.

We seek $u_m(t)$ in the form

$$u_m(t)=\sum_{j=1}^{m} g_{jm}(t)w_j$$

such that, for all w in V_m, $u_m(t)$ satisfies the approximate equation

$$\begin{aligned}(u_m''(t),w) &- (m^{-1}+|\nabla u_m(t)|^2)(\triangle u_m(t),w)\\ &+ (a(x)g(u_m'(t)),w)=0 \qquad (3.13)\end{aligned}$$

with the following initial conditions

$$u_m(0)=u_{0,m}\longrightarrow u_0 \text{ in } H_0^1(\Omega)\cap H^2(\Omega) \qquad (3.14)$$

$$u_m'(0)=u_{1,m}\longrightarrow u_1 \text{ in } H_0^1(\Omega)\cap L^{2q}(\Omega) \qquad (3.15)$$

Using (3.3) we deduce from (3.25) that $(g(u_{1m}))$ is bounded in $L^2(\Omega)$.

Under these conditions, the system (3.13) - (3.15) has a local solution $u_m(t)$ over the interval $[0,T_m[$. We shall see that $u_m(t)$ can be extended for all $t\geq 0$.

A priori Estimative I

For $w=2u_m'(t)$ in (3.13) we find

$$\frac{d}{dt}\left\{|u_m'(t)|^2+m^{-1}|\nabla u_m(t)|^2+\frac{1}{2}|\nabla u_m(t)|^4\right\}+$$
$$+2\int_\Omega a(x)g(u_m'(t))u_m'(t)dx=0$$

Integrate in $[0,t]$, $t<T_m$, to obtain

$$\begin{aligned}|u_m'(t)|^2+\frac{1}{2}|\nabla u_m(t)|^4 &+ 2\int_0^t\int_\Omega a(x)g(u_m')u_m'dxds \leq\\ &\leq |u_1|^2+|\nabla u_0|^2+|\nabla u_0|^4 \quad (3.16)\end{aligned}$$

It follows that

$$\begin{aligned}|u_m'(t)|&\leq k\\ |\nabla u_m(t)|&\leq k\end{aligned} \qquad (3.17)$$

Then we extend the approximate solution $u_m(t)$ to the interval $[0, T[$ for any $0 < T < \infty$.

From now on we denote by C various constants independent of m and t in $[0, T[$.

Also it follows from (3.16), (3.3) and (3.5) that

$$\int_0^t \int_\Omega a(x) g(u'_m) u'_m \, dx dt \le C \tag{3.18}$$

$$\int_0^t \int_\Omega |q(u'_m)|^{\frac{q+1}{q}} \, dx dt \le C \tag{3.19}$$

A priori Estimative II

Putting $w = -2\triangle u'_m(t)$ in (3.13) we have

$$\frac{d}{dt}\left\{|\nabla u'_m(t)|^2 + \left(m^{-1} + |\nabla u_m(t)|^2\right)|\triangle u_m(t)|^2\right\} +$$

$$+2\left(\nabla(a(x)g(u'_m)), \nabla u'_m)\right) = \left(\frac{d}{dt}|\nabla u_m(t)|^2\right)|\triangle u_m(t)|^2$$

Let us define

$$F_m(t) = \frac{|\nabla u'_m(t)|^2}{m^{-1} + |\nabla u_m(t)|^2} + |\triangle u_m(t)|^2 = f_m(t) + |\triangle u_m(t)|^2$$

A simple computation shows that

$$\begin{aligned} F'_m(t) &= \frac{-2(a(x)\nabla g(u'_m), \nabla u'_m) - (g(u'_m)\nabla a(x), \nabla u'_m)}{m^{-1} + |\nabla u_m(t)|^2} - \\ &- \frac{2(\nabla u_m(t), \nabla u'_m(t))|\nabla u'_m(t)|^2}{\left(m^{-1} + |\nabla u_m(t)|^2\right)^2} \end{aligned} \tag{3.20}$$

But:

$$(a\nabla g(u'), \nabla u') = (ag'(u')\nabla u', \nabla u') \ge \tau a_0 |\nabla u'|^2 \tag{3.21}$$

and using (3.3), the Sobolev embedding and (3.4) give

$$|(g(u')\nabla a, \nabla u')| \le C_0 C_{2q} |a|_{1,\infty} |\nabla u'|^{q+1} \tag{3.22}$$

From (3.22), (3.21), in (3.20) we get

$$\begin{aligned} F_m'(t) &\le -2\tau a_0 \frac{|\nabla u_m'|^2}{m^{-1}+|\nabla u_m|^2} + 2C_0C_{2q}|a|_{1,\infty}\frac{|\nabla u_m'|^{q+1}}{m^{-1}+|\nabla u_m|^2} \\ &+ 2\left[\frac{|\nabla u_m'|^2}{m^{-1}+|\nabla u_m|^2}\right]^{3/2} \end{aligned} \tag{3.23}$$

That is:

$$F_m'(t) \le 2\left[f_m^{1/2}(t) + \delta_0 f_m^{\frac{q-1}{2}} - \delta\right] f_m(t) \tag{3.24}$$

where $\delta_0 = C_0C_{2q}|a|_{1,\infty}(1+k)^{\frac{q-1}{2}}$, $\delta = \tau a_0$.

Integrating (3.24) from 0 to t we have

$$F_m(t) \le 2\int_0^t \left(f_m^{1/2}(s) + \delta_0 f_m^{\frac{q-1}{2}}(s) - \delta\right) f_m(s)ds + F_m(0) \tag{3.25}$$

Now, since $F_m(0) \to F(0)$, it follows of (3.6) that

$$F_m(0) < \epsilon_0 \tag{3.26}$$

for sufficiently large m.

We shall prove that

$$f_m^{1/2}(t) + \delta_0 f_m^{\frac{q-1}{2}}(t) < \frac{\delta}{2}, \quad \forall t \in [0,\infty[\tag{3.27}$$

for $\epsilon_0 = \min\left\{\left(\frac{\delta}{4}\right)^2, \left(\frac{\delta}{4\delta_0}\right)^{\frac{2}{q-1}}\right\}$. In fact

$$f_m^{1/2}(0) \le F_m^{1/2}(0) < \epsilon_0^{1/2} \le \frac{\delta}{4}$$

$$\delta_0 f_m^{\frac{q-1}{2}}(0) \le \delta_0 F_m^{\frac{q-1}{2}}(0) < \delta_0 \epsilon_0^{\frac{q-1}{2}} \le \frac{\delta}{4}$$

and thus we have

$$f_m^{1/2}(0) + \delta_0 f_m^{\frac{q-1}{2}}(0) < \frac{\delta}{2}$$

Suppose, then, that (3.27) does not hold for all $t \ge 0$.

Because of the continuity of $f_m(t)$, there is $t^* > 0$ such that

$$f_m^{1/2}(t) + \delta_0 f_m^{\frac{q-1}{2}}(t) < \frac{\delta}{2} \quad \text{for} \quad 0 \le t < t^* \tag{3.28}$$

$$f_m^{1/2}(t^*) + \delta_0 f_m^{\frac{q-1}{2}}(t^*) = \frac{\delta}{2} \tag{3.29}$$

(3.25), (3.26) and (3.28) gives:

$$F_m(t^*) \leq F_m(0) < \epsilon_0$$

This inequality yields

$$f_m^{1/2}(t^*) + \delta_0 f_m^{\frac{q-1}{2}}(t^*) < \frac{\delta}{2}$$

a contradiction to (3.19). Hence (3.27) is true.

From (3.25) and (3.17) we obtain

$$F_m(t) + \delta \int_0^t \frac{|\nabla u_m'(s)|^2}{m^{-1} + |\nabla u_m(s)|^2} ds \leq C$$

wich implies

$$|\nabla u_m(t)| \leq C \tag{3.30}$$

$$\frac{|\nabla u_m'(t)|^2}{m^{-1} + |\nabla u_m(t)|^2} \leq C \tag{3.31}$$

$$\int_0^t \frac{|\nabla u_m'(s)|^2}{m^{-1} + |\nabla u_m(s)|^2} ds \leq C \tag{3.32}$$

A priori Estimative III

Taking $w = u_m''(t)$ in (3.13) and choosing $t = 0$ we get

$$|u_m''(0)| \leq \left(\frac{1}{m} + |\nabla u_{0m}|^2 \right) |\Delta u_{0m}| + |a|_{1,\infty} |g(u_{1m})|$$

hence $u_m''(0)$ is bounded in $L^2(\Omega)$. Next, by differentiation of (3.13) and putting $w = 2u_m''(t)$ we find

$$\frac{d}{dt}\left\{ |u_m''(t)|^2 + \left(m^{-1} + |\nabla u_m(t)|^2\right) |\nabla u_m'(t)|^2 \right\} + 2\int_\Omega a(x) g'(u_m')(u_m'')dx$$

$$= 2\left(\nabla u_m, \nabla u_m'\right) |\nabla u_m'(t)|^2 + 4\left(\nabla u_m, \nabla u_m'\right) \int_\Omega (\Delta u_m) u_m'' dx$$

$$\leq |\nabla u_m| |\nabla u_m'|^3 + 4 |\nabla u_m| |\nabla u_m'| |u_m''| |\Delta u_m| \leq C + C|u_m''(t)|^2$$

where we have used the priori estimatives I and II.

Integrating from 0 a t we have, using the Gronwall's Inequality:

$$|u_m''(t)| \leq C \tag{3.33}$$

Passage to the limit.

The proof is essentially included in [8]. For completeness however we shall see the convergence of dissipative term. By applying the Dunford-Pettis and Banach-Bourbaki theorems we conclude from (3.17) - (3.19), (3.30) - (3.32) and (3.33), replacing the sequence u_m with a subsequence if needed, that

$$u_m \rightharpoonup u \qquad \text{weak-star in } L^\infty(0,T;H_0^1 \cap H^2) \tag{3.34}$$

$$u_m' \rightharpoonup u' \qquad \text{weak-star in } L^\infty(0,T;H_0^1) \tag{3.35}$$

$$u_m'' \rightharpoonup u'' \qquad \text{weak-star in } L^\infty(0,T;L^2) \tag{3.36}$$

$$u_m' \rightharpoonup u' \qquad \text{almost every where in } Q \tag{3.37}$$

$$g(u_m') \rightharpoonup \chi \quad \text{weak in } L^{\frac{q+1}{q}}(\Omega) \tag{3.38}$$

$$|\nabla u_m|^2 \Delta u_m \to \psi \quad \text{weak-star in} \quad L^\infty(0,T;L^2)$$

We have to show that u is a solution of (P). We shall prove only that

$$\int_Q a(x) g(u_m') v dx dt \to \int_Q a(x) g(u') v dx dt \tag{3.39}$$

for all $v \in L^{q+1}\left(0,T;H_0^1(\Omega)\right)$.

In fact, from (3.18) and Fatou's Lemma $u'g(u') \in L'(Q)$. This yield $g(u') \in L^1(Q)$. On the other hand (3.37) and the continuity of g we deduce that

$$g(u_m') \to g(u') \qquad \text{a.e. in } Q$$

Let $E \subseteq Q$ and set

$$E_1 = \{(x,t) \in E : g(u_m'(x,t)) \leq |E|^{-1/2}\}; \quad E_2 = E - E_1$$

when $|E|$ is the measure of E.

If $h(r) = \inf\{|x| : x \in \mathbb{R} \text{ and } |g(x)| \geq r\}$ then we have

$$\int_E |g(u_m')| dx dt \leq |E|^{1/2} + \left[h\left(|E|^{-1/2}\right)\right]^{-1} \int_{E_2} |u_m' g(u_m')| dx dt$$

Applying (3.18) we have that

$$\sup_{m\in\mathbb{N}}\int_E |g(u'_m(x,t))|dxdt \to 0 \quad \text{when} \quad |E|\to 0.$$

From Vitali's convergence theorem we get

$$g(u'_m)\to g(u') \quad \text{in} \quad L^1(Q)$$

Hence we have

$$\int_0^T\int_\Omega a(x)[g(u'_m)-g(u')]dxdt \leq |a|_{1,\infty}|g(u'_m)-g(u')|_{L'(Q)}\to 0$$

as $m\to\infty$.

So we get that

$$a(x)g(u'_m)\to a(x)g(u') \quad \text{in } L'(Q)$$

and from (3.38)

$$a(x)g(u'_m)\to a(x)g(u') \quad \text{weak-star in } L^{\frac{q+1}{q}}(Q)$$

this implies (3.39).

We now prove that $|\nabla u(t)|>0$ for all $t\geq 0$. We need the following lemma

Lemma. *If $v:[-T,T]\to H_0^1(\Omega)\cap H^2(\Omega)$ is a weak solution of*

$$\left|\begin{array}{l} v''(t)-|\nabla v(t)|^2\Delta v(t)+a(x)g(v'(t))=0, \quad -T\leq t\leq T \\ v(0)=0, \quad v'(0)=0 \end{array}\right.$$

then $v(t)=0$, for $t\in[-T,T]$.

Proof. Multiplying with $2v'(t)$ we have

$$\frac{d}{dt}\left\{|v'(t)|^2+\frac{1}{2}|\nabla v(t)|^4\right\}+2\int_\Omega a(x)g(v'(t))v'(t)dx=0$$

and integrating in $[0,t]$, using (3.2), gives

$$|v'(t)|^2+\frac{1}{2}|\nabla v(t)|^4 \leq 2a_0|\tau|\int_0^{|t|}|v'(s)|^2ds$$

Gronwall's Lemma assures $v'(t)=0$ and $v(t)=0$ for all $t\in[-T,T]$. This concludes the proof of this lemma. □

We now turn to the proof of $|\nabla u(t)| > 0, \quad \forall t \geq 0$. Suppose that there exists a number $T > 0$ such that $\nabla u(T) = 0$. Since the a priori estimatives imply that $\dfrac{|\nabla u'(T)|}{|\nabla u(T)|}$ is bounded, then

$$|\nabla u'(T)| \leq C|\nabla u(T)| = 0.$$

Hence, the above lemma implies that $u(t) = 0$, for $0 \leq t \leq T$, wich contradicts $u_0(x) \neq 0$.

The uniqueness is a consequence of the monotonicity of g and Gronwall's Inequality. We shall omit the proof. Since it can be obtained in a standard way.

Theorem 3.2. (Energy Decay) *In addition to (3.1) - (3.3), assume that*

$$g(s) \leq C_1|s| \text{ if } |s| \leq 1 \tag{3.40}$$

then the total energy

$$E(t) = |u'(t)|^2 + \frac{1}{2}|\nabla u(t)|^4$$

satisfies

$$E(t) \leq \frac{C_2}{(1+t)^2}, \qquad \text{for all } t \geq 0$$

where C_1 and C_2 are positive constants.

Proof. Taking the scalar product of the first equation of (P) with $2u'$ and integrating over Ω we obtain

$$E'(t) + 2\int_\Omega a(x)u'g(u')dx = 0 \tag{3.41}$$

Integrating (3.41) over $[t, t+1]$ we get

$$2\int_t^{t+1}\int_\Omega a(x)u'g(u')dxds = E(t) - E(t+1) \equiv D(t)^2$$

From this we obtain

$$\int_t^{t+1} |u'(s)|^2 ds \leq CD(t)^2 \tag{3.43}$$

By the Mean Value theorem there exist two points $t_1 \in \left[t, t+\frac{1}{4}\right]$ and $t_2 \in \left[t+\frac{3}{4}, t+1\right]$ such that

$$|u'(t_i)|^2 \leq CD(t)^2, \quad i = 1, 2 \tag{3.44}$$

thus, multiplying of the first equation in (P) by u and integrating it over $\Omega\times\,]t_1, t_2[$, we have from (3.42) - (3.44) and Lemma 2.1

$$\begin{aligned} \int_{t_1}^{t_2} |\nabla u(s)|^4 ds &= \int_{t_1}^{t_2} |u'(s)|^2 ds - (u'(t_1), u(t_1)) + (u'(t_2), u(t_2)) \\ &\quad - \int_{t_1}^{t_2} (a(x)g(u'), u) ds \\ &\leq C\left(D(t)^2 + D(t)E(t)^{1/4}\right) \equiv A(t)^2 \end{aligned} \tag{3.45}$$

From (3.43) and (3.45) we conclude that

$$E(t_2) \leq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} E(s)ds \leq CA(t)^2$$

there fore, we obtain

$$\begin{aligned} E(t) &= E(t_2) + 2\int_t^{t_2}\int_\Omega a(x)g(u')u' dx ds \\ &\leq CA(t) \leq C\{D(t)^2 + D(t)E(t)^{1/4}\} \end{aligned}$$

Using Young's inequality, noting that $D(t)^2 \leq E(t) \leq E(0)$ we get

$$\sup_{t\leq s\leq t+1} E(s)^{3/2} \leq CD(t)^2 = C(E(t) - E(t+1))$$

Hence, lemma 2.2 gives

$$E(t) \leq C(1+t)^{-2}, \quad \forall t \geq 0$$

this ends the proof of theorem. □

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