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GLOBAL EXISTENCE AND EXPONENTIAL DECAY TO THE WAVE EQUATION WITH LOCALIZED FRICTIONAL DAMPING*

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ABSTRACT.- *In this paper we show the global existence and the exponencial decay of solutions of the one-dimensional wave equation with localized frictional damping.*

1. INTRODUCTION

In this paper we study the existence of global smooth solution to the wave equations when the dissipation given by the frictional term is effective only in a part of the domain. Moreover we show that this local dissipation is strong enough to produce uniform decay of the solutions. Let us consider a string o length L. The mathematical model which defines the deformations of the string is given by

$$u_{tt} - \left[\sigma\left(u_x\right)\right]_x + a(x)u_t = 0 \quad \text{in} \quad \left]0, L\right[\, x \, \left]0, \infty\right[, \tag{1.1}$$

with initial data

$$u(x,0) = u_0(x) \quad u_t(x,0) = u_1(x) \quad \text{in} \quad]0, L[\tag{1.2}$$

and boundary conditions

$$u(L,t) = u(0,t) = 0 \quad \forall t \geq 0 \tag{1.3}$$

where by $u = u(x,t)$ we denote the displacement, σ is a function satisfying

$$\sigma \in C^3(\mathbb{R}) \quad \text{such that} \quad \sigma'(0) > 0 \quad \text{and} \tag{1.4}$$

$$a \in C^1(0,L) \quad a \geq 0, \quad a(x) > 0 \quad \text{in} \quad]L_0, L] \quad \text{where} \quad 0 \leq L_0 < L. \tag{1.5}$$

It is well known by now that in non linear elasticity when $a \equiv 0$, the equation is conservative and the solution will blow up in a finite time, no matter how small and regular the initial data is. On the other hand, when the dissipation is effective in the whole domain, then when the initial data is taken small enough there exists only one global smooth solution of the system.

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The problem we consider here is intermediary between this two cases. The natural questions under in our case is about the strongness of the dissipation. We can ask, is the local dissipation strong enough to secure estimates and prevent blow up?. The mains result of this paper is to give a positive answer to that question. We will prove that the problem (1.1) - (1.5) is well posed, that is if $a \geq 0$, with a zero outside of a boundy of 0 or L, the problem has a global smooth solution which decays exponentially as time goes to infinity, provided the initial data is small enough.

The main result of this paper is summarized in the following Theorem.

Theorem 1.1 (Main Result) *Let us suppose that the initial data satisfies*

$$u_0 \in H^3(0,L) \cap H_0^1(0,L), \quad u_1 \in H^2(0,L) \cap H_0^1(0,L) \quad \text{such that}$$

$$\|u_0\|_{H^3(0,L)}^2 + \|u_1\|_{H^2(0,L)}^2 + \|u_{tt}(0)\|_{H^1(0,L)}^2 < \varepsilon^2 \tag{1.6}$$

for any $\varepsilon > 0$ *and small enough, where* $u_{tt}(0) = \sigma'(u_{0,x})u_{0,xx} - a(x)u_1$, *the there exists a global smooth solution of problem (1.1) - (1.5) satisfying* $u \in C(0,\infty; H^3) \cap C^1(0,\infty; H^2) \cap C^2(0,\infty; H^1)$, *which decays exponentially as time goes to infinity.*

The method we use is based on the multiplicative technique. We introduce a new multiplicator which allows us to get the global estimate that we need to show the global existence result.

The main ideia is to construct a Liapunov functional $\mathcal{L}$ satisfying

$$\frac{d}{dt}\mathcal{L}(t) \leq -\kappa \mathcal{L}(t)$$

where κ is a positive constant.

2. EXISTENCE AND ASYMPTOTIC BEHAVIOUR

Our first step is to stablish the local existence result.

Theorem 2.1 (Local Existence) *Let us take initial data satisfying*

$$u_0 \in H^3(0,L) \cap H_0^1(0,L), \quad u_1 \in H^2(0,L) \cap H_0^1(0,L)$$

then, there exists $T > 0$ *and a function*

$$u \in X := C(0,T; H^3) \cap C^1(0,T; H^2) \cap C^2(0,T; H^1)$$

satisfying problem (1.1) - (1.5).

Proof.- The main idea is to use the fixed point Theorem. To do this we define

$$W_\mu(0,T) := \Big\{ w \in C^0\big(0,T;H^2\big),\ w \in C^i\big(0,T;H^{3-i}\big)\ i=1,2$$
$$w(x,0)=u_0(x),\ w_t(x,0)=u_1(x),\quad \|w\|_W \le \mu \Big\}$$

$$T : W_\mu \to W_\mu$$
$$v \to Tv = u$$

where u is the solution of

$$\begin{cases} u_{tt} - \sigma'(v_x)u_{xx} + a(x)u_t = 0 \\ u(0)=u_0 \in H^2(0,L), \quad u_t(0)=u_1 \in H^1(0,L) \\ u(L,t)=u(0,t)=0 \end{cases} \qquad \blacksquare$$

To show the global existence, it is enough to show that

$$\lim_{t\to T_{max}} \|u(t)\|_X < C$$

Here we follows similar techniques as in [7]. Firts study the linearized problem.

$$u_{tt} - \sigma'(0)u_{xx} + a(x)u_t = f \quad \text{in} \quad]0,L[\ x\ ]0,T[\tag{2.1}$$

where

$$f = \big[\sigma'(u_x) - \sigma'(0)\big]u_{xx}$$

In general we consider

$$U_{tt} - \sigma'(0)U_{xx} + a(x)U_t = F \quad \text{in} \quad]0,L[\ x\ ]0,T[\tag{2.2}$$

$$U(L,t)=U(0,t)=0 \quad \forall t \ge 0 \tag{2.3}$$

where by U,F we are denoting $(U,F)=\{(u,f),(u_t,f_t),(u_{tt},f_{tt})\}$.

Under the above notations the energy associated is

$$E(t,U) = \frac{1}{2}\int_0^L U_t^2 + \sigma'(0)U_x^2\,dx. \tag{2.4}$$

Multiplying (2.2) by U_t, performing an integration by parts and using the boundary conditions we get

$$\frac{dE(t,U)}{dt} = -\int_0^L a(x)U_t^2\,dx + \int_0^L FU_t\,dx \tag{2.5}$$

Observation 1 The Lemas 2.1, 2.2, 2.3 that will be enunciates, are valid in the cases

$$(U, F) = \{(u, f), (u_t, f_t), (u_{tt}, f_{tt})\}.$$

Let us introduce the following functionals

$$\begin{aligned}
I(t,U) &= \int_0^L \alpha(x) U U_t \, dx + \frac{1}{2}\int_0^L a(x)\alpha(x)|U|^2(x)\, dx \\
J(t,U) &= \int_0^L U_t \, q(x) U_x \, dx \\
L(t,U) &= N\, E(t,U) + I(t,U) + J(t,U)
\end{aligned}$$

where

$$\begin{aligned}
&\alpha \in C^2([0,L]), \quad \alpha(0)<0, \, \alpha(L)>0, \quad \alpha'' \le 0, \\
&q \in C^1(0,L), \quad q(0)=q(L)=0 \quad \text{and } \alpha, q \text{ satisfying} \\
&\alpha(x) + \frac{1}{2} q'(x) \ge C > 0 \quad \text{and } \exists \ \alpha_0 > 0 \ \text{ such that} \\
&\alpha_0 a(x) - \alpha(x) + \frac{1}{2} q'(x) \ge D > 0
\end{aligned} \tag{2.6}$$

The above conditions over α imply that there exists $L^* \in (0, L_0)$ such that $\alpha(L^*)=0$, therefore we have

$$q'(x) := \begin{cases} \alpha(x) + C & \text{in } \left[L^*, L_0 + \delta_1\right] \\ -\alpha(x) + C & \text{in } [0, L^*] \cup \left[L_0 + \delta_2, L\right] \end{cases}$$

such that $q' \in C(0,L)$. Since $0 < \delta_1 < \delta_2$ taking $\int_0^L q'(x)\,dx = 0$.

Lema 2.1 *Let a, σ, α be as (1.5), (1.4), (2.6) and U be the strong solution of (2.2) - (2.3) then the following inequalities follows*

$$\frac{dI(t,U)}{dt} \le -\sigma'(0)\int_0^L \alpha(x)|U_x|^2\,dx + \int_0^L \alpha(x)|U_t|^2\,dx + \int_0^L \alpha(x) U F\,dx \tag{2.7}$$

Proof.- Multiplying (2.2) by $\alpha(x)U$ and integrating over $]0, L[$, we find that

$$\frac{1}{2}\frac{d}{dt}\int_0^L \alpha(x) a(x)|U|^2\,dx = \int_0^L F\alpha(x) U\,dx - \int_0^L \alpha(x) U U_{tt}\,dx + \sigma'(0)\int_0^L \alpha(x) U U_{xx}\,dx \tag{2.8}$$

Since

$$\int_0^L \alpha(x)UU_{xx}\,dx = -\int_0^L \alpha'(x)UU_x\,dx - \int_0^L \alpha(x)|U_x|^2\,dx \tag{2.9}$$
$$= \frac{1}{2}\int_0^L \alpha''(x)|U|^2\,dx - \int_0^L \alpha(x)|U_x|^2\,dx$$

Using (2.9) and $\alpha''(x)\leq 0$, (2.8) will be

$$\frac{1}{2}\frac{d}{dt}\int_0^L \alpha(x)a(x)|U|^2\,dx \leq \int_0^L F\alpha(x)U\,dx - \int_0^L \alpha(x)UU_{tt}\,dx - \sigma'(0)\int_0^L \alpha(x)|U_x|^2\,dx.$$

Since

$$\frac{d}{dt}\int_0^L \alpha(x)UU_t\,dx = \int_0^L \alpha(x)|U_t|^2\,dx + \int_0^L \alpha(x)UU_{tt}\,dx$$

Summing up the two above relations our conclusion follows. ■

Lemma 2.2 *Let a,σ,α be as in (1.5), (1.4) and U be the strong solution of (2.2) - (2.3) then the following inequality holds*

$$\frac{dJ(t,U)}{dt} = \frac{\sigma'(0)}{2}\underbrace{\left\{|U_x|^2(L,t)\varphi(L) - |U_x|^2(0,t)\varphi(0)\right\}}_{:=b(t)} \tag{2.10}$$
$$-\frac{1}{2}\int_0^L \varphi'(x)\left\{|U_t|^2 + \sigma'(0)|U_x|^2\right\}dx - \int_0^L a(x)U_t\,\varphi(x)U_x\,dx$$
$$+\int_0^L F\varphi(x)U_x\,dx$$

for any $\varphi \in C^1(0,L)$.

Proof.- Multiplying equation (2.2) by $\varphi(x)U_x$ and integrating over $]0,L[$ we find

$$\int_0^L U_{tt}\varphi(x)U_x dx = \sigma'(0)\int_0^L U_{xx}\varphi(x)U_x dx - \int_0^L a(x)\varphi(x)U_tU_x\,dx + \int_0^L F\varphi(x)U_x\,dx$$

integrating by parts

$$= \frac{-\sigma'(0)}{2}\int_0^L \varphi'(x)|U_x|^2\,dx + \frac{\sigma'(0)}{2}\varphi(x)|U_x|^2\Big|_0^L - \int_0^L a(x)\varphi(x)U_tU_x dx + \int_0^L F\varphi(x)U_x\,dx \tag{2.11}$$

But

$$\int_0^L U_{tt}\varphi(x)U_x dx = \frac{d}{dt}\int_0^L U_t\varphi U_x dx - \int_0^L U_t\varphi(x)U_{xt}\, dx$$

integrating by parts

$$= \frac{d}{dt}\int_0^L U_t\varphi U_x dx + \frac{1}{2}\int_0^L \varphi'(x)|U_t|^2\, dx \tag{2.12}$$

From relations (2.11) and (2.12), our conclusion follows. ■

Lemma 2.3 *Let a, σ, α, q be as in (1.5), (1.4), (2.6) and let us take N large enough, then there exists $\hat{\delta} > 0$ such that*

$$\frac{dL(t,U)}{dt} \le -\hat{\delta}E(t,U) + \int_0^L F\{NU_t + \alpha(x)U + q(x)U_x\}\, dx \tag{2.13}$$

Moreover, there exist positive constants k_1, k_2 such that

$$k_1 E(t,U) \le L(t,U) \le k_2 E(t,U) \tag{2.14}$$

Proof.- From (2.5), (2.7), (2.10) and since $q(0) = q(L) = 0$ (that is $b(t) = 0$) we have

$$\frac{dL(t,U)}{dt} \le$$

$$\underbrace{-N\int_0^L a(x)|U_t|^2 dx - \sigma'(0)\int_0^L \alpha(x)|U_x|^2 dx - \frac{1}{2}\int_0^L q'(x)\left\{|U_t|^2 + \sigma'(0)|U_x|^2\right\}dx}_{:=I_{11}} \tag{2.15}$$

$$\underbrace{+\int_0^L \alpha(x)|U_t|^2\, dx}_{:=I_{12}} \underbrace{-\int_0^L a(x)U_t q(x)U_x\, dx}_{:=I_2} + \int_0^L F\{NU_t + \alpha(x)U + q(x)U_x\}dx$$

Let us find some estimates to I_1, I_2, where $I_1 = I_{11} + I_{12}$

$$I_1 = -\int_0^L \left[Na(x) - \alpha(x) + \frac{1}{2}q'(x)\right]|U_t|^2\, dx - \sigma'(0)\int_0^L \left[\alpha(x) + \frac{1}{2}q'(x)\right]|U_x|^2\, dx$$

$$I_2 = -\int_0^L \left(\frac{1}{\sqrt{2\varepsilon}}\sqrt{a}U_t\right)\left(\sqrt{2\varepsilon}\sqrt{a}qU_x\right)dx$$

Summing up I_1 with I_2 we get

$$I_1 + I_2 \le -\int_0^L \left[\left(N - \frac{1}{4\varepsilon}\right)a(x) - \alpha(x) + \frac{1}{2}q'(x)\right]|U_t|^2\, dx -$$

$$\sigma'(0)\int_0^L \left[\alpha(x) + \frac{1}{2}q'(x)\right]|U_x|^2\, dx + \varepsilon \underbrace{\int_0^L aq^2|U_x|^2\, dx}_{:=I_3}.$$

Since,

$$I_3 = \int_{L_0}^L a(x)q^2|U_x|^2\, dx \le \underbrace{\left(\max_{x\in[L_0,L]} |q(x)|^2\right)}_{:=M_0}\int_{L_0}^L |U_x|^2\, dx$$

we can write

$$I_1 + I_2 \le -\int_0^L \left[\left(N - \frac{1}{4\varepsilon}\right)a(x) - \alpha(x) + \frac{1}{2}q'(x)\right]|U_t|^2\, dx - \int_{L_0}^L \sigma'(0)\left[\alpha(x) + \frac{1}{2}q'(x)\right]|U_x|^2\, dx$$

$$-\int_0^{L_0} \sigma'(0)\left[\alpha(x) + \frac{1}{2}q'(x)\right]|U_x|^2\, dx + \varepsilon M_0 \int_{L_0}^L |U_x|^2\, dx \tag{2.16}$$

where α and q must satisfy $\alpha(x) + \frac{1}{2}q'(x) \ge C > 0$ in $[0, L]$. Let us take, ε such that $\varepsilon < \frac{C}{M_0}\sigma'(0)$ and N such that

$$N > \frac{1}{4\varepsilon} \quad \text{and} \quad \left[\left(N - \frac{1}{4\varepsilon}\right)a(x) - \alpha(x) + \frac{1}{2}q'(x)\right] \ge D > 0.$$

where,

$$C = \min_{x\in[0,L]} \left\{\alpha(x) + \frac{1}{2}q'(x)\right\}$$

$$D = \min_{x\in[0,L]} \left\{\left(N - \frac{1}{4\varepsilon}\right)a(x) - \alpha(x) + \frac{1}{2}q'(x)\right\}$$

Under the above notations we can rewrite inequality (2.16) in the following way

$$I_1 + I_2 \le -D\int_0^L |U_t|^2\,dx - \left(\sigma'(0)C - \varepsilon M_0\right)\int_{L_0}^L |U_x|^2\,dx - \sigma'(0)C\int_0^{L_0}|U_x|^2\,dx \tag{2.17}$$

$$\le -D\int_0^L |U_t|^2\,dx - \left(\sigma'(0)C - \varepsilon M_0\right)\int_{L_0}^L |U_x|^2\,dx$$

$$\le -2\min\left\{D, C - \frac{\varepsilon M_0}{\sigma'(0)}\right\}\frac{1}{2}\int_0^L |U_t|^2 + \sigma'(0)|U_x|^2\,dx$$

$$\le \underbrace{-2\min\left\{D, C - \frac{\varepsilon M_0}{\sigma'(0)}\right\}}_{:=\delta} E(t,U)$$

We finish the proof of (2.13) using relation (2.17) in (2.15). To show (2.14), we proced as follows

$$\left|\int_0^L \alpha(x)UU_t\,dx\right| \le |\alpha|_\infty \frac{1}{2}\int_0^L |U|^2 + |U_t|^2\,dx \tag{2.18}$$

$$\left|\int_0^L a(x)\alpha(x)|U|^2(x)dx\right| \le |\alpha|_\infty \frac{1}{2}\int_0^L |U|^2\,dx \tag{2.19}$$

$$\left|\int_0^L q(x)U_xU_t\,dx\right| \le |q|_\infty \frac{1}{2}\int_0^L |U_x|^2 + |U_t|^2\,dx. \tag{2.20}$$

From (2.18), (2.19), (2.20) we have

$$\left|\underbrace{\int_0^L \alpha(x)UU_t\,dx + \frac{1}{2}\int_0^L a(x)\alpha(x)|U|^2\,dx + \int_0^L q(x)U_xU_t\,dx}_{:=R}\right|$$

$$\le |\alpha|_\infty \frac{1}{2}\int_0^L 2|U|^2 + |U_t|^2\,dx + |q|_\infty \frac{1}{2}\int_0^L |U_x|^2 + |U_t|^2\,dx$$

Using Poincare's inequality we have

$$\begin{aligned} |R| &\le |\alpha|_\infty \frac{1}{2}\int_0^L 2C_p|U_x|^2 + |U_t|^2\,dx + |q|_\infty \frac{1}{2}\int_0^L |U_x|^2 + |U_t|^2\,dx \\ &\le b_2 E(t,U) \end{aligned} \tag{2.21}$$

where b_2 is a positive constant. From (2.21) it follows that

$$\int_0^L \alpha(x)UU_t dx + \frac{1}{2}\int_0^L a(x)\alpha(x)|U|^2 dx + \int_0^L q(x)U_xU_t dx \le b_2E(t,U) \quad (2.22)$$

Recalling the definition of $L(t,U)$ and using (2.22) we arrive at

$$L(t,U) \le \underbrace{\{N+b_2\}}_{:=k_2} E(t,U)$$

on the other hand, using (2.21) we get

$$L(t,U) \ge \underbrace{\{N-b_2\}}_{:=k_1} E(t,U)$$

Taking N such that $N > b_2$ our conclusion follows. ■

Let us define the following functional

$$\mathcal{M}(t) = \|u(t)\|^2_{H^3(0,L)} + \|u_t(t)\|^2_{H^2(0,L)} + \|u_{tt}(t)\|^2_{H^1(0,L)}$$

To obtain the global existence result we use the following hypotheses

$$\mathcal{M}(0) = \|u_0\|^2_{H^3(0,L)} + \|u_1\|^2_{H^2(0,L)} + \|u_{tt}(0)\|^2_{H^1(0,L)} < \varepsilon^2$$

for $\varepsilon > 0$ small enough. Since the local solution is continuous it follows that there exists $t_0 > 0$ such that

$$\mathcal{M}(t) < \varepsilon^2 \quad \forall t \in [0, t_0]$$

Let us define

$$T^* = \sup\{t_1, \mathcal{M}(t) < d\varepsilon^2, t \in [0, t_1[\},$$

and let us suppose that $T^* < \infty$. Therefeore

$$\mathcal{M}(t) < d\varepsilon^2 \quad \forall t \in [0, T^*[. \quad (2.23)$$

Using Gagliardo Niremberg's inequality we get

$$|u_x(x,t)| \le c_1\varepsilon. \quad (2.24)$$

By the mean value theorem, (2.23) and the continuity of σ'' we have

$$|\sigma'(u_x)-\sigma'(0)|\le c_2\varepsilon. \tag{2.25}$$

Using Gagliardo Niremberg's inequality and (2.23) we get

$$|u_{xx}|<c\varepsilon \quad , \quad |u_{xt}|<c\varepsilon. \tag{2.26}$$

From equation (2.1) we have

$$\int_0^L |u_{xx}|^2\,dx \le 2c_3\int_0^L |u_{tt}|^2+|u_t|^2\,dx \tag{2.27}$$

$$\int_0^L |u_{xxt}|^2\,dx \le 3c_3\hat{c}^2\int_0^L |u_{ttt}|^2+|u_{tt}|^2+|u_t|^2\,dx \tag{2.28}$$

where

$$\begin{aligned} c_3 &:= \left\{\frac{1}{\sigma'(0)-\varepsilon c_2}\right\}^2 \\ \hat{c} &:= \left\{c\varepsilon \max_{[-1,1]}|\sigma''(x)|\frac{1}{\sigma'(0)-\varepsilon c_2}+1\right\} \end{aligned}$$

Taking $\varepsilon>0$ such that

$$\sigma'(0)>\varepsilon c_2.$$

Let us define

$$\begin{aligned} L_1(t) &= L(t,u)+\frac{N}{2}\int_0^L\left[\sigma'(u_x)-\sigma'(0)\right]|u_x|^2\,dx \\ L_2(t) &= L(t,u_t)+\frac{N}{2}\int_0^L\left[\sigma'(u_x)-\sigma'(0)\right]|u_{xt}|^2\,dx \\ L_3(t) &= L(t,u_{tt})+\frac{N}{2}\int_0^L\left[\sigma'(u_x)-\sigma'(0)\right]|u_{xtt}|^2\,dx \\ S(t) &= L_1(t)+L_2(t)+L_3(t) \end{aligned}$$

Lemma 2.4 *With initial data (1.6) and* u *local solution of problem (1.1) - (1.5) there exists positive constants* k_3 *and* k_4 *such that*

$$k_3E(t,u) \leq L_1(t) \leq k_4E(t,u) \tag{2.29}$$

$$k_3E(t,u_t) \leq L_2(t) \leq k_4E(t,u_t) \tag{2.30}$$

$$k_3E(t,u_{tt}) \leq L_3(t) \leq k_4E(t,u_{tt}) \tag{2.31}$$

where

$$k_3 = k_1 - \frac{N}{2}c_2\frac{\varepsilon}{\sigma'(0)}$$

$$k_4 = k_2 + \frac{N}{2}c_2\frac{\varepsilon}{\sigma'(0)}$$

for ε *small enough.*

Proof.- This result follows easily from relations (2.14) and (2.25). ■

Lemma 2.5 *Under the same hypotheses of Lemma 2.4, there exists a positive constant* k *such that*

$$\frac{dS(t)}{dt} \leq -kS(t). \tag{2.32}$$

Moreover, there exist positive constants k_5 *and* k_6 *such that*

$$k_5\mathcal{M}(t) \leq S(t) \leq k_6\mathcal{M}(t). \tag{2.33}$$

Proof.-
To show (2.32) we need to estimate the following expression

$$\int_0^L F\{NU_t + \alpha U + qU_x\}\,dx$$

where $(F,U) = \{(f,u),(f_t,u_t),(f_{tt},u_{tt})\}$,

(*i*) **Caso** $(F,U) = (f,u)$

$$\begin{aligned}
\int_0^L fu_t &= \int_0^L \left[\sigma'(u_x) - \sigma'(0)\right]u_{xx}u_t\,dx \\
&= -\int_0^L \left\{\left[\sigma'(u_x) - \sigma'(0)\right]u_t\right\}_x u_x\,dx \\
&= -\int_0^L \sigma''(u_x)u_{xx}u_xu_t\,dx - \int_0^L \left[\sigma'(u_x) - \sigma'(0)\right]u_{tx}u_x\,dx \\
&= -\int_0^L \sigma''(u_x)u_{xx}u_xu_t\,dx - \frac{1}{2}\frac{d}{dt}\int_0^L \left[\sigma'(u_x) - \sigma'(0)\right]\left|u_x\right|^2 dx + \frac{1}{2}\int_0^L \sigma''(u_x)u_{xt}\left|u_x\right|^2 dx
\end{aligned}$$

From (2.26), $|\sigma''(u_x)| < M$ and Young's inequality it follows

$$\begin{aligned}
&\leq \frac{Mc\varepsilon}{2}\int_0^L |u_t|^2 + |u_x|^2\,dx + \frac{Mc\varepsilon}{2}\int_0^L |u_x|^2\,dx - \frac{1}{2}\frac{d}{dt}\int_0^L [\sigma'(u_x)-\sigma'(0)]|u_x|^2\,dx \\
&\leq Mc\varepsilon \max\left\{\frac{2}{\sigma'(0)},1\right\}E(t,u) - \frac{1}{2}\frac{d}{dt}\int_0^L [\sigma'(u_x)-\sigma'(0)]|u_x|^2\,dx \qquad (2.34)
\end{aligned}$$

From (2.26), (2.27) and Poincare's inequality we have

$$\int_0^L f\alpha u\,dx \leq \varepsilon b_3\{E(t,u)+E(t,u_t)\} \tag{2.35}$$

where b_3 is a positive constant. From (2.25) and (2.27) we get

$$\int_0^L fqu_x\,dx \leq \varepsilon b_4\{E(t,u)+E(t,u_t)\} \tag{2.36}$$

where b_4 is a positive constant.

(*ii*) **Caso** $(F,U)=(f_t,u_t)$

$$\int_0^L f_t u_{tt}\,dx = \int_0^L \sigma''(u_x)u_{xt}u_{xx}u_{tt}\,dx + \int_0^L [\sigma'(u_x)-\sigma'(0)]u_{xxt}u_{tt}dx$$

Using (2.26) we get

$$\int_0^L f_t u_{tt}\,dx \leq \frac{Mc}{2}\varepsilon\int_0^L u_{xt}^2 + u_{tt}^2 dx + \underbrace{\int_0^L [\sigma'(u_x)-\sigma'(0)]u_{xxt}u_{tt}dx}_{:=I_1} \tag{2.37}$$

Performing an integration by parts and using (2.26) we have

$$I_1 \leq \frac{Mc\varepsilon}{2}\int_0^L |u_{tt}|^2 + |u_{xt}|^2\,dx + \frac{Mc\varepsilon}{2}\int_0^L |u_{xt}|^2\,dx - \frac{1}{2}\frac{d}{dt}\int_0^L [\sigma'(u_x)-\sigma'(0)]|u_{xt}|^2\,dx \tag{2.38}$$

Sustituting of (2.38) in (2.37), we have

$$\int_0^L f_t u_{tt}\,dx \leq Mc\varepsilon \max\left\{\frac{3}{2\sigma'(0)},1\right\}E(t,u_t) - \frac{1}{2}\frac{d}{dt}\int_0^L [\sigma'(u_x)-\sigma'(0)]|u_{xt}|^2\,dx \tag{2.39}$$

From (2.26) and Young's inequality we get

$$\int_0^L f_t\alpha u_t\,dx \leq \frac{Mc}{2}\varepsilon|\alpha|_\infty\int_0^L |u_{xt}|^2 + |u_t|^2\,dx + \underbrace{\int_0^L [\sigma'(u_x)-\sigma'(0)]u_{xxt}\alpha(x)u_t dx}_{:=I_2} \tag{2.40}$$

To estimate I_2 we make an integration by parts

$$
\begin{aligned}
I_2 &= -\int_0^L \{[\sigma'(u_x)-\sigma'(0)]\alpha(x)u_t\}_x\, u_{xt}dx \\
&= -\int_0^L \sigma''(u_x)\alpha(x)u_t u_{xt}dx - \int_0^L [\sigma'(u_x)-\sigma'(0)]\alpha'(x)u_t u_{xt}dx \\
&\quad -\int_0^L [\sigma'(u_x)-\sigma'(0)]\alpha(x)|u_{xt}|^2\, dx
\end{aligned}
$$

Using (2.26) and (2.25) once more

$$
\begin{aligned}
I_2 \le{}& \frac{Mc}{2}\varepsilon|\alpha|_\infty \int_0^L |u_{xt}|^2 + |u_t|^2\, dx + \frac{c_2}{2}\varepsilon|\alpha'|_\infty \int_0^L |u_{xt}|^2 + |u_t|^2\, dx + \\
& c_2\varepsilon|\alpha|_\infty \int_0^L |u_{xt}|^2\, dx
\end{aligned} \tag{2.41}
$$

Inserting (2.41) into (2.40) we conclude that

$$\int_0^L f_t\alpha u_t\, dx \le \varepsilon b_5\{E(t,u_t)+E(t,u)\} \tag{2.42}$$

where b_5 is a positive constant. Using (2.26), (2.25) and (2.28) we have that

$$\int_0^L f_t q u_{xt}\, dx \le b_6\varepsilon\{E(t,u_t)+E(t,u_{tt})+E(t,u)\} \tag{2.43}$$

where $b_6>0$.

(iii) **Caso** $(F,U)=(f_{tt},u_{tt})$

$$
\begin{aligned}
\int_0^L f_{tt}u_{ttt}\, dx ={}& \int_0^L \sigma'''(u_x)|u_{xt}|^2 u_{xx}u_{ttt}\, dx + \int_0^L \sigma'' u_{xtt}u_{xx}u_{ttt}\, dx + \\
& 2\int_0^L \sigma''(u_x)u_{xt}u_{xxt}u_{ttt}\, dx + \int_0^L [\sigma'(u_x)-\sigma'(0)]u_{xxtt}u_{ttt}\, dx
\end{aligned}
$$

Using (2.26), $|\sigma'''(u_x)| \le M_1$ and Young's inequality

$$
\begin{aligned}
&\le M_1 c\frac{\varepsilon^2}{2}\int_0^L |u_{xx}|^2 + |u_{ttt}|^2\, dx + M\, c\frac{\varepsilon}{2}\int_0^L |u_{xtt}|^2 + |u_{ttt}|^2\, dx + \\
&\quad Mc\varepsilon \int_0^L |u_{xxt}|^2 + |u_{ttt}|^2\, dx + \underbrace{\int_0^L [\sigma'(u_x)-\sigma'(0)]u_{xxtt}u_{ttt}\, dx}_{:=I_3}
\end{aligned} \tag{2.44}
$$

Integrating by parts

$$\begin{aligned}
I_3 &= -\int_0^L \{[\sigma'(u_x)-\sigma'(0)]u_{ttt}\}_x u_{xtt}\,dx \\
&= -\int_0^L \sigma''(u_x)u_{xx}u_{ttt}u_{xtt}\,dx - \int_0^L [\sigma'(u_x)-\sigma'(0)]u_{tttx}u_{xtt}\,dx \\
&= -\int_0^L \sigma''(u_x)u_{xx}u_{ttt}u_{xtt}\,dx - \frac{1}{2}\frac{d}{dt}\int_0^L [\sigma'(u_x)-\sigma'(0)]|u_{xtt}|^2 \\
&\quad +\frac{1}{2}\int_0^L \sigma''(u_x)u_{xt}|u_{xtt}|^2\,dx
\end{aligned}$$

Using (2.26) we get

$$I_3 \le Mc\frac{\varepsilon}{2}\int_0^L 2|u_{xtt}|^2+|u_{ttt}|^2\,dx - \frac{1}{2}\frac{d}{dt}\int_0^L [\sigma'(u_x)-\sigma'(0)]|u_{xtt}|^2\,dx \qquad (2.45)$$

Substitution of (2.45) into (2.44) yields

$$\begin{aligned}
\int_0^L f_{tt}u_{ttt} &\le \varepsilon \max\left\{M_1 c\frac{\varepsilon}{2}, Mc\frac{3}{2}\right\}\int_0^L |u_{xtt}|^2+|u_{xxt}|^2+|u_{xx}|^2+3|u_{ttt}|^2\,dx \\
&\quad -\frac{1}{2}\frac{d}{dt}\int_0^L [\sigma'(u_x)-\sigma'(0)]|u_{xtt}|^2\,dx
\end{aligned}$$

Using (2.27) e (2.28) we get

$$\begin{aligned}
\int_0^L f_{tt}u_{ttt} &\le \varepsilon b_7\{E(t,u_{tt})+E(t,u_t)+E(t,u)\} \\
&\quad -\frac{1}{2}\frac{d}{dt}\int_0^L [\sigma'(u_x)-\sigma'(0)]|u_{xtt}|^2\,dx
\end{aligned} \qquad (2.46)$$

where $b_7>0$. On the other hand, using (2.26), we have

$$\begin{aligned}
\int_0^L f_{tt}\alpha u_{tt} &\le \frac{M_1 c\varepsilon}{2}|\alpha|_\infty \int_0^L |u_{xx}|^2+|u_{tt}|^2\,dx + \frac{Mc\varepsilon}{2}|\alpha|_\infty \int_0^L |u_{xtt}|^2+|u_{tt}|^2\,dx \\
&\quad + Mc\varepsilon|\alpha|_\infty \int_0^L |u_{xxt}|^2+|u_{tt}|^2\,dx + \underbrace{\int_0^L \alpha(x)[\sigma'(u_x)-\sigma'(0)]u_{xxtt}u_{tt}}_{:=I_4}
\end{aligned} \qquad (2.47)$$

Integrating by parts

$$
\begin{aligned}
I_4 &= -\int_0^L \{\alpha(x)[\sigma'(u_x)-\sigma'(0)]u_{tt}\}_x u_{xtt}\,dx \\
&= -\int_0^L \sigma''(u_x)u_{xxx}u_{tt}\alpha(x)u_{xtt}\,dx - \int_0^L \alpha(x)[\sigma'(u_x)-\sigma'(0)]|u_{ttx}|^2\,dx \\
&\quad -\int_0^L \alpha'(x)[\sigma'(u_x)-\sigma'(0)]u_{tt}u_{xtt}\,dx
\end{aligned}
$$

Using (2.26) and (2.25) we get

$$
\begin{aligned}
I_4 \le\ & \frac{Mc\varepsilon}{2}|\alpha|_\infty \int_0^L |u_{xtt}|^2 + |u_{tt}|^2\,dx + c_2\varepsilon|\alpha|_\infty \int_0^L |u_{xtt}|^2\,dx + \\
& \frac{c_2\varepsilon}{2}|\alpha'|_\infty \int_0^L |u_{xtt}|^2 + |u_{tt}|^2\,dx
\end{aligned}
\tag{2.48}
$$

Subsititution of (2.48) into (2.47) yields

$$
\int_0^L f_{tt}\alpha u_{tt}\,dx \le 4\varepsilon c_4.\int_0^L |u_{xx}|^2 + |u_{tt}|^2 + |u_{xtt}|^2 + |u_{xxt}|^2\,dx
$$

Where

$$
c_4 := \max\left\{\frac{M_1 c}{2}|\alpha|_\infty, Mc|\alpha|_\infty, c_2|\alpha|_\infty, \frac{c_1}{2}|\alpha'|_\infty\right\}
$$

Using (2.27) e (2.28) we have

$$
\int_0^L f_{tt}\alpha u_{tt}\,dx \le \varepsilon b_8\{E(t,u_{tt}) + E(t,u_t) + E(t,u)\}
\tag{2.49}
$$

for some $b_8 > 0$.

Using (2.26) we get

$$
\begin{aligned}
\int_0^L f_{tt}q(x)u_{xtt}\,dx \le\ & \frac{M_1 c}{2}\varepsilon|q|_\infty \int_0^L |u_{xx}|^2 + |u_{xtt}|^2\,dx + \\
& Mc\varepsilon|q|_\infty \int_0^L |u_{xtt}|^2\,dx + \\
& Mc\varepsilon|q|_\infty \int_0^L |u_{xxt}|^2 + |u_{xtt}|^2\,dx + \\
& \underbrace{\int_0^L [\sigma'(u_x)-\sigma'(0)]q(x)\frac{1}{2}\frac{d}{dt}\{|u_{xtt}|^2\}\,dx}_{:=I_5}
\end{aligned}
\tag{2.50}
$$

Integrating by parts and using $q(0)=q(L)=0$ we get

$$
\begin{aligned}
I_5 &= -\frac{1}{2}\int_0^L \left[\sigma'(u_x)-\sigma'(0)\right]q(x)_x |u_{xtt}|^2\,dx \\
&= -\frac{1}{2}\int_0^L \sigma''(u_x)u_{xx}q(x)|u_{xtt}|^2\,dx - \frac{1}{2}\int_0^L \left[\sigma'(u_x)-\sigma'(0)\right]q'(x)|u_{xtt}|^2\,dx
\end{aligned}
$$

From (2.26) and (2.25) we get

$$
I_5 \leq \frac{Mc}{2}\varepsilon|q|_\infty \int_0^L |u_{xtt}|^2\,dx + \frac{c_2}{2}\varepsilon|q'|_\infty \int_0^L |u_{xtt}|^2\,dx \tag{2.51}
$$

Substitution of (2.51) into (2.50) produces

$$
\int_0^L f_{tt}q(x)u_{xtt}\,dx \leq \varepsilon c_5.\int_0^L \left\{|u_{xx}|^2 + 5|\dot{u}_{xtt}|^2 + |u_{xxt}|^2\right\}dx
$$

where

$$
c_5 := \max\left\{\frac{M_1c}{2}|q|_\infty, Mc|q|_\infty, \frac{c_2}{2}|q'|_\infty\right\}
$$

Using (2.27) and (2.28) we get

$$
\begin{aligned}
\int_0^L f_{tt}q(x)u_{xtt}\,dx \leq \varepsilon c_5.\max&\left\{2c_3, 2\hat{c}^2c_3, \frac{5}{\sigma'(0)}\right\} \\
&.\left\{E(t,u_{tt})+E(t,u_t)+E(t,u)\right\}
\end{aligned} \tag{2.52}
$$

Summing up (2.34), (2.35), (2.36), (2.39), (2.42), (2.43), (2.46), (2.49) and (2.52) we get

$$
\begin{aligned}
\sum_{i=0}^{2}\int_0^L F_i\left\{Nu_{i,t}+\alpha u_i+qu_{i,x}\right\}dx \leq \varepsilon\sum_{i=0}^{2}\gamma_i E(t,u_i) - \frac{N}{2}\frac{d}{dt}\int_0^L\left[\sigma'(u_x)-\sigma'(0)\right]|u_x|^2\,dx - \\
\frac{N}{2}\frac{d}{dt}\int_0^L\left[\sigma'(u_x)-\sigma'(0)\right]|u_{tx}|^2\,dx - \frac{N}{2}\frac{d}{dt}\int_0^L\left[\sigma'(u_x)-\sigma'(0)\right]|u_{ttx}|^2\,dx
\end{aligned} \tag{2.53}
$$

with the following notations

$$\begin{aligned} F_0 &= f \quad , u_0 = u \\ F_1 &= f_t \quad , u_1 = u_t \\ F_2 &= f_{tt} \quad , u_2 = u_{tt} \end{aligned}$$

where $\{\gamma_i\}_{i=0}^{2}$ are positive constants depending on $\varepsilon, c_p, \sigma'(0)$, of the L^∞ norm of the functions $\alpha, q, \sigma', q', \sigma''(u_x)$ and $\sigma'''(u_x)$.

Using Lemma 2.3 and (2.53) we have

$$\frac{dS(t)}{dt} \leq -\sum_{i=0}^{2} \left(\hat{\delta} - \varepsilon\gamma_i\right) E(t, u_i) \tag{2.54}$$

where $\left\{\hat{\delta} - \varepsilon\gamma_i\right\}_{i=0}^{2}$ is positive for ε small enough. From (2.54), (2.29), (2.30), (2.31) we have

$$\begin{aligned} \frac{dS(t)}{dt} &\leq -\frac{1}{k_4}\left\{\left(\hat{\delta} - \varepsilon\gamma_0\right) L_1(t) + \left(\hat{\delta} - \varepsilon\gamma_1\right) L_2(t) + \left(\hat{\delta} - \varepsilon\gamma_2\right) L_3(t)\right\} \\ &\leq -kS(t) \end{aligned}$$

where

$$k := \frac{1}{k_4} \min_{i=0,1,2} \left\{\hat{\delta} - \varepsilon\gamma_i\right\}$$

From where (2.32) follows. To show (2.33) we need the following estimates

$$\int_0^L |u_{ttt}|^2 \, dx \leq k_7 \int_0^L |u_{xxt}|^2 + |u_t|^2 + |u_{xx}|^2 \, dx \tag{2.55}$$

$$\int_0^L |u_{xxx}|^2 \, dx \leq k_8 \int_0^L |u_{xt}|^2 + |u_{xtt}|^2 + |u_{xx}|^2 + |u_t|^2 \, dx \tag{2.56}$$

where k_7 and k_8 positive constants. This inequalities follows differentiating relation (2.1) with respect to x and t, and using the same above reasoning. From the definition of S and relations (2.29), (2.30), (2.31) we get

$$S(t) \leq k_4 \left\{E(t,u) + E(t,u_t) + E(t,u_{tt})\right\}$$

Then, from (2.55) we can conclude that

$$S(t) \leq k_6 \mathcal{M}(t)$$

for some $k_6 > 0$. On the other hand, recalling the definition of S and keeping in mind relations (2.29), (2.30), (2.31) we have

$$S(t) \geq k_3 \left\{ E(t,u) + E(t,u_t) + E(t,u_{tt}) \right\}$$

Using (2.27), (2.28), (2.56) and Poincare's inequality we arrive at

$$S(t) \geq k_5 \mathcal{M}(t).$$

for $k_5 > 0$. Therefores (2.33) follows. ■

Integrating (2.32) from 0 to t and letting $t \to T^*$ we get

$$S(T^*) < S(0).$$

Applying (2.33) is follows that

$$k_5 \mathcal{M}(T^*) \leq S(T^*) < S(0) \leq k_6 \mathcal{M}(0),$$

that is

$$\mathcal{M}(T^*) \leq \frac{k_6}{k_5} \mathcal{M}(0) < \frac{k_6}{k_5} \varepsilon^2.$$

Taking $d = \dfrac{k_6}{k_5}$. There exists $T_1 > T^*$ such that $\mathcal{M}(t) < d\varepsilon^2$ for any t on $[T^*, T_1)$

which implies that there exists $T_1 > T^*$ such that $\mathcal{M}(t) < d\varepsilon^2$ for any t in the interval $[0, T_1)$, which is contradictory to the maximality of T^*. Therefore, we conclude that $T^* = \infty$ which implies that the solution is global in t. From (2.32) we get the exponential decay of S. Using (2.33) we get the exponential decay of $\mathcal{M}$. The proof is now complete.

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