A Transmission Problem with Nonlinear Damping and Source Terms

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Resumen

In this article we study the wave propagation over materials consisting of two components: one component is simple elastic while the other has a nonlinear internal damping with elastic coefficients dependent on time; both components having source terms. By using the potential well method we obtain the global existence, we also show that the energy of the system decays uniformly to zero.

Keywords: *stability, transmission problem, internal damping, source terms.*

Resumen

En este trabajo, estudiamos la propagación de ondas sobre materiales que consisten de dos componentes: un componente es simple elástico, mientras que el otro tiene un damping interno no lineal con coeficientes elásticos dependientes del tiempo; ambos componentes tienen términos fuente. Usando el método del potencial, obtenemos la existencia global, también mostramos que la energía del sistema decae uniformemente a cero.

Palabras Clave: *Estabilidad, problema de transmisión, damping interno, término fuente.*

1. Introduction

The main purpose of this work is to study the asymptotic behavior of solutions of the following nonlinear transmission problem

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$$
\begin{vmatrix}
\rho_1 u_{tt} - b u_{xx} = \mu_1 f_1(u) & \text{, in }]0, L_0[\times \mathbb{R}^+ & (1,1) \\
\rho_2 v_{tt} - a(t) v_{xx} + g(v_t) = \mu_2 f_2(v) & \text{, in }]L_0, L[\times \mathbb{R}^+ & (1,2) \\
u(x,0) = 0 = v(L,t) & , t > 0 & (1,3) \\
u(L_0, t) = v(L_0, t) & , t > 0 & (1,4) \\
b u_x(L_0, t) = a(t) v_x(L_0, t) & , t > 0 & (1,5) \\
u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x) & , x \epsilon]0, L_0[& (1,6) \\
v(x,0) = v^0(x), \quad v_t(x,0) = v^1(x) & , x \epsilon]L_0, L[& (1,7)
\end{vmatrix}
$$

where ρ_1, ρ_2 are different densities of the material, $\mu_i \in \mathbb{R}$, $i = 1, 2$, $b > 0$, g is a non-decreasing C^1 function, a is a elastic coefficient dependent on time and f_i is a function like $-|u|^{p_i-1} u$, $p_i \geq 1$, $i = 1, 2$.

Transmission problem or diffraction problems arise in several applications in physics and biology. The stability of conservative system by means of a internal damping has been studied by many authors (see $[1-7]$, among others). For the transmission problem there exists several works about controllability and stabilization by mens of feedback fuctions on a part of the boundary (see $[8], [11], [12]$).

When the coefficients dependent on time and $f_i(s) s \geq 0$, $\mu_i = -1$, Muñoz Rivera and Cabanillas L.[14] showed that solutions converge to zero exponentially. In our case we have $\mu_i \epsilon \mathbb{R}$, $|f_i(s)| \leq |s|^{p_i}$, $\forall s \epsilon \mathbb{R}$, with non linear damping g.

The first part of this paper is to study the global existence of regular and weak solution to problem (1,1)-(1,7), here we have there some technical difficulties that we need over come because of the coefficient dependent on time and the source term. Semigroup arguments are not suitable for finding solutions to (1,1)-(1,7); the method in [8] does not seen to be directly applicable to the function f_i, therefore, we make use of a Galerkin approximation and the potential well method.

The second part is to give energy decay estimates of the solution of (1,1)-(1,7) for a general non-linear damping g. We found that rate of decay of the solutions depend on behavior of the dissipative term in a neighborhood of zero, that is for a linear dissipation we obtain exponential decay while for a polynomial dissipation we obtain polynomial decay.

Our paper is organized as follows: In section 2, we present the notations and statement of results. In section 3, we prove solvability of (1,1)-(1,7), while section 4 deals with the asymptotic behavior of the solutions obtained in section 3.

2. Notations and statement of results

We denote
$$
(w, z) = \int_I w(x, z(x)) \, dx, \qquad |z|^2 = \int_I |z(x)|^2 \, dx
$$

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where $I =]0, L_0[$ or $]L_0, L[$ for $u's$ and $v's$ respectively. Assume that

(A1) We take $f_i \in C^1(\mathbb{R})$, $i = 1, 2$, $f_i(0) = 0$

$$|f_i'(s)| \leq |s|^{p_i - 1} \qquad 1 \leq p_1, p_2 < \infty$$

and without less of generality, we assume $p_1 \geq p_2$.

$$F_i(s) = \mu_i \int\limits_0^s f_i(\xi) \, d\xi$$

(A2) Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing C^1 function such that

$$g(s).s > 0 \qquad \text{for all } s \neq 0,$$

and there exist $c_i > 0$, $i = 1, 2, 3, 4$ such that

$$\begin{cases} c_1 |s|^p \leq |g(s)| \leq c_2 |s|^{1/p}, & \text{if } |s| \leq 1 \\ c_3 |s| \leq |g(s)| \leq c_4 |s|, & \text{if } |s| > 1 \end{cases}$$

where $p \geq 1$

(A3) $a \in W_{Loc}^{1,1}(0, \infty)$, $a(t) \geq a_0 > 0$, for some $a_0 > 0$
By V we denote the Hilbert Space

$$V = \left\{ (w, z) \in H^1(0, L_0) \times H^1(L_0, L) : w(0) = z(L) = 0; \ w(L_0) = z(L_0) \right\}$$

By E_1 and E_2 we denote the first order energy associated to each equation

$$E_1(t, u) = \tfrac{1}{2}\rho_1 |u_t(t)|^2 + \tfrac{b}{2} |u_x(t)|^2 - \int\limits_0^{L_0} F_1(u) \, dx$$

$$E_2(t, v) = \tfrac{1}{2}\rho_2 |v_t(t)|^2 + \tfrac{a(t)}{2} |v_x(t)|^2 - \int\limits_{L_0}^{L} F_2(v) \, dx$$

$$J_1(u) = \tfrac{1}{2}b |u_x|^2 - \tfrac{\mu_1}{p_1 + 1} |u|_{p_1+1}^{p_1+1}$$

$$J_2(v) = \tfrac{1}{2}a_0 |v_x|^2 - \tfrac{\mu_2}{p_2 + 1} |v|_{p_2+1}^{p_2+1}$$

$$I(u, v) = b |u_x|^2 + a_0 |v_x|^2 - \mu_1 |u|_{p_1+1}^{p_1+1} - \mu_2 |v|_{p_2+1}^{p_2+1}$$

$$E(t) \equiv E(t, u, v) = E_1(t, u) + E_2(t, v)$$

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and define the stable set as $W = \{(u,v) \in V : \ I(u,v) > 0 \} \cup \{0\}$

In order to show the decay property we will need the following lemma

Lemma 2.1 Let $E : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a nonincreasing function, and assume that there exist two constant $p > 0$ and $c > 0$ such that

$$\int_s^{+\infty} E^{\frac{p+1}{2}}(t)\, dt \leq cE(s) \qquad 0 \leq s < +\infty$$

The, we have

$$\begin{aligned} E(t) &\leq cE(0)(1+t)^{\frac{-2}{p-1}}, \quad \text{for all } t \geq 0 \text{ and } p > 1 \\ E(t) &\leq cE(0)e^{1-wt}, \quad \text{for all } t \geq 0 \text{ if } p = 1 \end{aligned}$$

where c and w are positive constants.

3. Existence and uniqueness of solutions

We begin this section defining what we mean by weak solution to the system (1,1)-(1,7).

Definition 3.1 We say that couple $\{u,v\}$ is a weak solution of (1,1)-(1,7) when

$$\{u,v\} \in L^\infty(0,T;V) \cap W^{1,\infty}(0,T,L^2(0,L_0) \times L^2(L_0,L))$$

and satisfies

$$-\rho_1 \int_0^{L_0} u^1(x)\varphi(x,0)\, dx - \rho_2 \int_{L_0}^{L} v^1(x)\psi(x,0)\, dx - \rho_1 \int_0^T \int_0^{L_0} u_t\varphi_t\, dxdt$$

$$-\rho_2 \int_0^T \int_{L_0}^{L} v_t\psi_t\, dxdt + b \int_0^T \int_0^{L_0} u_x\varphi_x\, dxdt + \int_0^T \int_0^{L_0} f_1(u)\varphi\, dxdt$$

$$+\int_0^T a(t) \int_{L_0}^{L} v_x\psi_x\, dxdt + \int_0^T \int_{L_0}^{L} f_2(v)\psi\, dxdt + \int_0^T \int_{L_0}^{L} g(v_t)\psi\, dxdt = 0$$

for any $\{\varphi, \psi\} \in C^2(0,T;V)$ such that,

$$\varphi(T) = \varphi_t(T) = \psi(T) = \psi_t(T) = 0.$$

In order to show the existence of strong solutions are need a regularity result for the elliptic

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system associated to the problem (1,1)-(1,7) whose proof can be obtained with little modifications in the book O.A. Ladyzhenkaya and N. N. Ural'tseva ([14],theorem 16.2)

Lemma 3.2 For any given functions $F \in L^2(0, L_0)$, $G \in L^2(L_0, L)$ there exists only solution $\{u, v\}$ of

$$
\left|
\begin{array}{l}
-b u_{xx} = F \quad \text{in} \quad]0, L_0[\\
-a(t) v_{xx} = G \quad \text{in} \quad]L_0, L[\\
u(0) = v(L) = 0 \\
u(L_0) = v(L_0), \quad b u_x(L_0) = a(t) v_x(L_0)
\end{array}
\right.
$$

with t a fixed value in $[0, T]$ satisfying $u \in H^2(0, L_0)$ and $v \in H^2(L_0, L)$

Now we are in a position to state the global existence results.

Theorem 3.3 Suppose that assumptions $(A1) - (A3)$ holds.

If $\{u^0, v^0\} \in W$, $\{u^1, v^1\} \in L^2(0, L_0) \times L^2(L_0, L)$ and

$$
\text{máx} \left\{ |\mu_1| b^{-1} c_x^{p_1+1} \left[\frac{2}{b} \frac{p_2+1}{p_2-1} E(0) \exp \int_0^T \frac{|a'(s)|}{a(s)} ds \right]^{\frac{p_1-1}{2}} , \right.
$$

$$
\left. |\mu_2| a_0^{-1} c_x^{p_2+1} \left[\frac{2}{a_0} \frac{p_2+1}{p_2-1} E(0) \exp \int_0^T \frac{|a'(s)|}{a(s)} ds \right]^{\frac{p_2-1}{2}} \right\} < 1 \tag{3.1}
$$

then there exists a unique local solution of (1,1)-(1,7) satisfying

$$
\{u, v\} \in C(0, T; V) \cap C^1(0, T; L^2(0, L_0) \times L^2(L_0, L))
$$

In addition, if $\{u^0, v^0\} \in W \cap (H^2(0, L_0) \times H^2(L_0, L))$, $\{u^1, v^1\} \in V$ and (3,1) holds, verifying the compatibility condition

$$
b u_x^0(L_0) = a(0) v_x^0(L_0) \tag{3.2}
$$

then there exists a strong solution $\{u, v\}$ satisfying

$$
\begin{aligned}
\{u, v\} \in \ & C(0, T; H^2(0, L_0) \times H^2(L_0, L)) \cap C^1(0, T; V) \cap \\
& C^2(0, T; L^2(0, L_0) \times L^2(L_0, L))
\end{aligned}
$$

Proof We employ the Galerkin Method to construct a solution.

Let $\{\{\varphi^i, \psi^i\}, \quad i = 1, 2, ...\}$ be a basis to V. We construct approximate solution

$$
\{u^m(t), v^m(t)\} = \sum_{i=1}^{m} h_{im}(t) \{\varphi^i, \psi^i\}
$$

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which is determinate by the ordinary differential equations

$$\rho_1\left(u_{tt}^m, \varphi^i\right) + b\left(u_x^m, \varphi_x^i\right) + \left(f_1\left(u^m\right), \varphi^i\right) + \rho_2\left(v_{tt}^m, \psi^i\right) +$$
$$a\left(t\right)\left(v_x^m, \psi_x^i\right) + \left(g\left(v_t^m\right), \psi^i\right) + \left(f_2\left(v^m\right), \psi^i\right) = 0 \tag{3.3}$$

where $i = 1, 2, 3...$
with the initial conditions

$$\left\{u^m\left(0\right), v^m\left(0\right)\right\} = \left\{u^0, v^0\right\}, \quad \left\{u_t^m\left(0\right), v_t^m\left(0\right)\right\} = \left\{u^1, v^1\right\} \tag{3.4}$$

By standard methods in differential equations we prove the existence of solutions to (3,3)-(3,4) on some interval $[0, T_m[, \quad 0 < T_m \leq \infty$. In order to extend the solution of (3,3)-(3,4) to the whole interval $[0, \infty[$ we need the priori estimate below.

Weak solutions. Multiplying (3.3) by $h'_{im}(t)$, integrating by parts and summing up on i we get

$$\frac{d}{dt}E\left(t, u^m, v^m\right) + \left(g\left(v_t^m\right), v_t^m\right) \leq \frac{|a'\left(t\right)|}{a\left(t\right)}E\left(t, u^m, v^m\right) \tag{3.5}$$

Integrating (3,5) over $]0, t[$, it hold that

$$E\left(t, u^m, v^m\right) + \int_0^t \left(g\left(v_t^m\left(s\right)\right), v_t^m\left(s\right)\right) ds$$

$$\leq E\left(0, u^0, v^0\right) + \int_0^t \frac{|a'\left(s\right)|}{a\left(s\right)}E\left(s, u^m\left(s\right), v^m\left(s\right)\right)$$

Employing gronwall´s lemma, from the last inequality we see that

$$E\left(t, u^m\left(t\right), v^m\left(t\right)\right) + \int_0^t \left(g\left(v_t^m\left(s\right)\right), v_t^m\left(s\right)\right) ds \tag{3.6}$$

$$\leq E\left(0, u^0, v^0\right)\exp\left(\int_0^t \frac{|a'\left(s\right)|}{a\left(s\right)}ds\right), 0 \leq t \leq T$$

Now to obtain a priori estimates, we need the following result

Lemma 3.4 Let $\left\{u^m\left(t\right), v^m\left(t\right)\right\}$ be the solution of $(3,3) - (3,4)$ with$\left\{u^0, v^0\right\} \in W \cap V$ and $\left\{u^1, v^1\right\} \in L^2\left(0, L_0\right) \times L^2\left(L_0, L\right)$.

If

$$\alpha = \text{máx}\left\{ |u_1| \, b^{-1} c_x^{p_1+1} \left[\frac{2}{b} \frac{P_2+1}{P_2-1} E(0) \, e \times y \int_0^T \frac{|a^1(s)|}{a(s)} ds \right]^{\frac{P_1-1}{2}}, \right.$$

$$\left. |\mu_2| \, a_0^{-1} c_x^{P_2+1} \left[\frac{2}{a_0} \frac{P_2+1}{P_2-1} E(0) \exp \int_0^T \frac{|a^1(s)|}{a(s)} ds \right]^{\frac{P_2-1}{2}} \right\} < 1$$

Then $\{u(t), v(t)\} \in W \cap V$ on $[0,T]$, that is, for all $t \in [0,T]$

$$I(u^m(t), v^m(t)) > 0$$

Proof. Since $I(u^0, v^0) > 0$ it follows from the continuity of $\{u^m(t), v^m(t)\}$ that

$$I(u^m(t), v^m(t)) \geq 0 \qquad (3.7)$$

for some interval near to $t = 0$

Let $t_{\text{máx}} > 0$ be a maximal time (possibly $t_{\text{máx}} = T_m$) such that $(3,7)$ holds on $[0, t_{\text{máx}}[$. In order to facilitate the notation, we will omit the index m of the solution of the approximate system. Note that

$$
\begin{aligned}
J(u(t), v(t)) &\geq \frac{b}{2}|u_x|^2 + \frac{a_0}{2}|v_x|^2 - \frac{\mu_1}{p_1+1}|u|_{p_1+1}^{p_1+1} - \frac{\mu_2}{p_2+1}|v|_{p_2+1}^{p_2+1} \\
&= \frac{1}{p_2+1} I(u(t); v(t)) + \frac{b(p_2-1)}{2(p_2+1)}|u_x|^2 + \frac{a_0(p_2-1)}{2(p_2+1)}|v_x|^2 \\
&\quad + \frac{\mu_1(p_1-p_2)}{(p_1+1)(p_2+1)}|u|_{p_1+1}^{p_1+1} \\
&\geq \frac{p_2-1}{2(p_2+1)}\left\{ b|u_x|^2 + a_0|v_x|^2 \right\}, \ \forall t \in [0, t_{\text{máx}}[
\end{aligned}
$$

Consequently, we get

$$
\begin{aligned}
b|u_x|^2 + a_0|v_x|^2 &\leq \frac{2(p_2+1)}{p_2-1} J(u(t), v(t)) \qquad (3.8) \\
&\leq \frac{2(p_2+1)}{p_2-1} E(t, u, v) \\
&\leq \frac{2(p_2+1)}{p_2-1}\left[E(0, u^0, v^0)\exp\int_0^T \frac{|a^1(s)|}{a(s)}ds \right] \text{ in } [0, t\,\text{máx}[
\end{aligned}
$$

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It follows from the Sobolev-Poincare inequality and (3,8) that

$$\mu \, |v|_{p_1+1}^{p_1+1} \leq |\mu_1| \, c_x^{p_1+1} \, |u_x|^{p_1+1} = \frac{|\mu_1|}{b} c_x^{p_1+1} |u_x|^{p_1-1} \left(b \, |u_x|^2 \right) \tag{3.9}$$

$$\leq \frac{|\mu_1|}{b} c_x^{p_1+1} \left[\frac{2}{b} \frac{p_2+1}{(p_2-1)} E(0) \exp \int_0^T \frac{|a^1(s)|}{a(s)} ds \right]^{\frac{p_1-1}{2}} b \, |u_x|^2$$

Similarly

$$\mu_2 \, |v|_{p_2+1}^{p_2+1} \leq \frac{|\mu_2|}{a_0} c_x^{p_2+1} \left[\frac{2}{a_o} \frac{p_2+1}{p_2-1} E(0) \exp \int_0^T \frac{|a^1(s)|}{a(s)} ds \right]^{\frac{p_2-1}{2}} a_0 \, |v_x|^2 \tag{3.10}$$

$$< a_0 \, |v_x|^2$$

this from (3,9) and (3,10) we obtain

$$\mu_1 \, |u|_{p_1+1}^{p_1+1} + \mu_2 + |v|_{p_2+1}^{p_2+1} < b \, |u_x|^2 + a_0 \, |v_x|^2 \tag{3.11}$$

Therefore we get $I\left(u\left(t\right); v\left(t\right)\right) > 0$ on $[0, t\,\text{máx}[$. This implies that we can take $t_{\text{máx}} = T_m$. This completes the proof of lemma 3.4.$\square$

Using lemma 3.4., we can deduce a priori estimate for $\{u(t), v(t)\}$. Lemma 3.4 implies that

$$E\left(t, u\left(t\right), v\left(t\right)\right) = \frac{1}{2} |u_t(t)|^2 + \frac{1}{2} |v_t(t)|^2 + J\left(u(t), v(t)\right) \tag{3.12}$$

$$\geq \frac{1}{2} |u_t(t)|^2 + \frac{1}{2} |v_t(t)|^2 + \frac{1}{p_2+1} I\left(u(t), v(t)\right) +$$

$$+ \frac{b(p_2-1)}{2(p_2+1)} |u_x|^2 + \frac{a_0(p_2-1)}{2(p_2+1)} |v_x|^2 + \frac{\mu_1(p_1-p_2)}{(p_1+1)(p_2+1)} |u|_{p_1+1}^{p_1+1}$$

$$\geq \frac{1}{2} |u_t(t)|^2 + \frac{1}{2} |v_t(t)|^2 + \frac{p_2-1}{2(p_2+1)} \left(b \, |u_x|^2 + a_0 \, |v_x|^2 \right)$$

From (3,6) and (3,12), we get

$$\frac{1}{2} |u_t(t)|^2 + \frac{1}{2} |v_t(t)|^2 + \frac{p_2-1}{2(p_2+1)} \left(b \, |u_x|^2 + a_0 \, |v_x|^2 \right) + \int_0^t \left(g\left(v_t(s) \right), v_t(s) \right) ds$$

$$\leq E\left(0, u^0, v^0\right) \exp \left(\int_0^T \frac{|a'(s)|}{a(s)} ds \right) \leq L_1,$$

where L_1 is a positive constant independent of $m \in \mathbb{N}$ and $t \in [0, T]$.
Thus, we deduce that

$$\{u^m, v^m\} \text{ is bounded in } L^\infty(0, T; V)$$

$$\{u_t^m, v_t^m\} \text{ is bounded in } L^\infty\left(0, T; L^2\left(0, L_0\right) \times L^2\left(L_0, L\right)\right)$$

wich imply that

$$\{u^m, v^m\} \rightarrow \{u, v\} \text{ weakly * in } L^\infty\left(0, T; V\right)$$
$$\{u_t^m, v_t^m\} \rightarrow \{u_t, v_t\} \text{ weakly * in } L^\infty\left(0, T; L^2\left(0, L_0\right) \times L^2\left(L_0, L\right)\right)$$

Using Aubin-Lions compactness lemma we have

$$\{u^m, v^m\} \rightarrow \{u, v\} \text{ strongly in } L^2\left(0, T; L^2\left(0, L_0\right) \times L^2\left(L_0, L\right)\right)$$

and consequently

$$u^m \rightarrow u \text{ a.e in }]0, L_0] \text{ and } f_1\left(u^m\right) \rightarrow f_1\left(u\right) \text{ a.e. in }]0, L_0]$$
$$v^m \rightarrow v \text{ a.e in }]L_0, L] \text{ and } f_2\left(v^m\right) \rightarrow f_2\left(v\right) \text{ a.e. in }]L_0, L]$$

Besides, from the growth condition in $(A1)$ we have that

$$f_1\left(u^m\right) \text{ is bounded in } L^2\left(0, T; L^2\left(0, L_0\right)\right)$$
$$f_2\left(v^m\right) \text{ is bounded in } L^2\left(0, T; L^2\left(L_0, L\right)\right)$$

and therefore

$$\{f_1\left(u^m\right), f_2\left(v^m\right)\} \rightarrow \{f_1\left(u\right), f_2\left(v\right)\} \text{ in } L^2\left(0, T; L^2\left(0, L_0\right) \times L^2\left(L_0, L\right)\right)$$

Now, we note that from (3,13) and the assumption $(A2)$, we get

$$\int_0^t \left|g\left(v_t^m\left(s\right)\right)\right|^2 ds \leq L$$

where L is a positive constant independent of m and t.
So, we can take a subsequence, still denote by (v^m) such that

$$g\left(v_t^m\right) \rightarrow \chi \text{ weakly in } L^2\left(]L_0, L[\times]0, T[\right)$$

Returning to (3,8) and using standard argument we can show, from the convergence above that

$$\rho_1 u_{tt} - b u_{xx} = \mu_1 f_1\left(u\right) \text{ in } L^2\left(0, T; H^{-1}\left(0, L_0\right)\right)$$
$$\rho_2 v_{tt} - a\left(t\right) v_{xx} + \chi = \mu_2 f_2\left(v\right) \text{ in } L^2\left(0, T; H^{-1}\left(L_0, L\right)\right)$$

Our goal is to prove that

$$\chi = g\left(v_t\right)$$

but this relation follows from a standard theory of monotone and semicontinuous operators ($cf.$ [09]) and the proof is omitted.

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Therefore, $\{u, v\}$ satisfies $(1,1)-(1,5)$.

Regularity of solutions: To get the regularity result, we take a basis

$$B = \left\{\left\{\varphi^i, \psi^i\right\}, \ i \in \mathbb{N}\right\}$$

such that

$$\left\{u^0, v^0\right\}, \left\{u^1, v^1\right\} \in \text{span}\left\{\left\{\varphi^0, \psi^0\right\}, \left\{\varphi^1, \psi^1\right\}\right\}$$

Let us differenciate the approximate equation and multiply by $h'_{im}(t)$. Using a similar argument as before, we obtain

$$\frac{d}{dt}E_2(t, u, v) + \int_{L_0}^{L} g'(v_t^m(x,t))(v_{tt}^m(x,t))^2 dx$$

$$= \mu_1(f_1'(u_m)u_t^m, u_{tt}^m) + \mu_2(f_2'(v_m)v_t^m, v_{tt}^m)$$

$$-a_t(t)(v_x^m, v_{xtt}^m) + \frac{1}{2}a_t(t)|v_{tt}|^2$$

where

$$E_2(t, u, v) = \frac{\rho_1}{2}|u_{tt}|^2 + \frac{b}{2}|u_{xt}|^2 + \frac{\rho_2}{2}|v_{tt}|^2 + \frac{a(t)}{2}|v_{xt}|^2$$

Note that

$$-a_t(v_x^m, v_{xtt}^m) = -(a_t(v_x^m, v_{xt}^m))_t + a_{tt}(v_x^m, v_{xt}^m) + a_t|v_{xt}^m|^2 \qquad (3.15)$$

$E_2(0, u^m, v^m)$ in bounded, because on choice of the basis.

Now, from the growth condition $(A1)$ and the Sobolev imbedding we have

$$\int_0^{L_0} f_1'(u^m)u_t^m u_{tt}^m dx \leq c\left[\int_0^{L_0}|u_x^m|^2 dx\right]^{\frac{p_1-1}{2}}|u_{xt}^m||u_{tt}^m| \qquad (3.16)$$

and similarly

$$\int_{L_0}^{L} f_2'(v^m)v_t^m v_{tt}^m dx \leq c\left[\int_{L_0}^{L}|v_x^m|^2 dx\right]^{\frac{p_2-1}{2}}|v_{xt}^m||v_{tt}^m| \qquad (3.17)$$

Taking into account the first estimate $(3,13)$, $(3,15)$, $(3,17)$, from $(3,14)$ and the Gronwall inequality we conclude that

$$E_2(t, u^m, v^m) \leq c \qquad (3.18)$$

which imply that

$$\{u_t^m, v_t^m\} \ \rightharpoonup \ \{u_t, v_t\} \ \text{weakly} * \text{ in } L^\infty\left(0, T; H^1(0, L_0) \times H^1(L_0, L)\right)$$

$$\{u_{tt}^m, v_{tt}^m\} \ \rightharpoonup \ \{u_t, v_t\} \ \text{weakly} * \text{ in } L^\infty\left(0, T; L^2(0, L_0) \times L^2(L_0, L)\right)$$

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Therefore we have $\{u, v\}$ satisfies (1,1)-(1,5) and we have

$$
\left|
\begin{array}{l}
-bu_{xx} = -\rho_1 u_{tt} + \mu_1 f_1(u) \in L^2(0, L_0) \\
-a(t) v_{xx} = -\rho_2 v_{tt} - g(v_t) + \mu_2 f_2(v) \in L^2(L_0, L) \\
u(L_0, t) = v(L_0, t), \quad bu_x(L_0, t) = a(t) v_x(L_0, t) \\
u(0, t) = 0 = v(L, t)
\end{array}
\right.
$$

then using lemma 3.2 we have the required regularity to $\{u, v\}$. $\square$

4. Exponential Decay

In this section we study the asymptotic behavior of the solution of system (1,1)-(1,7). In the remainder of this paper we denote by c a positive constant which takes we shall suppose that

$$\rho_1 \le \rho_2 \quad \text{and} \tag{4.1}$$

$$a(t) \le b, \quad a_t(t) \le 0, \quad \forall t \in]0, \infty[\tag{4.2}$$

Theorem 4.1

Let $\{u, v\}$ be the weak solution obtained in theorem 3.3.
Suppose that (4,1)-(4,2) and $(A2)$ hold with $p = 1$.
If in addition, the initial data satisfy

$$v_x^0(L_0) = 0 \tag{4.3}$$

then there exists positive constants γ and c such that

$$E(t) \le cE(0) e^{-\gamma t} \quad \forall t \ge 0. \tag{4.4}$$

We shall prove this theorem for strong solutions; our conclusion follow by standard density arguments.
The dissipative property of system (1,1)-(1,7) is given by the following lemma.

Lemma 4.2

The first order energy satisfies

$$\frac{d}{dt} E_1(t, u, v) = -(g(v_t), v_t) + a_t |v_x|^2 \tag{4.5}$$

Proof

Multiplying equation (1,1) by u_t, equation (1,2) by v_t and performing an integration by parts we get the result. $\square$

Let $\psi \in C_0^\infty(0, L)$ be such that $\psi = 1$ in $]L_0 - \delta, L_0 + \delta[$ for some $\delta > 0$, small constant. Let us introduce the following functional

$$I(t) = \int_0^{L_0} \rho_1 u_t q u_x \, dx + \int_{L_0}^L \rho_2 v_t \psi q v_x \, dx$$

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where $q(x) = x$.

Lemma 4.2   There exists c_1 such that

$$\frac{d}{dt} I(t) \leq -\frac{L_0}{2} \left\{ (\rho_2 - \rho_1) v_t^2(L_0, t) + a(t) \left[1 - \frac{a(t)}{b} \right] \right\}$$

$$-L_0 \left[F_2(v(L_0, t)) - F_1(u(L_0, t)) \right] - \frac{1}{2} \int_0^{L_0} \left(\rho_1 v_t^2 + b v_x^2 - 2\mu_1 F_1(u) \right) dx$$

$$-\frac{a(t)}{4} \int_{L_0}^{L} v_x^2 dx + c_1 \left\{ \int_{L_0+\delta}^{L} (v_t^2 + v_x^2) dx + \int_{L_0}^{L} g(v_t)^2 dx + |v|_{p_2+1}^{p_2+1} \right\}$$

Proof

Multiplying equation (1,1) by $q u_x$, equation (1,2) by $\psi q v_x$, integrating by parts and using the corresponding boundary conditions, we have

$$\frac{d}{dt} (\rho_1 u_t, q\psi u_x) = \frac{L_0}{2} \left[\rho_1 u_t^2(L_0, t) + b u_x^2(L_0, t) \right] + L_0 F_1(u(L_1, t)) \qquad (4.6)$$

$$-\frac{1}{2} \int_0^{L_0} \left[\rho_1 u_t^2 + b u_x^2 - 2F_1(u) \right] dx$$

$$\frac{d}{dx} (\rho_2 v_t, q\psi v_x) \leq -\frac{L_0}{2} \left[\rho_2 v_t^2(L_0, t) + a(t) v_x^2(L_0, t) - L_0 F_2(v(L_0, t)) \right] \qquad (4.7)$$

$$-\frac{a(t)}{4} \int_{L_0}^{L_0+\delta} v_x^2 dx + c_1 \left[\int_{L_0+\delta}^{L} (v_t^2 + v_x^2) dx + \int_{L_0}^{L} \left(g(v_t)^2 + |F_2(v)|_{\mathbb{R}} \right) dx \right]$$

Summing up (4,5) with (4,7) our conclusion follows.$\square$

Let $\varphi \in C^\infty(\mathbb{R})$ a nonnegative function such that $\varphi = 0$ in $I_{\delta/2} = \left] L_0 - \frac{\delta}{2}, L_0 + \frac{\delta}{2} \right[$ and $\varphi = 1$ in $\mathbb{R} \backslash I_\delta$ and consider the functional

$$J(t) = \int_{L_0}^{L} \rho_2 v_t \varphi v \, dx$$

We have the following lemma

Lemma 4.4

Given $\varepsilon > 0$, there exists a positive constant c_ε such that

$$\frac{d}{dt} J(t) \leq -\frac{a(t)}{2} \int_{L_0+\delta}^{L} v_x^2 dx + \varepsilon a(t) \int_{L}^{L_0+\delta} v_x^2 dx + c_\varepsilon \int_{L_0}^{L} \left(v_t^2 + g(v_t)^2 \right) dx + |v|_{p_2+1}^{p_2+1}$$

Proof

Multiplying equation (1,2) by φv and integrating by parts we get

$$\frac{d}{dt} J(t) = -a(t)(v_x, \varphi v_x) - a(t)(v_x, \varphi_x v) + \rho_2(v_t, \varphi v_t) - (g(v_t), \varphi v)$$
$$+\mu_2(f_2(v), \varphi v)$$

Applying Young's inequality and hypothesis $(A,1)$ we conclude our assertion.$\square$
Let us consider the following functional

$$K(t) = I(t) + (2c_1 + 1) J(t)$$

Using lemma 4.3 and fixing $\varepsilon = \varepsilon_1$ in lemma 4.4, where ε_1 is the solution of the equation

$$(2c_1 + 1)\varepsilon_1 = \frac{1}{8}$$

we have that there exists a positive constant c_2 such that

$$\frac{d}{dt} K(t) \leq -E_1(t, u) - \frac{1}{8} \int_{L_0}^{L} a(t) v_x^2 dx + \varepsilon E(t, u, v) + \tag{4.8}$$
$$c_2 \int_{L_0}^{L} \left(v_t^2 + g(v_t)^2 + (v)^{p_2+1}\right) dx$$

To arrive at the desired inequality on $E(t)$ we must estimate the last term in (4,8), we need the following result.

Lemma 4.5

Let $\{u, v\}$ be a solution in theorem 3.3.
Then there exist $T_0 > 0$ such that if $T \geq T_0$ we have

$$\int_s^T |v|^{p_2+1} dx \leq \varepsilon \left[\int_s^T \left(b|u_x|^2 + |u_t|^2\right) ds + \int_s^T \left|a^{1/2} v_x\right|^2 ds \right] + \tag{4.9}$$
$$+c_\varepsilon \int_s^T |v_t|^2 ds$$

for any $\varepsilon > 0$ and c_ε is a constant depending on T and ε, for any initial data $\{u^0, v^0\}$, $\{u^1, v^1\}$ satisfying $E(0, u, v) \leq k$ where $k > 0$ is fixed and $0 < S < T < \infty$.
Proof.
The proof is given quite similarly as in the proof of (4,11) in [8] and we will omit it.$\square$

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Proof of theorem 4.1.
From hypothesis on the function g, with $p = 1$, we get

$$\int_{L_0}^{L} \left(v_t^2 + g(v_t)^2 \right) dx \leq \left(c_1^{-1} + c_2 \right) \int_{L_0}^{L} g(v_t) v_t dx \qquad (4.10)$$

Let us denote

$$L(t) = NE(t) + k(t)$$

with $N > 0$. Using Young's Inequality and taking N large enough we find that

$$\theta_0 E(t) \leq L(t) \leq \theta_1 E(t) \qquad (4.11)$$

for some positive constants θ_0 and θ_1.
Substituting $(4,9), (4,10), (4,11)$ in $(4,8)$, using the identity of lemma 4.2 and integrating from S to T, where $0 \leq S < T < \infty$ we obtain

$$\int_{S}^{T} E(t) \, dt \leq cE(s)$$

This inequality, by lemma 2.1, implies that

$$E(t) \leq cE(0) e^{-\gamma t}$$

The proof of theorem 4.1 is completed. $\square$

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