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EL PROBLEMA DE TRANSMISIÓN CON COEFICIENTES DEPENDIENTES DEL TIEMPO CON AMORTIGUAMIENTO INTERNO NO LINEAL

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Resumen.- *En este artículo consideramos el problema no lineal de transmisión para una ecuación de onda con coeficientes dependientes del tiempo y amortiguamiento interno no lineal. Se prueba la existencia global y se estudia propiedades de decaimiento de las soluciones. El resultado se alcanza usando técnicas de multiplicadores y el teorema continuación única conveniente para una ecuación de onda.*

Palabras claves: *Problema de transmisión, coeficientes dependientes del tiempo, estabilidad.*

THE TRANSMISSION PROBLEM WITH TIME DEPENDENT COEFFICIENTS WITH NONLINEAR INTERNAL DAMPING

Abstract.- *In this paper we consider the nonlinear transmission problem for the wave equation with time dependent coefficients and nonlinear internal damping. We prove global existence and study decay properties of the solutions. The result is achieved by using the multiplier technique and suitable unique continuation theorem for the wave equation.*

Key words: *Transmission Problem, time dependent coefficients, stability.*

1. Introduction

In this work, we consider the transmission problem

$$\rho_1 u_{tt} - (b(x,t)u_x)_x + f_1(u) = 0 \text{ in }]0, L_0[\times \mathbb{R}^+ \tag{1.1}$$

$$\rho_2 v_{tt} - (a(x,t)v_x)_x + g(v_t) + f_2(v) = 0 \text{ in }]L_0, L[\times \mathbb{R}^+ \tag{1.2}$$

$$u(0,t) = v(L,t) = 0, \ t > 0 \tag{1.3}$$

$$u(L_0,t) = v(L_0,t), \ b(L_0,t)u_x(L_0,t) = a(L_0,t)v_x(L_0,t), \ t > 0 \tag{1.4}$$

$$u(x,0) = u^0(x), \ u_t(x,o) = u^1(x), \ x \in]0, L_0[\tag{1.5}$$

$$v(x,0) = v^0(x), \ v_t(x,o) = v^1(x), \ x \in]L_0, L[\tag{1.6}$$

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where ρ_1, ρ_2, are different constants, f_1, f_2, g are nonlinear functions and $a(x,t)$, $b(x,t)$ are positive functions. Controllability and Stability for transmission problem has been studied by many authors (see for example J. L. Lions [9], J. Lagnese [7], W. Liu and G. Williams [10], J. Muñoz Rivera and H. Portillo Oquendo [11], D. Andrade, L. H. Fatori and J. Muñoz Rivera [1]). E. Cabanillas L and J. Muñoz Rivera [2] considered the problem (1.1) - (1.6) with $b(x,t) = b > 0$ and $g(s) = as$.

The goal of this work is to study the existence and uniqueness of global solutions of (1.1) - (1.6) and the asymptotic behavior of the energy.

In general, the dependence on spatial and time variables of the coefficients causes difficulties, semigroups arguments are not suitable for finding solutions to (1.1)-(1.6); therefore, we make use of a Galerkin's process. Note that the timedependent coefficients also appear in the second boundary condition, thus there are some technical difficulties that we need to overcome. To prove the decay rates, the main difficulty is that the dissipation only works in $[L_0, L]$ and we need to estimate over the whole domain $[0, L]$; we overcome this problem introducing suitable multipliers and a compactness/uniqueness argument.

2. Notations and Preliminaries

We denote

$$(w,z) = \int_I w(x)z(x)dx\,,\quad |z|^2 = \int_I |z(x)|^2 dx$$

where $I =]0, L_0[$ or $]L_0, L[$ for u's and v's respectively. Now, we state the general hypotheses.

(A.1) The function $f_i \in C^1(\mathbb{R})$, $i = 1, 2$, satisfy

$$f_i(s)s \geq 0,\ \forall s \in \mathbb{R}$$
$$\left|f_i^{(j)}(s)\right| \leq c(1+|s|)^{\rho-j},\ \forall s \in \mathbb{R},\ j = 0, 1$$

for some $c > 0$ and $\rho \geq 1$.

$$f_1(s) \geq f_2(s)$$
$$F_i(s) = \int_0^s f_i(\xi)d\xi,\ i = 1, 2$$

(A.2) Assumptions on the coefficient a

$$b,\ a \in W^{1,\infty}(o,\infty;C^1(I)) \cap W^{2,\infty}(0,\infty;L^\infty(I))$$

$$b_t,\ a_t \in L^1(0,\infty;L^\infty(I))$$

$$b(x,t) \geq bo > 0,\ a(x,t) \geq a_0 > 0,\ \forall (x,t) \in I \times]0,\infty[$$

(A.3) Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing C^1 function such that

$$g(s).s > 0, \text{ for all } s \neq 0$$

and there exist $c_i > 0$, $i = 1,2,3,4$ such that

$$\begin{cases} c_3 |s|^p \leq |g(s)| \leq c_4 |s|^{1/p} & , \text{ if } |s| \leq 1 \\ \\ c_1 |s| \leq |g(s)| \leq c_2 |s| & , \text{ if } |s| > 1 \end{cases}$$

where $p \geq 1$

By V we denote the Hilbert space

$$V = \{(w,z) \in H^1(0,L_0) \times H^1(L_0,L) : w(0) = z(L) = 0;\ w(L_0) = z(L_0)\}$$

By E_1 and E_2 we denote the first order energy associated to each equation,

$$\begin{aligned} E_1(t,u) &= \frac{1}{2}\left\{\rho_1 |u_t|^2 + (b,u_x^2) + 2\int_0^{L_0} F_1(u)dx\right\} \\ E_2(t,v) &= \frac{1}{2}\left\{\rho_2 |v_t|^2 + (a,v_x^2) + 2\int_{L_0}^{L} F_2(v)dx\right\} \\ E(t) &= E_1(t,u,v) = E_1(t,u) + E_2(t,v). \end{aligned}$$

We conclude this section with the following lemma which will play essential role when establishing the asymptotic behavior.

Lemma 2.1 *Let $E : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a non-increasing function and assume that there exists two constants $p > 0$ and $c > 0$ such that*

$$\int_s^{+\infty} E^{\frac{p+1}{2}}(t)dt \leq cE(s)\ ,\ \ 0 \leq s < +\infty$$

then we have

$$
\begin{aligned}
E(t) &\leq cE(0)(1+t)^{-\frac{2}{p-1}}, \ \textit{for all } t \geq 0 \textit{ if } p > 1 \\
E(t) &\leq cE(0)e^{1-wt}, \ \textit{for all } t \geq 0 \textit{ if } p = 1
\end{aligned}
$$

where c and w are positive constants.

Proof. See reference [[4] Lema 9.1].

3. Existence and Uniqueness of solutions

First of all, we define what we will understand of weak solutions of problem (1.1) - (1.6).

Definición 3.1 *We say that the couple $\{u, v\}$ is a weak solution of (1.1) - (1.6) when*

$$
\{u, v\} \in L^\infty(0,T;V) \cap W^{1,\infty}(0,T;L^2(0,L_0) \times L^2(L_0,L))
$$

and satisfies

$$
\begin{aligned}
&-\rho_1 \int_0^{L_0} u^1(x)\varphi(x,0)dx + \rho_1 \int_0^{L_0} u^0(x)\varphi_t(x,0)dx - \rho_2 \int_{L_0}^{L} v^1(x)\psi(x,0)dx \\
&+\rho_2 \int_{L_0}^{L} v^0(x)\psi_t(x,0)dx + \rho_1 \int_0^T \int_0^{L_0} (u\varphi_{tt} + b(x,t)u_x\varphi_x + f_1(u)\varphi)dxdt \\
&+\rho_2 \int_0^T \int_{L_0}^{L} (v\psi_{tt} + a(x,t)v_x\psi_x + g(v_t)\psi + f_2(v)\psi)dxdt = 0
\end{aligned}
$$

for any $\{\varphi, \psi\} \in C^2(0,T;V)$ such that $\varphi(T) = \varphi_t(T) = 0 = \psi(T) = \psi_t(T)$

In order to show the existence of strong solutions we need a regularity result for the elliptic system associated to the problem (1.1) - (1.6) whose proof can be obtained, with little modifications, in the book by O.A. Ladyzhenskaya and N. N. Ural'tseva; ([5], theorem 16.2).

Lema 3.2 *For any given functions* $F \in L^2(0, L_0)$, $G \in L^2(L_0, L)$, *there exists only one solution* $\{u, v\}$ *of*

$$\begin{aligned} -(b(x,t)v_x)_x &= F \text{ in }]0, L_0[\\ -(a(x,t)v_x)_x &= G \text{ in }]L_0, L[\\ u(0,t) &= v(L,t) = 0 \\ u(L_0,t) &= v(L_0,t), \;\; b(L_0,t)u_x(L_0) = a(L_0,t)v_x(L_0) \end{aligned}$$

with t *a fixed value in* $[0, T]$ *satisfying*

$$u \in H^2(0, L_0) \text{ and } v \in H^2(L_0, L)$$

The existence result to the system (1.1) - (1.6) is summarized in the following theorem.

Theorem 3.3 *Suppose that* $\{u^0, v^0\} \in V$, $\{u^1, v^1\} \in L^2(0, L_0) \times L^2(L_0, L)$ *and that assumptions (A.1) - (A.3) hold. Then there exists a unique weak solutiont of (1.1) - (1.6) satisfying*

$$\{u, v\} \in C(0, T; V) \cap C^1(0, T; L^2(0, L_0) \times L^2(L_0, L)).$$

In addition, if $\{u^0, v^0\} \in H^2(0, L_0) \times H^2(L_0, L)$, $\{u^1, v^1\} \in V$, *verifying the compatibility condition below*

$$b(L_0, 0)u_x^0(L_0) = a(L_0, 0)v_x^0(L_0)$$

Then

$$\{u, v\} \in \bigcap_{k=0}^{2} W^{k,\infty}(0, T, H^{2-k}(0, L_0) \times H^{2-k}(L_0, L))$$

Proof. The main idea is to use the Galerkin Method.

4. Main Result: Exponential Decay.

In this section we prove that the solution of the system (1.1) - (1.6) decay exponentially as time goes to infinity. In the remainder of this paper we denote by c a positive constant which takes different values in different places. We shall suppose that $\rho_1 \leq \rho_2$ and

$$\begin{aligned} b(x,t) &= b > o, \ \forall (x,t) \in]0, L_0[\times]0, \infty[\\ a(x,t) &\leq b, \ a_t(x,t) \leq 0, \ \forall (x,t) \in]L_0, L[\times]0, \infty[\\ a_x(x,t) &\leq 0 \end{aligned}$$

Theorem 4.1. *Take* $p = 1$, $\{u^0, v^0\} \in V$ *and* $\{u^1, v^1\} \in L^2(0, L_0) \times L^2(L_0, L)$ *with*

$$u_x^0(L_0) = 0$$

then there exisis positive constants γ *and* c *such that*

$$E(t) \leq cE(0)e^{-\gamma t} \ , \ \forall t \geq 0.$$

We shall prove this theorem for strong solutions; our conclusion follow by standard density arguments.

The dissipative property of system (1.1) - (1.6) is given by the following lemma.

Lemma 4.2. *The first order energy satisfies*

$$\frac{d}{dt} E_1(t, u, v) = -(g(v_t), v_t) + (a_t, v_t^2)$$

Proof. Multiplying equation (1.1) by u_t, equation (1.2) by v_t and performing an integration by parts we get the result.

Let $\psi \in C_0^\infty(0, L)$ be such that $\psi = 1$ in $]L_0 - \delta, L_0 + \delta[$ for some $\delta > 0$, small constant.

Let us introduce the following functional

$$I(t) = \int_0^{L_0} \rho_1 u_t q u_x dx + \int_{L_0}^{L} \rho_2 v_t \psi q v_x dx$$

where $q(x) = x$.

Lemma 4.3. *There exists $c_1 > 0$ such that*

$$\frac{d}{dt}I(t) \leq -\frac{L_0}{2}\left\{(\rho_2-\rho_1)v_t^2(L_0,t)+a(L_0,t)\left[1-\frac{a(L_0,t)}{b}\right]v_x^2(L_0,t)\right\}$$

$$-L_0(F_1(u(L_0,t)) - F_2(v(L_0,t))) - \frac{1}{2}\int_0^{L_0}(\rho_1 u_t + bu_x^2 + 2F(u))dx$$

$$-\frac{1}{4}\int_{L_0}^{L_0+\delta} av_x^2 dx + c_1\left(\int_{L_0+\delta}^{L}\left(v_t^2+v_x^2\right)dx + \int_{L_0}^{L}(v^2+g(v_t)^2).dx + \int_0^{L_0} u^2 dx\right) + \epsilon E(t,u,v)$$

for any $\epsilon > 0$.

Proof. Multiplying equation (1.1) by qu_x, equation (1.2) by $\psi q v_x$ integrating by parts and using the corresponding boundary conditions we obtain the lemma.

Let $\varphi \in C^\infty(\mathbb{R})$ a nonnegative function such that $\varphi = 0$ in $I_{\delta/2} =]L_0 - \frac{\delta}{2}, L_0 + \frac{\delta}{2}[$ and $\varphi = 1$ in $\mathbb{R} \setminus I_\delta$ and consider the functional

$$J(t) = \int_{L_0}^{L} \rho_2 v_t \varphi v dx.$$

We have the following lemma

Lemma 4.4. *Given $\epsilon > 0$, there exists a positive constant c_ϵ such that*

$$\frac{d}{dt}J(t) \leq -\frac{1}{2}\int_{L_0+\delta}^{L} av_x^2 dx + \epsilon\left[\int_{L_0}^{L_0+\delta} av_x^2 dx + E(t,u,v)\right] + c_\epsilon \int_{L_0}^{L}\left(v_t^2 + g(v_t)^2 + v^2\right)dx$$

Let us consider the following functional

$$K(t) = I(t) + (2c_1 + 1)J(t)$$

and we take $\epsilon = \epsilon_1$ in lemma 4.4, where ϵ_1 is the solution of the equation

$$(2c_1 + 1)\epsilon_1 = \frac{1}{8}$$

taking in consideration (A.1) in lemma 4.3 we obtain

$$\frac{d}{dt}K(t) \leq -E_1(t,u) - \frac{1}{8}\int_{L_0}^{L}\left(av_x^2 + 2F_2(v)\right)dx + \epsilon E(t,u,v) + c_2\left(\int_{L_0}^{L}\left(v_t^2 + v^2\right)dx + \int_0^{L_0} u^2 dx\right) \tag{4.1}$$

Now in order to estimate the last two terms of (4.1) we need the following result

Lemma 4.5. *Let $\{u, v\}$ be a solution in theorem 3.3 Then there exists $T_0 > 0$ such that if $T \geq T_0$ we have*

$$\int_S^T \left(|v|^2 + |u|^2\right) ds \leq \epsilon \left[\int_S^T \left(\left|b^{1/2}u_x\right|^2 + |u_t|^2\right) ds + \int_S^T \left|a^{1/2}v_x\right|^2 ds\right] + c_\epsilon \int_S^T |v_t|^2 ds$$

for any $\epsilon > 0$ and c_ϵ is a constant depending on T and ϵ, by independent of $\{u, v\}$, for any initial data $\{u^0, v^0\}$, $\{u^1, v^1\}$ sattsfying $E(0, u, v) \leq R$, where $R > 0$ is fixed and $0 < S < T < +\infty$.

Proof. We use a contradiction method. (to see [3].)

Proof of theorem 4.1 Let us introduce the functional

$$L(t) = NE(t) + K(t)$$

with $N > 0$. Using Young's Inequality and taking N large enough we find that

$$\theta_0 E(t) \leq L(t) \leq \theta_1 E(t) \tag{4.2}$$

for some positive constants θ_0 and θ_1.

Applying the inequalities (4.1) and (4.2), along with the ones in Lemma 4.5 and integrating from S to T where $0 \leq S \leq T < \infty$ we obtain

$$\int_S^T E(t)dt \leq cE(S).$$

In this condition, lemma 2.1 implies that

$$E(t) \leq cE(0)e^{-rt}$$

this completes the proof.

5. Polynomial Decay.

In this section we study the asymptotic behavior of the solutious of system (1.1) - (1.6) when the function $g(s)$ is non-linear in a neighborhood to zero like s^p with $p > 1$. In this case we shall prove that the solution decays like $(1+t)^{-2(p-1)}$.

Theorem 5.1 *With the hypotheses in theorem 4.1 and* $p > 1$ *the weak solution decays polynomially, i. e.*

$$E(t) = CE(0)(1+t)^{-2/(p-1)}, \quad \forall t \geq 0$$

Proof. From (A.3) and making use of Hölder's inequality, theorem follows.

Remark. If we consider, in (1.2), a linear localized dissipation $\alpha = \alpha(x) \in C^2(]L_0, L[)$, $\alpha(x) = 1$ in $]L_0, L_0 + \delta[$, $\alpha(x) = 0$ in $]L_0 + 2\delta, L[$ our situation is very delicate and we need a new unique continuation theorem for the wave equation with variable coefficients. This is in preparation by the authors.

REFERENCES

[1] Andrade D., Fatori, L. H. and Rivera, J. E. - *Nonlinear transmission Problem with a Dissipative boundary condition of memory type*, Elect. J. DifF. Eq, vol 2006 (2006) N 53, pp. 1-16.

[2] Cabanillas, L. E. and Rivera, J. E. - *A Nonlinear Transmission Problem with Time Dependent Coefficients*, Elect. J. Diff. Eq, vol 2007 (2007) N 131, pp. 1-13.

[3] Cabanillas L.E. - *A Transmission Problem with Time Dependent Coefficients.*

[4] Komornik, V. - *Exact Controllability and Stabilization; The multiplier method*, Masson. Paris, 1994.

[5] Ladyzhenskaya, O. - *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

[6] Lasiecka, I. and Tataru, D. - *Uniform boundary Stabilization of Semilinear wave Equations wiht Nonlinear Boundary Damping*, Differential Integ. Eq. 6(3)(1993), pp. 507-533.

[7] Lagnese, J. - *Boundany Controllability in Problem of Transmission for a class of second order Hyperbolit Systems*, ESAIM: Control, Optim and Cal. Var. 2(1997), pp. 343-357.

[8] Lions, J. L. - *Quelques Methodes de Résolution dés Résolution dés problémes aux limites Nonlineaires*, Dunod, Gaulthier - Villars, Paris, 1969.

[9] Lions, J. L. - *Controlabilité Exacte, Perturbations et stabilization de Systems Distribués (tome I)*, collection RMA, Masson, Paris 1988.

[10] Liu, W. - *The exponential, The exponential stability of the problem of transmission of the wave equation*, Bulletin of the Austral. Math. Soc. 57(1998), pp. 305-327.

[11] Munoz R. and Oquendo H. P. - *The transmission probiem of Viscoelastic Waves*, Act. Appl. Math. 62 (2000), pp. 1-21.

[12] Ruiz A. - *Unique Continuation for weak Solutions of the wave Equation plus a Potential*, J. Math. Pures Appl. 71 (1992), pp. 455-467.